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Optimal stopping for a Cauchy problem without uniqueness [1]

by

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The optimal stopping time problem for a deterministic system described by the Cauchy problem $\dot{y}(t)=g(y(t))$, $y(0)=x$, where g is assumed to be only uniformly continuous, is considered. Under lower semicontinuity assumptions on the cost, it is proved that a solution exists and the Hamilton-Jacobi function still is lower semicontinuous. A convergence result for a discrete time version of the optimal stopping problem is given.

1. Introduction

In this paper we are concerned with the optimal stopping problem for a dynamical system in R^P whose evolution is given by the solution trajectories of the Cauchy problem

$$\begin{cases} \dot{y}(t)=g(y(t)) & t\in[0,T] \\ y(0)=x\in\bar{O} \end{cases} \tag{1}$$

where g is a uniformly continuous bounded vector field on R^P, O is an open bounded subset of R^P with a C^1 boundary ∂O. We shall make the following assumption

$$\langle g(x), n(x)\rangle>0 \qquad \forall x\in\partial O \tag{2}$$

where $n(x)$ is the outward normal to $\bar{O}$ at x. The optimal stopping problem for a dynamical system of type (1) where g is Lipschitz continuous is a classical problem of control theory, we refer, for the general results mainly to Bensoussan [1], Bensoussan-Lions [2] and Menaldi [10].

Under our assumptions, we have not a uniqueness result for the solution trajectories, however, as it is well known, the set of solutions is a compact subset of $C^0([0,T]; R^P)$ (see e.g. [9], [11]). This particular situation leads us to modify

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the classical stopping problem, since we have to find not only the time but also the trajectory which guarantee the minimum of the cost functional.

Section 1 contains the assumptions and notations we use all over the paper. In Section 2 we give an existence theorem for the optimal problem (P). The proof relies on the lower semicontinuity (l.s.c.) assumption of the transition cost f and the stopping cost ψ, joined with the compactness of the set of solution trajectories of (1) in $C^0([0, T]; R^p)$. We refer, for a similar problem, dealing with, a multivalued dynamical system in discrete time, to Capuzzo Dolcetta-Falcone [3], where the lack of uniqueness is due to the fact that g is a multivalued mapping.

In Section 3 we give the characterization of the optimal solution by the Hamilton-Jacobi function $u(x)$. Section 4 is devoted to the problem of the regularity of u. We prove that u is l.s.c. if f and ψ are l.s.c. and an example shows that we cannot expect the continuity of u even if the costs are very regular. We notice, however, that the continuity of f and ψ implies the continuity of u whenever the Cauchy problem (1) has a unique solution, since, in this case, we have continuous dependence on the data without any regularity assumption on g. In Section 5 we describe a discrete time version of problem (1), which is a suitable generalization of the classical Euler scheme. We consider in Section 6 a discrete time version of the optimal stopping problem for the solution trajectories of that iterative scheme. We prove that the related Hamilton-Jacobi functions $u^{\Delta t}$ uniformly converge as $\Delta t \to 0$, to u in $\bar{O}$, when f and ψ are continuous. A similar kind of discretization for the classical deterministic optimal stopping problem was proposed by Capuzzo Dolcetta-Matzeu in [4], [5], (see also Capuzzo Dolcetta-Matzeu-Menaldi [6], Gawronski [7], Goletti [8] for other deterministic control problems).

2. Assumptions and notations

— O is a *bounded open* subset of R^p with a C^1 boundary ∂O; $n(x)$ is the outward normal to $\bar{O}$ at $x \in \partial O$.

— g is a *uniformly continuous bounded* vector field on R^p.

— Y_x is the set of solutions trajectories of the Cauchy problem (1).

— $\tau_x(y_x(\cdot)) = \begin{cases} \inf\{t \in [0, T]: y_x(t) \notin \bar{O}\} =: \inf \tau_x \text{ if } \tau_x \neq \emptyset \\ T \text{ if } \tau_x = \emptyset \end{cases}$
for any $y_x(\cdot) \in Y_x$.

— f and ψ are real *bounded measurable* functions on $\bar{O}$.

— $N^{\Delta t} = [T/\Delta t]$, where $0 < \Delta t < T$ and $[\cdot]$ denotes the integer part

— $Y^{\Delta t}$ is the set of the $N^{\Delta t}$-sequences defined by the discrete iterative scheme

$$\left| \begin{array}{l} y_x^{\Delta t}((k+1)\Delta t) = y_x^{\Delta t}(k\Delta t) + \Delta t\, g(\tilde{y}_x^{\Delta t}(k\Delta t)), \quad k = 0, 1, \dots, N^{\Delta t} - 1 \\ y_x^{\Delta t}(0) = x \in \bar{O} \end{array} \right. \tag{3}$$

where $\tilde{y}_x^{\Delta t}(k\Delta t)$ satisfies the condition

$$|\tilde{y}_x^{\Delta t}(k\Delta t)-y_x^{\Delta t}(k\Delta t)|\leqslant M\Delta t, \qquad \text{with } M=\max_{x\in\bar{O}}|g(x)| \tag{4}$$

— $y_x^{\Delta t}(\cdot)$ is the *piecewise linear* function defined as follows

$$y_x^{\Delta}(t)=\begin{cases} y_x^{\Delta t}(k\Delta t)+\left(\dfrac{t-k\Delta t}{\Delta t}\right)\left(y_x^{\Delta t}((k+1)\Delta t)-y_x^{\Delta t}(k\Delta t)\right), \\ \qquad\qquad \forall\, t\in[k\Delta t,(k+1)\Delta t], \quad k=0,\dots,N^{\Delta t}-1 \\ y_x^{\Delta t}(N^{\Delta t}\Delta t)\ \forall\, t\in[N^{\Delta t}\Delta t, T] \end{cases} \tag{5}$$

— $C^0([0,T])$ will denote both the spaces $C^0([0,T],\boldsymbol{R})$ and $C^0([0,T],\boldsymbol{R}^P)$ provided with their maximum norm $\|\cdot\|_\infty$

— $C^{0,\beta}(\bar{O})$, with $\beta\in[0,1]$, will denote the space of β-Hölder continuous functions defined on $\bar{O}$ with values in $\boldsymbol{R}$.

3. Formulation of the problem and existence of solutions

Let us define for any $x\in O$ the cost functional $J_x: Y_x\times[0,T]\to\boldsymbol{R}$ as

$$J_x(y_x(\cdot),t)=\int_0^{t\wedge\tau_x(y_x(\cdot))} f(y_x(s))e^{-\alpha s}ds+\chi_{t<\tau_x(y_x(\cdot))}\,\psi(y_x(t))\times$$
$$\times e^{-\alpha t}\ \forall\, y_x(\cdot)\in Y_x,\ t\in[0,T].$$

We notice that the cost functional does not depend on the values of f and ψ on ∂O, so we assume

$$f(x)=\psi(x)=0 \qquad \forall x\in\partial O. \tag{6}$$

The tangential condition (2) allows us to write the cost functional as

$$J_x(y_x(\cdot),t)=\int_0^t f(y_x(s))e^{-\alpha s}ds+\psi(y_x(t))e^{-\alpha t},$$

simply by extending f and ψ as zero functions in $\boldsymbol{R}^P\setminus\bar{O}$. Since the Cauchy problem (1) has not a unique solution, we shall define the optimal stopping problem as follows:

$$(P)\quad \left|\begin{array}{l}\text{For any } x\in O \text{ find a pair } (y_x^*(\cdot),t^*)\in Y_x\times[0,T] \text{ such that}\\ J_x(y_x^*(\cdot),t^*)=\operatorname*{Inf}_{y_x(\cdot)\in Y_x}\ \operatorname*{Inf}_{t\in[0,T]} J_x(y_x(\cdot),t)\end{array}\right.$$

Let us prove now an existence result.

THEOREM 1. *Let (2), (6) be verified and let f and ψ be lower semicontinuous (l.s.c.) on $\bar{O}$. Then, for any $x\in\bar{O}$, there exists a solution $(y_x^*(\cdot),t^*)$ of (P).*

Proof. Since Y_x is a compact subset of $C^0([0, T])$, it is enough to show that $J_x(y_x(\cdot), t)$ is lower semicontinuous on $Y_x \times [0, T]$. In fact, let $t_n \to t$ in $[0, T]$ and $y_n \to y$ in Y_x, then

$$J_x(y_n(\cdot), t_n) - J_x(y(\cdot), t) = \int_0^{t_n} f(y_n(s)) e^{-\alpha s} ds - \int_0^t f(y(s)) e^{-\alpha s} ds +$$
$$+ \psi(y_n(t_n)) e^{-\alpha t_n} - \psi(y(t)) e^{-\alpha t} \geqslant -\|f\|_\infty |t_n - t| +$$
$$+ \int_0^t (f(y_n(s)) - f(y(s))) e^{-\alpha s} ds - \|\psi\|_\infty | e^{-\alpha t} - e^{-\alpha t}| +$$
$$+ (\psi(y_n(t_n)) - \psi(y(t))) e^{-\alpha t}$$

therefore, by Fatou's Lemma, J_x is l.s.c. ■

4. Characterization of the optimal solution via Hamilton-Jacobi

Let us give a characterization of the optimal pair $(y^*(\cdot), t^*)$ in terms of the Hamilton-Jacobi function

$$u(x) = \operatorname*{Inf}_{y_x(\cdot) \in Y_x} \operatorname*{Inf}_{t \in [0,T]} J_x(y_x(\cdot), t). \tag{7}$$

The results of this section stand on the dynamic programming principle, which can be easily proved.

PROPOSITION 1 (DP Principle). *For any* $x \in O$, $y_x(\cdot) \in Y_x$ and t, $\bar{t} \in [0, T]$, *with* $t \geqslant \bar{t}$, *the following equality holds true*

$$J_x(y_x(\cdot), t) = \int_0^{\bar{t}} f(y_x(s)) e^{-\alpha s} ds + J_{y_x(\bar{t})}(y_x(\cdot), t - \bar{t}) e^{-\alpha \bar{t}}$$

where we identify the piece of trajectory $y_x(t)$, $t > \bar{t}$, *and the trajectory* $y_{y_x(\bar{t})}(\cdot)$.

THEOREM 2. *Let (2), (6) be verified and* f, ψ *be l.s.c. Then* u *is the maximum solution of the inequality system*

$$\begin{cases} u(x) \leqslant \psi(x) & \forall x \in \bar{O} \\ u(x) \leqslant \operatorname*{Inf}_{y_x(\cdot) \in Y_x} \left(\int_0^t f(y_x(s)) e^{-\alpha s} ds + u(y_x(t)) e^{-\alpha t} \right) & \forall x \in \bar{O}, \quad \forall t \in [0, T] \\ u(x) = 0 & \forall x \in \partial O \end{cases} \tag{8}$$

Proof. The first inequality is trivial, since, for any $y_x(\cdot) \in Y_x$, $x \in \bar{O}$, it results

$$J_x(y_x(\cdot), 0) = \psi(x)$$

Furthermore, by Proposition 1, for any $y_x(\cdot)\in Y_x$, $x\in\bar{O}$ and $t\in[0,T]$, we have

$$u(x)\leqslant\int_0^t f(y_x(s))e^{-\alpha s}ds+\operatorname*{Inf}_{y(\cdot)\in Y_{y_x(t)}}\ \operatorname*{Inf}_{\tau\in[t,T]} J_{y_x(t)}(y(\cdot),\tau-t)e^{-\alpha t}=$$

$$=\int_0^t f(y_x(s))e^{-\alpha s}ds+u(y_x(t))e^{-\alpha t}$$

that is the second inequality in (8).

By the tangential condition (2), one has

$$J_x(y_x(\cdot),t)=0 \quad \forall\, x\in\partial O, \quad y_x(\cdot)\in Y_x, \quad t\in[0,T],$$

hence the third relation follows.

Finally, let us prove that u is the maximum solution of (8). Let w be a solution of (8), then, for any $y_x(\cdot)\in Y_x$ and $t\in[0,T]$, we have

$$w(x)\leqslant\int_0^t f(y_x(s))e^{-\alpha s}ds+\psi(y_x(t))e^{-\alpha t}=J_x(y_x(\cdot),t),$$

then $w(x)\leqslant u(x)$, $\forall\, x\in\bar{O}$. ■

Let now $y_x^*(\cdot)$ denote an *optimal trajectory* for $J_x(\cdot,\cdot)$ in the sense that there exists some $t\in[0,T]$ such that $u(x)=J_x(y_x^*(\cdot),t)$, and let C be defined as

$$C=\{t\in[0,T]:\ u(y_x^*(t))=\psi(y_x^*(t))\}.$$

We can state the following

THEOREM 3. *Let (2), (6) be verified and f, ψ be l.s.c. If the function Ψ: $[0,T]\to R$ defined as*

$$\Psi(t)=\psi(y_x^*(t)) \quad \forall\, t\in[0,T]$$

is continuous, then C is not empty and the time

$$t_x^*=\operatorname{Inf}\{t\in C\}$$

is a minimum and is optimal on $y_x^*(\cdot)$, that is $u(x)=J_x(y_x^*(\cdot),t_x^*)$.

Proof. First of all, let us observe that any $t\in[0,T]$ such that $u(y_x^*(t))<\psi(y_x^*(t))$ cannot be optimal on $y_x^*(\cdot)$ since the second relation in (8) gives

$$u(x)<\int_0^t f(y_x^*(s))e^{-\alpha s}ds+\psi(y_x^*(t))e^{-\alpha t}=J_x(y_x^*(\cdot),t).$$

Therefore, by the optimality of $y_x^*(\cdot)$, C cannot be empty.

Now we want to prove that t_x^* is the minimum of C. It is enough to show that the function

$$U(t)=u(y_x^*(t))$$

is continuous in $[0, \bar{t}_x]$, where $\bar{t}_x \in [0, T]$ is an arbitrarily fixed optimal stopping time for the trajectory $y_x^*(\cdot)$ (so one has $\bar{t}_x \geq t_x^*$).

Indeed, for any $y(\cdot) \in Y_{y_x^*(t)}$ and $\Theta \in [t, T]$, we have:

$$J_{y_x^*(t)}(y(\cdot), \Theta - t) = \int_0^{\Theta - t} f(y(s)) e^{-\alpha s} ds + \psi(y(\Theta - t)) e^{-\alpha(\Theta - t)}.$$

Let us define

$$\bar{y}_x(\eta) = \begin{cases} y_x^*(\eta) & \text{for } \eta \in [0, t] \\ y(\eta - t) & \text{for } \eta > t \end{cases}$$

It is easy to verify that

$$J_{y_x^*(t)}(y(\cdot), \Theta - t) = \left[J_x(\bar{y}_x(\cdot), \Theta) - \int_0^t f(y_x^*(\eta)) e^{-\alpha\eta} d\eta \right] e^{\alpha t}.$$

Now, for any $t \in C$, such that $t < \bar{t}_x$,

$$u(y_x^*(t)) = \left[\operatorname*{Inf}_{y(\cdot) \in y_x^*(t)} \left(\operatorname*{Inf}_{\theta > t} J_x(y(\cdot), \Theta) \right) - \int_0^t f(y_x^*(\eta)) e^{-\alpha\eta} d\eta \right] e^{\alpha t} =$$

$$= \left[u(x) - \int_0^t f(y_x^*(\eta)) e^{-\alpha\eta} d\eta \right] e^{\alpha t},$$

that is $u(y_x^*(\cdot))$ is continuous for any $t \in [0, \bar{t}_x]$, then t_x^* is a minimum.

At this point, since $u(x) < J_x(y_x^*(\cdot), t)$ $\forall\, t < t_x^*$, then

$$u(x) = \operatorname*{Min}_{t \in [t_x^*, T]} J_x(y_x^*(\cdot), t) = \int_0^{t_x^*} f(y_x^*(s)) e^{-\alpha s} ds + \operatorname*{Min}_{t \in [t_x^*, T]} \left(\int_{t_x^*}^t f(y_x^*(s)) e^{-\alpha s} ds + \right.$$

$$\left. + \psi(y_x^*(t)) e^{-\alpha t} \right) = \int_0^{t_x^*} f(y_x^*(s)) e^{-\alpha s} ds + u(y_x^*(t_x^*)) e^{-\alpha t_x^*} = J_x(y_x^*(\cdot), t_x^*)$$

that is $(y_x^*(\cdot), t_x^*)$ is an optimal pair. ■

5. Regularity of the Hamilton-Jacobi function

One can easily show that if $\psi(t)$ is continuous then $U(t)$ is also continuous. An interesting question is: does the continuity of f and ψ on O imply the same property for u? The answer is no and the following example shows that in general u is not continuous, also if f and ψ are very regular.

Example 1.

Let us consider the Cauchy problem in R

$$\dot{y}(t) = \sqrt{|y(t)|} \qquad \forall\, t > 0$$
$$y(0) = x_0 \in [-1, 1]$$

and the associated cost functional related to $\psi(x)=0\ \forall x\in[-1,1]$ and to an arbitrary $f\in C^0([-1,1])$ such that $f(x)=-x\ \forall x\in[-1/2,1/2]$ and $f(-1)=f(1)=0$.

For any $x_0\neq 0$ there exists a unique solution trajectory. The solution is given by

$$y_{x_0}(t)=\frac{1}{4}(t+2\sqrt{x_0})^2 \qquad \text{when } x_0>0$$

and

$$y_{x_0}(t)=\begin{cases} -\frac{1}{4}(t-2\sqrt{-x_0})^2 & \text{for } 0\leqslant t\leqslant 2\sqrt{-x_0} \\ 0 & \text{for } t>2\sqrt{-x_0} \end{cases}$$

when $x_0<0$.

When $x_0=0$, there exist infinite solution trajectories: in fact, for any $b\geqslant 0$,

$$y_0(t)=\begin{cases} 0 & \text{for } 0\leqslant t\leqslant b \\ \frac{1}{4}(t-b)^2 & \text{for } t>b \end{cases}$$

is a solution of the Cauchy problem. In particular $y_0(t)\equiv 0$ and $y_0(t)=\frac{1}{4}t^2$ are two solution trajectories (they are respectively the minimal and maximal solution for our Cauchy problem). Therefore, for any $x_0\in\left[-\frac{1}{2},0\right]$, $J_{x_0}(y_{x_0}(\cdot),t)>0$ $\forall t>0$, hence

$$u(x_0)=J_{x_0}(y_{x_0}(\cdot),0)=\psi(x_0)=0.$$

Since obviously we have $u(0)<0$, $u(x)$ is left-discontinuous at 0.

Nevertheless, when f and ψ have the "minimal" regularity which guarantees the existence of an "optimal couple", i.e. lower semicontinuity, then also u has the same property.

THEOREM 4. *Let (2), (6) be verified and f ψ be l.s.c. Then u is l.s.c.*

Proof. Since $u(x)=\underset{t\in[0,T]}{\mathrm{Min}}\ \underset{y_x\in Y_x}{\mathrm{Min}}\ J_x(y_x(\cdot),t)$, it suffices to show that, for any $t\in[0,T]$, the function

$$u_t(x)=\underset{y_x\in Y_x}{\mathrm{Min}}\ J_x(y_x(\cdot),t)$$

is l.s.c. in x. The result will follow from the general fact that, if $\{F_t(x)\}_{t\in K}$ is a family of functions l.s.c. with respect to the pair (t,x) and K is a compact space, then the function $G(x)=\underset{t\in K}{\mathrm{Min}}\ F_t(x)$ is l.s.c. (see e.g. [1]).

Let $t_n\to t$, $x_n\to x$ in $\bar{O}$ and y_n be an optimal trajectory defined in $[0,T]$ and starting at x_n. Since $\{y_n\}$ is a compact subset of $C^0([0,T])$, there exists at least a subsequence $\{y_{n_k}\}$ which converges to an element $\tilde{y}\in C^0([0,T])$. It is easy to show that $\tilde{y}\in Y_x$, then by the l.s.c. of f and ψ, we get

$$\underset{k\to+\infty}{\underline{\lim}}\ u_{t_{n_k}}(x_{n_k})=\underset{k\to+\infty}{\underline{\lim}}\ J_{x_{n_k}}(y_{n_k}(\cdot),t_{n_k})\geqslant \underset{k\to+\infty}{\underline{\lim}}\int_0^{t_{n_k}} f(y_{n_k}(s))e^{-\alpha s}\,ds+$$

$$+\underset{k\to+\infty}{\underline{\lim}}\ \psi(y_{n_k}(t_{n_k}))e^{-\alpha t}\geqslant\int_0^t f(\tilde{y}(s))e^{-\alpha s}\,ds+\psi(\tilde{y}(t))e^{-\alpha t}\geqslant u_t(x).$$

We can repeat the same argument for the sequence $\{x_k'\}=\{x_n\}/\{x_{n_k}\}$, so

$$\lim_{n\to+\infty} u_{tn_k}(x_n)\geqslant u_t(x)$$

hence $u_t(x)$ is l.s.c. with respect to (t,x) and u is l.s.c. too. ■

Remark 1. As we said in the introduction, the continuity of u comes out from the continuity of f and ψ whenever (1) has a unique solution. This result is easy to prove, since it is well known (see e.g. [9], [11]) that the hypotheses guaranteeing the uniqueness of the solution imply the continuous dependence on the data, then, if we are in this situation, the continuity of u simply follows by the continuity of the cost functional with respect to x.

6. Discretization of the dynamical system and convergence of solutions

Let $\Delta t>0$ and consider the iterative scheme (3), (4), that we recall for reader's convenience

$$y_x^{\Delta t}((k+1)\Delta t)=y_x^{\Delta t}(k\Delta t)+\Delta t g(\tilde{y}_x^{\Delta t}(k\Delta t)),\qquad \mathrm{p}=0,1,\dots,N^{\Delta t}-1$$
$$y_x^{\Delta t}(0)=x\in\bar{O},$$

where $\tilde{y}_x^{\Delta t}(k\Delta t)$ is an arbitrary element satisfying the condition

$$|\tilde{y}_x^{\Delta t}(k\Delta t)-y_x^{\Delta t}(k\Delta t)\leqslant M\Delta t,\ \text{with}\ M=\|g\|_\infty.$$

The relation between the solutions of problem (1) and those of the iterative scheme is stated by the following

Theorem 5. *For any $y_x(\cdot)\in Y_x$, there exists $y_x^{\Delta t}\in Y_x^{\Delta t}$ such that*

$$y_x(k\Delta t)=y_x^{\Delta t}(k\Delta t)\qquad k=0,\dots,N^{\Delta t}$$

Conversely, for every family $\{y_x^{\Delta t}\}_{\Delta t>0}$, with $y_x^{\Delta t}\in Y_x^{\Delta t}$, there exists a sequence $\Delta t_n\to 0$ such that the piecewise linear function $y_x^{\Delta t_n}(t)$ defined in (5) uniformly converges in $[0,T]$, as $n\to+\infty$, to an element $y_x(\cdot)\in Y_x$.

Proof. As far as the first statement is concerned, we notice that there exists $\vartheta\in(k,k+1)$ such that

$$y_x((k+1)\Delta t)=y_x(k\Delta t)+\Delta t\dot{y}_x(\vartheta\Delta t)=y_x(k\Delta t)+\Delta t g(y_x(\vartheta\Delta t)).$$

so $y_x^{\Delta t}$ can be defined by choosing

$$\tilde{y}_x^{\Delta t}(k\Delta t)=y_x(\vartheta\Delta t),$$

and condition (4) is satisfied, since, for every $k=0,\dots,N^{\Delta t}$,

$$|\tilde{y}_x^{\Delta t}(k\Delta t)-y_x^{\Delta t}(k\Delta t)|=|y_x(\vartheta\Delta t)-y_x(k\Delta t)|=|\int_{k\Delta t}^{\vartheta\Delta t} g(y_x(s))\,ds|\leqslant M\Delta t$$

Let $\{y_x^{\Delta t}\}_{\Delta t>0}$ be a family of elements in $Y_x^{\Delta t}$. The family of functions $\{y_x^{\Delta t}(\cdot)\}_{\Delta t>0}$, where $y_x^{\Delta t}(\cdot)$ is defined in (5), is a compact subset of $C^0([0,T])$. In fact it is bounded in $C^0([0,T])$, since

$$\max_{t\in[0,T]} |y_x^{\Delta t}(t)| = \max_{k\in\{0,\dots,N^{\Delta t}\}} |y_x^{\Delta t}(k\Delta t)| \leqslant |x| + MN^{\Delta t}\Delta t \leqslant |x| + MT$$

Moreover it is equicontinuous; in fact let t, $s\in[0,T]$, $t<s$ and let j, $k\in\{0,\dots,\dots,N^{\Delta t}\}$ such that $t\in[k\Delta t,(k+1)\Delta t]$, $s\in[j\Delta t,(j+1)\Delta t]$, then we have

$$|y_x^{\Delta t}(t)-y_x^{\Delta t}(s)| \leqslant |y_x^{\Delta t}(k\Delta t)+(t-k\Delta t)\,g(\tilde{y}_x^{\Delta t}(k\Delta t))-y_x^{\Delta t}(j\Delta t)+(s-j\Delta t)\,g(\tilde{y}_x^{\Delta t}(j\Delta t))|$$

If $k=j$, it easily follows that

$$|y_x^{\Delta t}(t)-y_x^{\Delta t}(s)| \leqslant M\,|t-s|,$$

whereas $k<j$ implies

$$\begin{aligned}|y_x^{\Delta t}(t)-y_x^{\Delta t}(s)| &\leqslant |y_x^{\Delta t}(k\Delta t)+(t-k\Delta t)\,g(\tilde{y}_x^{\Delta t}(k\Delta t))-y_x^{\Delta t}((k+1)\Delta t)|+\\ &+\sum_{h=k+1}^{j-1}|y_x^{\Delta t}(h\Delta t)-y_x^{\Delta t}((h+1)\Delta t)|+|y_x^{\Delta t}(j\Delta t)+(s-j\Delta t)\,g(\tilde{y}_x^{\Delta t}(j\Delta t))-\\ &-y_x^{\Delta t}(j\Delta t)| \leqslant ((k+1)\Delta t-t)M+(j-k-1)\Delta tM+(s-j\Delta t)M=M\,|t-s|\end{aligned}$$

By the compactness of $\{y_x^{\Delta t}(\cdot)\}_{\Delta t>0}$ in $C^0([0,T])$, it follows that a sequence $y_x^{\Delta t_n}(\cdot)$ converges to some $y_x(\cdot)$ uniformly in $[0,T]$.
Let us prove that $y_x(\cdot)\in Y_x$. We define

$$\tilde{g}(\tilde{y}_x^{\Delta t_n}(s)) = \begin{cases} g(\tilde{y}_x^{\Delta t_n}(k\Delta t)) & \text{if } s\in[k\Delta t,(k+1)\Delta t] \quad k=0,1,\dots,N^{\Delta t}-1\\ 0 & \text{if } s\in[N^{\Delta t}\Delta t, T]\end{cases}$$

so that, for $n\in\mathbb{N}$ and $t\in[0,T]$,

$$y_x^{\Delta t_n}(t)=x+\int_0^t \tilde{g}(\tilde{y}_x^{\Delta t_n}(s))\,ds.$$

Since $\dot{y}_x(\cdot)\in Y_x$ if and only if

$$y_x(t)=x+\int_0^t g(y_x(s))\,ds \qquad \forall\, t\in[0,T],$$

it is enough to prove that

$$\lim_{n\to+\infty}\int_0^t (\tilde{g}(\tilde{y}_x^{\Delta t_n}(s))-g(y_x^{\Delta t_n}(s))\,ds=0 \qquad \forall\, t\in[0,T] \tag{9}$$

Actually, if $t\in[0,T]$ and $k=\left[\dfrac{t}{\Delta t_n}\right]$, one has

$$|\int_0^t (\tilde{g}(\tilde{y}_x^{\Delta t_n}(s)) - g(y_x^{\Delta t_n}(s)))\,ds| = |\sum_{j=0}^{k-l} \int_{j\Delta t_n}^{(j+l)\Delta t_n} (g(\tilde{y}_x^{\Delta t_n}(j\Delta t_n)) - g(y_x^{\Delta t_n}(s)))\,ds +$$

$$+ \int_{k\Delta t_n}^{t} \{g(\tilde{y}_x^{\Delta t_n}(k\Delta t_n)) - g(y_x^{\Delta t_n}((k\Delta t_n)) - (s-k\Delta t_n)\,g(\tilde{y}_x^{\Delta t_n}(k\Delta t_n)))\}\,ds|$$

$$\leqslant N^{\Delta t_n}\Delta t_n \sup_{\substack{j\in\{0,\dots,N^{\Delta t_n}-1\\ s\in[j\Delta t_n,(j+1)\Delta t_n]}} |g(\tilde{y}_x^{\Delta t_n}(j\Delta t_n)) - g(y_x^{\Delta t_n}(s))| + 2M\Delta t_n \tag{10}$$

Let us notice that, for every $j\in\{0,\dots,N^{\Delta t_n}-1\}$ and $s\in[j\Delta t_n,(j+1)\Delta t_n]$, we have by (4)

$$|\tilde{y}_x^{\Delta t_n}(j\Delta t_n) - y_x^{\Delta t_n}(s)| = |\tilde{y}_x^{\Delta t_n}(j\Delta t_n) - y_x^{\Delta t_n}(j\Delta t_n) - (s-j\Delta t_n)\,g(\tilde{y}_x^{\Delta t_n}(j\Delta t_n))| \leqslant 2M\Delta t_n,$$

so

$$|\int_0^t (\tilde{g}(\tilde{y}_x^{\Delta t_n}(s)) - g(y_x^{\Delta t_n}(s)))\,ds| \leqslant T \sup_{\substack{x,y\in\mathbf{R}^p\\ |x-y|\leqslant 2M\Delta t_n}} |g(x)-g(y)| + 2M\Delta t_n$$

By the uniform continuity of g we obtain the statement. ∎

7. Convergence of $u^{\Delta t}(x)$

Let us give now the "discrete time" version of problem (P), that is suggested by the statement of Theorem 5. For every $\Delta t>0$, we define the cost functional $J_x^{\Delta t}$ on $Y_x^{\Delta t}\times\{0,\dots,N^{\Delta t}\}$ as follows

$$J_x^{\Delta t}(y_x^{\Delta t},k) = \Delta t \sum_{j=0}^{k-1} f(y_x^{\Delta t}(j\Delta t))(1-\alpha\Delta t)^j + \psi(y_x^{\Delta t}(k\Delta t))(1-\alpha\Delta t)^k.$$

We shall consider in this section the optimal stopping problem in discrete time

$(P^{\Delta t})$ | For any $x\in\bar{O}$ find a pair $(y_x^{*\Delta t},k_x^*)$ in $Y_x^{\Delta t}\{0,\dots,N^{\Delta t}\}$ such that
$$u^{\Delta t}(x) = \operatorname*{Inf}_{y_x^{\Delta t}\in Y_x^{\Delta t}} \operatorname*{Inf}_{k\in\{0,\dots,N^{\Delta t}\}} J_x^{\Delta t}(y_x^{\Delta t},k) = J_x^{\Delta t}(y_x^{*\Delta t},k_x^*)$$

The following existence theorem for $(P^{\Delta t})$ can be easily proved by the same arguments as used in Theorem 1.

THEOREM 6. *Let f, ψ be l.s.c. on $\bar{O}$. Then, for any $x\in\bar{O}$, there exists at least a pair $(y_x^{*\Delta t},k_x^*)$, solution of $(P^{\Delta t})$.*

Our aim is to show that the solutions of $(P^{\Delta t})$ in some sense approximate the solutions of (P). We begin by proving a regularity result for $u^{\Delta t}$ which does not hold for u.

THEOREM 7. *Let f, $\psi \in C^{0,\beta}(\bar{O})$ and let g be γ-Hölder continuous on R^P namely*

$$|g(x)-g(y)| \leqslant c_g |x-y|^\gamma \qquad \forall\, x, y \in R^P,$$

then $u^{\Delta t}$ is δ-Hölder continuous with $\delta = \gamma^{N^{\Delta t}}\beta$.

Proof. Let $x_1, x_2 \in \bar{O}$ and let $y_{x_1}^{\Delta t} \in Y_{x_1}^{\Delta t}$. By simply choosing

$$\tilde{y}_{x_2}^{\Delta t}(k\Delta t) = y_{x_2}^{\Delta t}(k\Delta t) + \tilde{y}_{x_1}^{\Delta t}(k\Delta t) - y_{x_1}^{\Delta t}(k\Delta t) \quad k=0, \ldots, N^{\Delta t}-1$$

we can define a solution of the discrete iterative scheme (3), (4) starting at x_2, i.e. $y_{x_2}^{\Delta t} \in Y_{x_2}^{\Delta t}$. Moreover, we get the following estimate, for any $x_1, x_2 \in \bar{O}$ such that $|x_1 - x_2| < 1$:

$$|y_{x_2}^{\Delta t}(k\Delta t) - y_{x_1}^{\Delta t}(k\Delta t)| \leqslant (1 - c_g \Delta t)^{-k} |x_1 - x_2|^{\gamma^k} \tag{11}$$

In fact (11) is true for $k=1$; let (11) hold for $k=h>1$, then

$$\begin{aligned} &|y_{x_2}^{\Delta t}((h+1)\Delta t) - y_{x_1}^{\Delta t}((h+1)\Delta t)| \leqslant |y_{x_2}^{\Delta t}(h\Delta t) - y_{x_1}^{\Delta t}(h\Delta t)| + \\ &+ \Delta t\, |g(\tilde{y}_{x_2}^{\Delta t}(h\Delta t)) - g(\tilde{y}_{x_1}^{\Delta t}(h\Delta t))| \leqslant (1 - c_g \Delta t)^{-h} |x_1 - x_2|^{\gamma^h} + \\ &+ \Delta t c_g (1 - c_g \Delta t)^{-h} |x_1 - x_2|^{\gamma^{h+1}} \leqslant (1 - c_g \Delta t)^{-h-1} |x_1 - x_2|^{\gamma^{h+1}} \end{aligned}$$

By the β-Hölder continuity of f and ψ (we denote c_f and c_ψ respectively their Hölder constants), we obtain, applying (11),

$$\begin{aligned} |J_{x_1}^{\Delta t}(y_{x_1}^{\Delta t}, k) - J_{x_2}^{\Delta t}(y_{x_2}^{\Delta t}, k)| &\leqslant c_f \Delta t \sum_{j=0}^{k-1} \left(\frac{1-\alpha\Delta t}{(1-c_g \Delta t)} \beta \right)^j |x_1 - x_2|^{\gamma^j \beta} + \\ &+ c_\psi \left(\frac{1-\alpha\Delta t}{(1-c_g \Delta t)} \beta \right)^k |x_1 - x_2|\, \gamma^k \beta \leqslant C\, |x_1 - x_2|^{\gamma^{N^{\Delta t}} \beta} \end{aligned}$$

In the last estimate the constant C depends only on $N^{\Delta t}$ so that, by a simple argument, we get the $\gamma^{N^{\Delta t}}$ β-Hölder continuity of $u^{\Delta t}$. ■

REMARK. Let us notice that $\{u^{\Delta t}\}_{\Delta t > 0}$ in general is not equicontinuous in $\bar{O}$, even if $u^{\Delta t}$ is Hölder continuous for every $\Delta t > 0$. In [4], under Lipschitz continuity assumptions on g, a classical Euler scheme defining $y_x^{\Delta t}$ was used to show that the assumption $\alpha > c_g$ leads to the equi-Hölder continuity of $u^{\Delta t}$. In our case, $\alpha > c_g$ still gives an upper bound independent on $N^{\Delta t}$ for the constraint $C(N^{\Delta t})$, but unfortunately the Hölder exponent of $u^{\Delta t}$, namely $\gamma^{N^{\Delta t}}\beta$, goes to 0 as $\Delta t \to 0$. However, even if $\{u^{\Delta t}\}_{\Delta t > 0}$ is not equicontinuous in $C^0([0, T])$, we can state a convergence result for the approximated problems $(P^{\Delta t})$ to problem (P).

THEOREM 8. *Let (2), (6) be verified and let f, ψ be continuous. Then, for every $x \in \bar{O}$, there exists $\Delta t_n \to 0$, $h_n \in \{0, \ldots, N^{\Delta t_n}\}$, $y_n^{\Delta t} \in Y_x^{\Delta t_n}$, with $u^{\Delta t_n}(x) = J_x^{\Delta t_n}(y_n^{\Delta t_n}, h_n)$ such that $(y_x^{\Delta t_n}(\cdot), h_n^{\Delta t_n})$ converges in $C^0([0, T]) \times [0, T]$ to an optimal couple $(y_x(\cdot), t) \in Y_x \times [0, T]$. Furthermore, if $u(x) = J_x(y_x(\cdot), t)$, then, $u^{\Delta t}(x) \to u(x)$ uniformly in $\bar{O}$.*

The proof relies on the following

LEMMA 1. *Let the assumptions of Theorem 8 be verified and let* $\{y_x^{\Delta t_n}\} \subset Y_x^{\Delta t_n}$, $\{h_n\} \subset N$ *be such that* $y_x^{\Delta t_n}(\cdot) \to y_x(\cdot)$ *in* $C^0([0,T])$ *and* $h_n \Delta t_n \to t$ *in* R. *Then* $y_x(\cdot) \in Y_x$ *and* $J_x^{\Delta t_n}(y_x^{\Delta t_n}, h_n)$ *converges to* $J_x(y_x(\cdot), t)$.

Proof. The first statement is contained in the proof of Theorem 5. For the second statement,

$$|J_x^{\Delta t_n}(y_x^{\Delta t_n}, h_n) - J_x(y_x(\cdot), t)| \leq |\Delta t_n \sum_{j=0}^{h_n-1} f(y_x^{\Delta t_n}(j\Delta t_n))(1-\alpha\Delta t_n)^j -$$

$$-\int_0^t f(y_x(s))e^{-\alpha s}\,ds| + |\psi(y_x^{\Delta t_n}(h_n\Delta t_n))(1-\alpha\Delta t_n)^{h_n} - \psi(y_x(t))e^{-\alpha t}| \leq$$

$$\leq \sum_{j=0}^{h_n-1} \int_{j\Delta t_n}^{(j+1)\Delta t_n} |f(y_x^{\Delta t_n}(j\Delta t_n))(1-\alpha\Delta t_n)^j - f(y_x(s))e^{-\alpha s}|\,ds +$$

$$+ \int_{h_n\Delta t_n}^{t} |f(y_x(s))e^{-\alpha s}|\,ds + |\psi(y_x^{\Delta t_n}(h_n\Delta t_n)(1-\alpha\Delta t_n)^{h_n} + \psi(y_x(t))e^{-\alpha t}| \leq$$

$$\leq \|f\|_\infty \sum_{j=0}^{h_n-1} \int_{i\Delta t_n}^{(j+1)\Delta t_n} |(1-\alpha\Delta t_n)^j - e^{-\alpha s}|\,ds + \sum_{j=0}^{h_n-1} \int_{j\Delta t_n}^{(j+1)\Delta t_n} |f(y_x^{\Delta t_n}(j\Delta t_n)) -$$

$$f(y_x(s))|\,e^{-\alpha s}\,ds + \|f\|_\infty |t - h_n\Delta t_n| + \|\psi\|_\infty |(1-\alpha\Delta t_n)^{h_n} - e^{-\alpha t}| +$$

$$+ |\psi(y_x^{\Delta t_n}(h_n\Delta t_n)) - \psi(y_x(t))|$$

Since $\lim_{n\to\infty} (1-\alpha\Delta t_n)^{[s/\Delta t_n]} = e^{-\alpha s}$ uniformly in $[0, T]$, the second statement of the lemma follows from the continuity of f and ψ and the convergence of $y_x^{\Delta t_n}(\cdot)$ to $y_x(\cdot)$ in $C^0([0,T])$. ■

Proof of Theorem 8. The family $A = \{(y_x^{\Delta t}(\cdot), h\Delta t)\}_{\Delta t>0}$, where $(y_x^{\Delta t}(\cdot), h\Delta t)$ is an arbitrary optimal couple of $(P^{\Delta t})$, is obviously a compact subset of $C^0([0,T]) \times [0,T]$, so a sequence $\{(y_x^{\Delta t_n}(\cdot), h_n\Delta t_n)\}_{n\in N}$, converges to a couple $(y_x(\cdot), t)$ with $t \in [0,T]$, and $y_x(\cdot) \in Y_x$ (see also the proof of Theorem 5). By Lemma 1, $J_x^{\Delta t_n}(y_x^{\Delta t_n}, h_n)$ converges to $J_x(y_x(\cdot), t)$. At this point it is enough to prove that

$$J_x(t, y_x(\cdot)) \leq J_x(\tilde{t}, \tilde{y}_x(\cdot)) \quad \forall (\tilde{y}_x(\cdot), \tilde{t}) \in Y_x \times [0,T] \tag{12}$$

Actually $\tilde{t} = \lim_{n\to\infty} \tilde{h}_n \Delta t_n$, with $\tilde{h}_n = \left[\frac{\tilde{t}}{\Delta t_n}\right]$, and $\tilde{y}_x(\cdot) = \lim_{n\to\infty} \tilde{y}_x^{\Delta t_n}(\cdot)$, where $\{\tilde{y}_x^{\Delta t_n}(\cdot)\}$ is a suitable sequence in $C^0([0,T])$, (see Theorem 5). Then, the optimality of $(y_x^{\Delta t_n}, h_n)$ i.e.

$$J_x^{\Delta t_n}(y_x^{\Delta t_n}, h_n) \leq J_x^{\Delta t_n}(\tilde{y}_x^{\Delta t_n}, \tilde{h}_n)$$

implies (12) by passing to the limit as $n \to \infty$. ■

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Optymalizacja czasu zatrzymania dla problemu Cauchy'ego w przypadku braku jednoznaczności rozwiązania

Rozważa się zadanie optymalizacji zatrzymania dla układu deterministycznego opisywanego problemem Cauchy'ego $\dot{y}(t)=g(y(t))$, $y(0)=x$, gdzie o funkcji g zakłada się jedynie, że jest jednostajnie ciągła. Przy założeniu lokalnej półciągłości funkcjonału jakości dowodzi się, że rozwiązanie istnieje i że funkcja Hamiltona-Jacobiego zachowuje własność półciągłości z dołu. Podano rezultaty dotyczące zbieżności dla dyskretnej w czasie wersji zadania optymalizacji czasu zatrzymania.

Оптимизация времени задержки для задачи Коши в случае отсутствия однозначности решения

Рассматривается задача оптимизации задержки для детерминированной системы, описываемой задачей Коши $\dot{y}(t)=g(y(t))$, $y(0)=x$, где о функции g предполагается лишь, что она является равномерно непрерывной. При предпосылке о локальной полунепрерывности функционала качества доказывается, что решение существует и что функция Гамильтона-Якоби сохраняет свойство полунепрерывности снизу. Приведены результаты, касающиеся сходимости для дискретного во времени варианта задачи оптимизации времени задержки.