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A numerical method for a parabolic bang-bang problem

by

W. HACKBUSCH

Institut für Informatik
Universität Kiel
Olshausenstr. 40, D-2300 Kiel 1

TH. WILL

Schässburger Gasse 6
D-5276 Wiehl 3 F.R. of Germany

The final state of a parabolic initial-boundary value problem is controlled by means of the boundary data. The solution is of bang-bang type. Replacing the parabolic equation by a difference scheme we obtain a discrete control problem. For its numerical solution we develop a multi-grid iteration. Numerical examples show that this method works efficiently.

1. Introduction

In Section 2 we consider the solution $y=y(x, t; u)$ of a parabolic initial-boundary value problem depending on the boundary data u. In order to approximate a given function $z \in L^2(\Omega)$ by the final state $y(\cdot, T; u)$, one can try to minimize the cost function

$$J(u)=\|y(\cdot, T; u)-z\|^2_{L^2(\Omega)}+\delta\|u\|^2_{L^2(O,T)}. \tag{1.1}$$

The problem is well-posed if either δ is positive or the admissible controls u are bounded. Often pointwise bounds u_1, u_2 are prescribed:

$$u_1(t)\leqslant u(t)\leqslant u_2(t) \quad \text{for a.a. } t\in(0, T). \tag{1.2}$$

If u is pointwise bounded and if δ vanishes in (1.1), the bang-bang principle applies: for almost all $t \in (0, T)$ the optimal control $u(t)$ equals one of the bounds $u_1(t)$ or $u_2(t)$.

The discrete control problem is obtained by replacing the partial differential equation with a difference scheme. In case of $\delta>0$ (cf (1.1)) the discrete optimal control u_n can be represented by

$$\delta u_n=K_n u_n+f_n,$$

where K_n is "smoothing". Such equations "of the second kind" can be solved numerically by the multi-grid method of the second kind described in [2, 3]. The iteration is very fast if δ is not too small, but its convergence rate deteriorates with δ tending to zero. Thus, the limit $\delta=0$ can be regarded as the worst case. The failure of the mentioned algorithm motivates the study of the bang-bang problem, i.e. of minimizing the cost function (1.1) for $\delta=0$.

Section 3 contains the definition of the difference scheme, of the discrete cost function, and the discrete optimal control. We formulate some discrete counterparts of the bang-bang principle in the continuous case. In particular we prove the uniqueness of the discrete solution.

The new multi-grid iteration described in Section 4 is neither a generalization nor a modification of the multi-grid method mentioned above. It is also not related to multi-grid algorithms used for elliptic problems (cf [4]). The algorithm presented in this paper is not restricted to bang-bang problems (i.e. to the case of $\delta=0$ in (1.1)) as is for instance the numerical method of Glashoff and Sachs [1].

2. The Continuous Problem

2.1. The Parabolic Control Problem

Let $\Omega=(0, 1)$ be a space interval and $(0, T)$, $T>0$, a time interval. The lateral boundary of $Q=\Omega_t\times(0, T)$ is denoted by $\Sigma=\Gamma\times(0, T)$, where $\Gamma=\{0, 1\}$ is the boundary of Ω.

We consider the parabolic initial-boundary value problem

$$y_t+Ay=0 \quad \text{in } Q \tag{2.1a}$$

$$By|_{x=0}=u \quad \text{on } (0, T) \tag{2.1b$_0$}$$

$$By|_{x=1}=0 \quad \text{on } (0, T) \tag{2.1b$_1$}$$

$$y|_{t=0}=0 \quad \text{in } \Omega, \tag{2.1c}$$

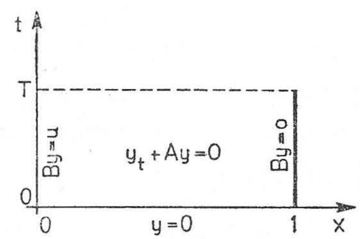

Fig. 1

where

$$A=-\beta(x)\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial}{\partial x}\right), \qquad \alpha(x), \beta(x)\geqslant\varepsilon>0,$$

$$B=\frac{\partial}{\partial n} \quad \left(\frac{\partial}{\partial n}=+\frac{\partial}{\partial x} \quad \text{at } x=1, \quad \frac{\partial}{\partial n}=-\frac{\partial}{\partial x} \quad \text{at } x=0\right).$$

It is only for simplicity that we assume the special forms of A and B. The coefficients α, β may depend on (x, t) and B may be a mixed boundary operator. The Eqs (2.1a, b_1, c) can be replaced by inhomogeneous ones. The considerations of this paper hold for more-dimensional domains $\Omega \in R^d$, $d>1$, too.

If α and β are sufficiently smooth and $u \in L^2(0, T)$, the solution $y=y(x, t; u)$ of (2.1) is uniquely determined and belongs to $H^{3/2, 3/4}(Q)$. The trace satisfies

$$y(\cdot, T; u)=y(u)|_{t=T} \in L^2(\Omega).$$

Hence, a bounded linear operator

$$S: u \in L^2(0, T) \to y(\cdot, T; u) \in L^2(\Omega) \tag{2.2}$$

is defined.

2.2. Control Problem

Let $z \in L^2(\Omega)$ be a given function. The control problem consists in minimizing the cost function

$$J(u)=(Su-z, Su-z)_{L^2(\Omega)}=\int_\Omega |y(x, T; u)-z(x)|^2\, dx \tag{2.3}$$

over the set

$$U_{ad}=\{u \in L^2(0, T): |u(t)| \leq 1 \text{ a.e. on } (0, T)\}$$

of admissible controls. Since U_{ad} is weakly closed we have

THEOREM 2.1. *The problem $J(u)=Min$, $u \in U_{ad}$, has at least one solution (optimal control) $u^* \in U_{ad}$.*

COROLLARY 2.2. *In case of two optimal controls u^*, $u^{**} \in U_{ad}$, the equality $Su^*=$ $=Su^{**}$ holds.*

Proof. Set $u(\tau):=u^*+\tau(u^{**}-u^*)$ and $j(\tau);=J(u(\tau))$. Here, j is a parabola with minima at $\tau=0, 1$. Thus, $j''(\tau)=2\,\|S(u^{**}-u^*)\|^2_{L^2(\Omega)} \leq 0$ must hold implying $S(u^{**}-u^*)=0$. ■

2.3. Bang-Bang Principle

The adjoint differential equation reads as

$$-p_t+A^*p=0 \quad \text{in } Q \tag{2.4a}$$

$$Cp \quad =0 \quad \text{on } \Sigma \tag{2.4b}$$

$$p|_{t=T} \quad =w \quad \text{in } \Omega, \tag{2.4c}$$

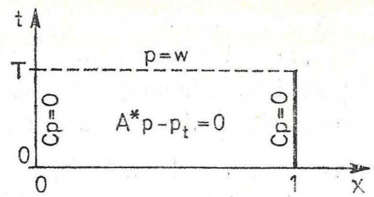

Fig. 2

where

$$A^*p=-\frac{\partial}{\partial x}\alpha\frac{\partial}{\partial x}(\beta p), \qquad Cp=\alpha\frac{\partial}{\partial n}(\beta p).$$

The "final value" $w\in L^2(\Omega)$ determines a unique solution $p=p(x,t;w)\in H^{1,1/2}(Q)$ with trace $p|_{x=0}=p(0,\cdot;w)\in L^2(\Omega)$. Correspondingly,

$$S^*: w\in L^2(\Omega)\to\alpha(0)\,\beta(0)\,p(0,\cdot\,;w)\in L^2(0,T) \tag{2.5}$$

describes a bounded linear mapping. Repeated partial integration proves

THEOREM 2.3. *S^* is adjoint to S: $(Su,w)_{L^2(\Omega)}=(u,S^*w)_{L^2(0,T)}$.*

The property (2.6) described in the next theorem is a characterization of an optimal control.

THEOREM 2.4. *Let $u^*\in U_{ad}$ be an optimal control. Then a.e. on $(0,T)$ one of the following two equations holds:*

$$[S^*(Su^*-z)](t)=0 \quad \text{or} \quad u^*(t)=-\operatorname{sign}[S^*(Su^*-z)](t). \tag{2.6}$$

If the first equation of (2.6) is valid on a set of measure zero, u^* is called to satisfy the *bang-bang principle*. Hence, a bang-bang solution fulfils $u^*=-\operatorname{sign}[S^*(Su^*-z)]$ a.e. on $(0,T)$. Under usual conditions (*cf* Theorem 6 of Glashoff and Sachs [1]) the following alternative holds: Either there is an (optimal) control $u^*\in U_{ad}$ with $Su^*=z$ or any optimal u^* satisfies the bang-bang principle. The latter property is important because of

THEOREM 2.5. *The bang-bang principle implies uniqueness of the optimal control.*

Proof. Set $q(u):=S^*(Su-z)$ and assume that u^* and u^{**} are two optimal controls. Corollary 2.2 implies $q(u^*)=q(u^{**})$. By the bang-bang principle $u^*(t)=-\operatorname{sign}q(u^*)(t)=-\operatorname{sign}q(u^{**})(t)=u^{**}(t)$ is valid for a.a. $t\in(0,T)$. Thus, $u^*=u^{**}$ proves uniqueness. ■

3. The Discrete Problem

3.1. The Discrete Parabolic Initial-Boundary Value Problem

Let $\Delta x=1/m$ and $h=T/n$ be the equidistant step widths of the intervals $\bar{\Omega}=[0,1]$ and $[0,T]$, respectively. The counterparts of $L^2(\Omega)$ and $L^2(0,T)$ are the vector spaces R^{m+1} and R^n with the scalar products

$$(w,z)_{m+1}:=\Delta x \sum_{i=0}^{m} w(i)\,z(i) \qquad \text{for } w,z\in \mathbf{R}^{m+1},$$

$$(u,v)_n:=h \sum_{j=0}^{n-1} u(j)v(j) \qquad \text{for } u,v\in \mathbf{R}^{n}.$$

The components $z(i)$ and $u(j)$ correspond to the values at $x=i*\Delta x$ and $t=j*h$, respectively. The coefficient functions $\alpha(x)$ and $\beta(x)$ of the differential operator A become

$$\alpha_m(i):=\alpha\big((i-1/2)\Delta x\big) \quad (0\leqslant i\leqslant m+1),$$
$$\beta_m(i):=\beta(i\Delta x) \quad (0\leqslant i\leqslant m),$$

where α is extended to the interval $[-\Delta x/2,\, 1+\Delta x/2]$.

The spatial forward and backward differences are

$$\partial_{\Delta x}^{+} y(i,j)=[y(i+1,j)-y(i,j)]/\Delta x,$$
$$\partial_{\Delta x}^{-} y(i,j)=[y(i,j)-y(i-1,j)]/\Delta x,$$

while the time differences are

$$\partial_h^{+} y(i,j)=[y(i,j+1)-y(i,j)]/h,$$
$$\partial_h^{-} y(i,j)=[y(i,j)-y(i,j-1)]/h.$$

The discrete analogues of the differential operator A and the boundary operator B are

$$A_{\Delta x}=-\beta_m\,\partial_{\Delta x}^{-}\,\alpha_m\,\partial_{\Delta x}^{+},$$

$$B_{\Delta x}y(i,j)=\begin{cases}-\partial_{\Delta x}^{-} y(i,j) & \text{for } i=0,\\ +\partial_{\Delta x}^{+} y(i,j) & \text{for } i=m.\end{cases}$$

Let $u_n=(u_n(j))_{j=0}^{n-1}\in \mathbf{R}^n$ be a given control. The discrete initial-boundary value problem

$$\partial_h^{-} y(i,j)+A_{\Delta x}y(i,j)=0 \quad (0\leqslant i\leqslant m,\ 1\leqslant j\leqslant n), \tag{3.1a}$$

$$B_{\Delta x}y(0,j)=u_n(j-1) \quad (1\leqslant j\leqslant n), \tag{3.1b$_0$}$$

$$B_{\Delta x}y(m,j)=0 \quad (1\leqslant j\leqslant n), \tag{3.1b$_1$}$$

$$y(i,0)=0 \quad (0\leqslant i\leqslant m) \tag{3.1c}$$

determines a unique solution $y(i,j;u_n)$ for $-1\leqslant i\leqslant m+1$, $1\leqslant j\leqslant n$. (3.1) is the implicit difference scheme. Note that auxiliary values $y(-1,j)$ and $y(m+1,j)$ corresponding to $x=-\Delta x$ and $x=1+\Delta x$ are involved. However, they can be eliminated immediately by means of (3.1b$_{0,1}$). Hence, one obtains the following.

REMARK 3.1. The vectors $y_j=\big(y(i,j)\big)_{i=0}^{m}\in \mathbf{R}^{m+1}$ are to be computed from

$$y_0=0,\quad Ty_j=y_{j-1}+t\begin{bmatrix}u_n(j-1)\\ 0\\ \vdots\\ 0\end{bmatrix} \tag{3.2}$$

where $t>0$ is a given constant and T is a given irreducible tridiagonal matrix having positive eigenvalues.

3.2. Discrete Analogues of S and S^*

The discrete counterpart of S $(cf\,(2.2))$ is the linear mapping

$$S_{nm}: u_n \in R^n \to (y(i,n;u_n))_{i=0}^m \in R^{m+1}.$$

Consider the discretization of the adjoint problem (2.4) by

$$\partial_n^+ p(i,j) - A_{\Delta x}^* p(i,j) = 0 \quad (0 \leqslant i \leqslant m,\ 0 \leqslant j \leqslant n-1), \tag{3.3a}$$

$$C_{\Delta x} p(i,j) = 0 \quad (i=0 \text{ or } m,\ 0 \leqslant j \leqslant n-1), \tag{3.3b}$$

$$p(i,n) = w_m(i) \quad (0 \leqslant i \leqslant m), \tag{3.3c}$$

where

$$A_{\Delta x}^* = -\partial_{\Delta x}^- \alpha_m \partial_{\Delta x}^+ \beta_m, \qquad C_{\Delta x} p = \alpha_m B_{\Delta x}(\beta_m p).$$

The solution of (3.3) corresponding to the final values $w_m \in R^{m+1}$ is denoted by $p(i,j;w_m)$. According to (2.5) we define

$$S_{nm}^*: w_m \in R^{m+1} \to \alpha_m(0)\,\beta_m(0) \quad (p(0,j;w_m))_{j=0}^{n-1} \in IR^n).$$

Note that $\alpha_m(0) = \alpha(-\Delta x/2)$ and $\beta_m(0) = \beta(0)$. The schemes (3.1) and (3.3) are chosen in such a way that Theorem 2.3 remains valid for S_{nm}.

THEOREM 3.2. *S_{nm}^* is the operator adjoint to S_{nm}:*

$$(S_{nm} u_n, w_m)_{m+1} = (u_n, S_{nm}^* w_m)_n \text{ for all } w_m \in R^{m+1}, u_n \in R^n$$

3.3. Discrete Optimal Control Problem

The function z of (2.3) is to be replaced by a vector $z_m \in R^{m+1}$, e.g. defined by $z_m(i) = z(i*\Delta x)$, $0 \leqslant i \leqslant m$. The set of admissible controls is

$$U_{ad}^n = \{v \in R^n: |v(i)| \leqslant 1,\ 0 \leqslant i \leqslant n-1\}.$$

Defining the discrete cost function by

$$J_{nm}(u_n) := (S_{nm} u_n - z_m, S_{nm} u_n - z_m)_{m+1} \qquad (u_n \in R^n)$$

we seek the solution $u_n^* \in U_{ad}^n$ of

$$J_{nm}(u_n) = \text{Min}, \ u_n \in U_{ad}^u. \tag{3.4}$$

The minimizer $u_n^* \in U_{ad}^u$ is called (discrete) optimal control. As in Theorem 2.1 and in Corollary 2.2 we have

THEOREM 3.3. *Problem* (3.4) *has at least one solution* $u_n^* \in U_{ad}^n$. *Two optimal controls* $u_n^*, u_n^{**} \in U_{ad}^n$ *satisfy* $S_{nm} u_n^* = S_{nm} u_n^{**}$.

Similarly, the discrete version of Theorem 2.4 holds:

THEOREM 3.4. *Let* u_n *be optimal. One of the equations*

$$S_{nm}^* (S_{nm} u_n^* - z_m)(i) = 0 \quad \text{or} \quad u_n^*(i) = -\operatorname{sign}[S_{nm}^*(S_{nm} u_n^* - z_m)(i)] \tag{3.5}$$

hold for every $i = 0, 1, \ldots, n-1$.

Proof. For fixed i define the unit vector e by $e(i) = 1$ and $e(k) = 0$ for $k \neq$ and set $j(\tau) := J_{nm}(u_n + \tau e)$. Since $\frac{d}{d\tau} j(0) = 2 \Delta x\, S_{nm}^* (S_{nm} u_n^* - z_m)(i)$, Eq (3.5) is necessary for a minimum at $\tau = 0$. ∎

The bang-bang principle implies $u^*(t) = \pm 1$ for a.a. $t \in (0, T)$ in the continuous case. The discrete counterpart "$u_n^*(i) = \pm 1$ for all $i = 0, \ldots, n-1$" is wrong! In general, there are indices i with $u_n^*(i) \in (-1, +1)$ and $S_{mn}^* (S_{nm} u_n - z_m)(i) = 0$ according to (3.5). However, their number is bounded by m independently of n. Besides that, uniqueness holds as in Theorem 2.5.

THEOREM 3.5. *Assume that* J_{nm} *does not vanish for the optimal control* u_n^*, *i.e.* $S_{nm} u_n^* \neq z_m$. *Further, suppose that the states* $y_j = (y(i,j))_{i=0}^m \in R^{m+1}$, $0 \leq j \leq n$, *satisfy the second equation of* (3.2) *with a irreducible tridiagonal matrix* T *having only positive eigenvalues. Then the discrete optimal control* u_n *is uniquely determined and there are at most* m *indices* $i_1, \ldots, i_m$ *with* $-1 < u_n(i_\nu) < +1$.

The assumption that T has only positive eigenvalues is not restrictive as can be seen from

LEMMA 3.6. *Let* T *be irreducible and tridiagonal. Then*

(i) *All eigenvalues of* T *are simple. In particular* T *is diagonizable.*

(ii) *If* T *is a real matrix, all eigenvalues are real.*

(iii) *If* T *is real and positive definite* (*i.e.* $(u, Tu) > 0$ *for* $u = 0$, *but not necessarily symmetric*), *all eigenvalues are positive.*

The proof of Theorem 3.5 is prepared by several lemmas.

LEMMA 3.7. *Let* $i_0 < i_1 < \ldots < i_r$ *be integers. An interpolating function of the form* $f(x) = \sum_{\nu=0}^{r} a_\nu x^{i_\nu}$ *is uniquely determined by* $r+1$ *values at different arguments* $x_k > 0$, $0 \leq k \leq r$. *In particular,* $f(x_k) = 0$ *for all* $0 \leq k \leq r$ *implies* $a_\nu = 0$ *for* $0 \leq \nu \leq r$.

Proof. It suffices to prove the latter part. Assume $f(x_k) = 0$ for $0 \leq k \leq r$. If $r = 0$, the result $a_0 = 0$ is obvious. Suppose that Lemma 3.7 holds for $r-1$ instead of r and apply it to $g(x) := \frac{d}{dx}(f(x)/x^{i_0})$. By Rolle's theorem g vanishes at r different arguments $\xi_k \in (x_k, x_{k+1})$, $0 \leq k \leq r-1$. By the inductive assumption, the r coefficients $b_\nu = a_\nu (i_\nu - i_0)$ of g, $1 \leq \nu \leq r$, vanish. Thence, $a_\nu = 0$ follows for $1 \leq \nu \leq r$. Then, $a_0 = 0$ is immediate. ∎

LEMMA 3.8. *Let T be the matrix of Theorem 3.5 and e be the unit vector $(1, 0, \ldots \ldots, 0) \in R^{m+1}$, $i_0 < i_1 < \ldots < i_r$ be $r+1$ integers, where $r \leqslant m$. Then the vectvrs $\{T^{-i_\nu} e : 0 \leqslant \nu \leqslant r\}$ are linearly independent.*

Proof. Assume that $\sum_{\nu=0}^{r} a_\nu T^{-i_\nu} e = 0$. Let λ be an eigenvalue of T^* with corresponding eigenvector $v = (v_i)_{i=0}^m \in R^{m+1}$. By assumption,

$$0 = \left(v, \sum_{\nu=0}^{r} a_\nu T^{-i_\nu} e\right)_{m+1} = \left(\sum_{\nu=0}^{r} (a_\nu \lambda^{-i_\nu} v, e\right)_{m+1} = \Delta x \sum_{\nu=0}^{r} a_\nu \lambda^{-i_\nu} v_0 \tag{3.6}$$

holds.

$T^* v = \lambda v$ yields $v_1 = (\lambda v_0 - T_{00} v_0)/T_{10}$, where $T_{10} \neq 0$ since T is strictly tridiagonal. T_{ij} denote elements of the matrix T. Thus, $v_0 = 0$ implies $v_1 = 0$. Similarly, the relation $v_{k+1} = (\lambda v_k - T_{kk} v_k - T_{k-1,k} v_{k-1})/T_{k+1,k}$ induces $v_2 = v_3 = \ldots = v_m = 0$ in case of $v_0 = 0$. Since $v \neq 0$, $v_0 \neq 0$ follows and (3.6) implies $f(1/\lambda) = 0$, where

$$f(x) = \sum_{\nu=0}^{r} a_\nu x^{i_\nu}.$$

By Lemma 3.6, there are $m+1$ different eigenvalues λ_k. The previous consideration shows $f(1/\lambda_k) = 0$, $0 \leqslant k \leqslant m$. Since $m \geqslant r$, one concludes from Lemma 3.7 that $a_\nu = 0$, $0 \leqslant \nu \leqslant r$. Therefore, the vectors $T^{-i_\nu} e$ must be linearly independent. ■

LEMMA 3.9. *Let $0 \neq w \in R^{m+1}$ and $q = S_{nm}^* w$. Then, $q(i) = 0$ holds for at most m indices $0 \leqslant i \leqslant n-1$.*

Proof. Assume that there are $r+1 > m$ indices $i_0 < i_1 < \ldots < i_r$ with $q(i_\nu) = 0$. Define u_n by

$$u_n(i_\nu) := a_\nu / t \text{ for } 0 \leqslant \nu \leqslant m, \quad u_n(i) = 0 \text{ otherwise},$$

where a_ν are arbitrary numbers and t is the same as in (3.2). Using the notation of Remark 3.1 we have $S_{nm} u_n = y_n$ and we conclude from (3.2) that

$$S_{nm} u_n = \sum_{\nu=0}^{m} a_\nu T^{-n+i_\nu} e.$$

By Lemma 3.8, the vectors $T^{-n+i_\nu} e$, $0 \leqslant \nu \leqslant m$, form a basis and we can choose the coefficients a_ν such that $S_{nm} u_n = w$. The assumption on q and the construction of u_n^* imply

$$0 = (u_n, q)_n = (u_n, S_{nm}^* w)_n = (S_{nm} u_n, w)_{m+1} = (w, w)_{m+1}$$

contradicting $w \neq 0$. Hence, the lemma is proved. ■

LEMMA 3.10. *$S_{nm} u_n^* \neq z_m$ implies uniqueness of an optimal control u_n^*.*

Proof. Let u_n^* and u_u^{**} be two optimal controls with difference $d_n = u_n^* - u_n^{**}$. Theorem 3.3 shows $q := S_{nm}^* (S_{nm} u_n^* - z_m) = S_{nm}^* (S_{nm} u_n^{**} - z_m)$. By Theorem 3.4, $u_n^*(i) = u_n^{**}(i)$ implying $d(i) = 0$ must hold for all i with $q(i) \neq 0$. We know from Lemma 3.9 that $q(i) = 0$ for at most m indices. Therefore, $d(i) \neq 0$ may occur for at most m indices, say for $i_0 < i_1 < \ldots < i_r$, $r < m$. As in the proof of Lemma 3.9, $S_{nm} d$ has the representation

$$S_{nm} d = t \sum_{\nu=0}^{r} d(i_\nu) T^{-n+i_\nu} e.$$

Since $S_{nm} d = 0$ by Theorem 3.3, Lemma 3.8 proves that all coefficients $d(i_\nu)$, $0 \leqslant \nu \leqslant r$, must vanish. Hence, $d = u_n^* - u_n^{**} = 0$ is demonstrated. ■

The proof of Theorem 3.5 is given by Lemma 3.10, (3.5) and Lemma 3.9.

3.4. Iterative Solution

In the following we consider only discrete problems and therefore we omit the indices n and m:

$$u = u_n, \quad z = z_m, \quad S = S_{nm}, \quad J = J_{nm}, \quad \text{etc.}$$

For the numerical treatment of the discrete problem (3.4) we shall develop an iterative method. Let $u \in R^n$ be a given approximation of the optimal u^*. We have to find a correction $v^* \in R^n$ with $u + v^* = u^*$ or equivalently

$$J(u+v^*) = \inf\{J(u+v) : v \in R^n, u+v \in U_{ad}^n\}, \quad u+v^* \in U_{ad}^n. \tag{3.6}$$

Since

$$J(u+v) = J(u) + (2q(u) + Kv, v)_n, \tag{3.7}$$

where

$$q(u) := S^*(Su - z), \tag{3.8a}$$

$$K := S^* S, \tag{3.8b}$$

the problem (3.6) can be rewritten as

$$I(v, q(u)) = \text{Min}, \quad v \in V(u) \tag{3.9}$$

with

$$I(v, q) := (2q + Kv, v)_n \tag{3.10}$$

and the set

$$V(u) := \{v \in R^n : u+v \in U_{ad}^n\} = \{v : -1 - u(i) \leqslant v(i) \leqslant 1 - u(i)\}$$

of admissible corrections.

These considerations suggest the following framework of an iterative process:

start: $u^0 \in U^n_{ad}$ given, (3.11a)

iteration: u^j given, (3.11b)

compute $\tilde{v}$: approximation to solution of (3.9) with $u = u^j$, (3.11c)

$u^{j+1} := u^j + \tilde{v}$. (3.11d)

In the step (3.11c) we need $q^j := q(u^j)$. Definition (3.8a) yields $q^{j+1} = q^j + K\tilde{v}$ with initial value $q^0 = q(u^0)$. This gives us the next formulation:

start: $u^0 \in U^n_{ad}$ given, $q^0 := q(u^0)$, (3.12a)

iteration: u^j, q^j given, (3.12b)

$\tilde{v}$: approximate solution of $I(v, q^j) = \mathrm{Min}$, $\quad v \in V(u^j)$, (3.12c)

$$u^{j+1} := u^j + \tilde{v}, \tag{3.12d}$$

$$q^{j+1} := q^j + K\tilde{v}. \tag{3.12e}$$

In Section 4 we shall propose how to perform step (3.12c). For step (3.12e) we refer to

REMARK 3.6. It is not necessary to know the entries of the matrix K explicitly. The multiplication $K*\tilde{v}$ is performed according to the definitions of $K = S^* S$ and S^*, S. Solve the discrete initial-boundary value problem (3.1) with $u_n := \tilde{v}$ resulting in $y(i, j; \tilde{v})$. Set $w_m = y(\cdot, n; \tilde{v})$ and solve the adjoint problem (3.3). Then, $K^*\tilde{v}(j) = \alpha_m(0)\, \beta_m(0)\, p(0, j; w_m)$.

4. The Multi-Grid Method

The original problem (3.4) is solved if one is able to treat (3.9), i.e. if one can find the minimizer of $I(v, q) = (2q + Kv, v)_n$. To find an approximation $\tilde{v}$ we shall replace the expression $(2q + Kv, v)_n$ with a simpler one. By cancelling the quadratic term completely one would be led to "minimize $I'(v, q) := (2q, v)_n$" which yields a gradient type method. The approximation $\tilde{v}$ should be more accurate if we do not omit the quadratic term completely but substitute $(Kv, v)_n$ by some $(K'v, v)_n$, where $K' \simeq K$. Such a matrix K' (and not only just one but a whole sequence of K_l s) will in the sequel be understood to be generated by auxiliary grids.

4.1. Auxiliary Grids

We define $l+1$ 'levels' $(i = 0, \dots, l)$ with step sizes

$$h_i = T/n_i \qquad (0 \leqslant i \leqslant l,\, n_0 < n_1 < \dots < n_l)$$

in the time direction, where for instance

$$n_i = 2^i n_0 \qquad (0 \leq i \leq l). \tag{4.1}$$

The number n of the preceding section corresponds to the finest step width n_l. For the spatial direction we choose

$$\Delta x_i = 1/m_i \text{ with } m_i \simeq m_0 \sqrt{n_i/n_0} \qquad (0 \leq i \leq l)$$

to ensure $h_i = 0(\Delta x_i^2)$. The matrices $K_{n_i m_i} = S^*_{n_i m_i} S_{n_i m_i}$ are defined as in Section 3.

The vector spaces R^{n_i} $(0 \leq i \leq l)$ are connected with R^{n_l} by prolongations $\pi_i : R^{n_i} \to R^{n_l}$ and restrictions $\rho_i : R^{n_l} \to R^{n_i}$. For $i = l$ we define formally

$$\pi_l = \rho_l = \text{identity}. \tag{4.2}$$

$$\begin{tikzcd}
R^{n_l} \arrow[rr, "K_l"] \arrow[rr, bend right, "K_i"'] \arrow[d, "\rho_i"] & & R^{n_l} \\
R^{n_i} \arrow[rr, "K_{n_i m_i}"] & & R^{n_i} \arrow[u, "\pi_i"']
\end{tikzcd}$$

Fig. 3

In case of n_i chosen according to (4.1) the prolongation π_i can be defined by piecewise constant interpolation:

$$(\pi_i v)(j) = v(k) \qquad \text{for} \qquad 0 \leq k \leq n_i - 1,\, 2^{l-i} k \leq j \leq 2^{l-i}(k+1) - 1.$$

The adjoint mapping can serve as restriction: $\rho_i = \pi_i^*$. Its explicit representation is

$$(\rho_i w)(k) = 2^{i-l} \sum_{j=0}^{2^{l-i}} w(2^{l-i} k + j) \qquad (0 \leq k \leq n_i - 1, w \in R^{n_l}).$$

The products

$$K_i := \pi_i K_{n_i m_i} \rho_i : R^{n_l} \to R^{n_l} \qquad (0 \leq i \leq l)$$

are operating in the finest grid. Definition (4.2) implies $K_l = K_{n_l m_l}$. Thus, K_l equals K from (3.8b) if we identify n_l with n from Section 3.

4.2. Two-Grid Iteration

Consider the problem

$$I_i(v, q) := (2q + K_i v, v)_{n_l} = \text{Min}, \quad v \in V(u). \tag{4.3}$$

This problem is of the same form as the original problem (3.9), with K replaced by K_i. Therefore, we can again apply the iteration (3.12) to (4.3). The undetermined part of the iteration (3.12) is the computation of $\bar{v}$ in step (3.12c). Since $K_i \simeq K_{i-1}$ $(i > 1)$ is assumed, the problem $I_i(v, q) = \text{Min}$, $v \in V(u)$, is close to $I_{i-1}(v, q) = \text{Min}$, $v \in V(u)$. Hence, the approximation $\bar{v}$ of $I_i(v, q) = \text{Min}$ can be defined as the (exact) minimizer of $I_{i-1}(v, q^j)=\text{Min}$. The resulting *two-grid method* at level $i>0$ for solving

$$I_i(v, q)=\text{Min}, \quad v \in V(u) \tag{4.4}$$

reads as

$$\textbf{start: } v^0:=0, \quad q^0=q \text{ given by (4.4);} \tag{4.5a}$$

$$\textbf{iteration: } v^j, q^j \text{ known;} \tag{4.5b}$$

$$\bar{v}: \text{minimizer of} \tag{4.5c}$$

$$I_{i-1}(v, q^j)=\text{Min}, \quad v \in V(u+v^j); \tag{4.6}$$

$$v^{j+1}:=v^j+\bar{v}; \tag{4.5d}$$

$$q^{j+1}:=q^j+K_i\bar{v}. \tag{4.5e}$$

REMARK 4.1. (i) Put $i=l$, $q:=q(u)$. Then the iterations (3.12) and (4.5) are equivalent up to step (3.12c/4.5c) if we identify $u+v^j$ from (4.5) with u^j from (3.12). As stated above, the problem (4.4) coincides with the original problem (3.9). (ii) Let $i \in \{0, 1, \ldots, l\}$. In step (4.5c) one approximates the solution of $I(v, q^j)= =\text{Min}$, $v \in V(u+v^j)$. The latter problem is equivalent to (4.4).

4.3. Convergence of the Two-Grid Iteration

The following lemma estimates the quality of the approximation $\bar{v}$ of step (4.5c)

LEMMA 4.2. *Let $i \in \{0, 1, \ldots, l\}$ and suppose*

$$\|K_i-K_{i-1}\| \leq \varepsilon_i, \tag{4.7}$$

where $\|\cdot\|$ is the spectral norm of $n_l \times n_l$ matrices. Then for all $u \in U_{ad}^n$ and all $q \in R^{n_l}$ the following estimate holds. The minimizers v^* and v'^* of the respective problems

$$I_i(v, q)=\text{Min}, \qquad I_{i-1}(v', q)=\text{Min}, \qquad v, v' \in V(u)$$

satisfy

$$I_i(v'^*, q) \leq I_i(v^*, q)+\varepsilon_i(\|v^*\|+\|v'^*\|)\,\|v^*-v'^*\| \leq I_i(v^*, q)+8\varepsilon_i \tag{4.8}$$

i.e. v'^* yields the minimum of I_i up to an error $\varepsilon_i(\|v^*\|+\|v'^*\|)\,\|v^*-v'^*\|$.

Proof. The inequality

$$\begin{aligned} I_i(v'^*, q)&=I_{i-1}(v'^*, q)+([K_i-K_{i-1}]v'^*, v'^*)_{n_l} \leq I_{i-1}(v^*, q)+\\ &+([K_i-K_{i-1}]v'^*, v'^*)_{n_l}=I(v^*, q)-([K_i-K_{i-1}](v^*-v'^*, v^*)_{n_l}+\\ &-([K_i-K_{i-1}]v'^*, v^*-v'^*)_{n_l} \end{aligned}$$

together with (4.7) implies $I_i(v'^*, q) \leq I_i(v^*, q)+\varepsilon_i(\|v^*\|+\|v'^*\|)\,\|v^*-v'^*\|$. The estimate by $8\varepsilon_i$ follows from the special choice of U_{ad}^n. ∎

One would expect $\varepsilon_i=0(h_i^K)$ for some $K>0$. Hence, the first iteration reduces the error of I_i to less than $8\varepsilon_i$. Note that in the next iteration step the old error v^* is replaced with the new error $\bar{v}^*-\bar{v}=v^*-v'^*$. Hopefully, the differences $v^*-v'^*$ is smaller than v' and v'^*. Thus, the right-hand side of (4.8) is expected to be smaller in the next iteration. Nevertheless, Lemma 4.2 does not imply convergence of the iteration (4.5). But convergence can be ensured by a slight modification:

REMARK 4.3. Replace the steps (4.5d, e) by

$$v^{j+1} := v^j + \lambda^* \bar{v} \text{ with } \lambda^* \in [0, 1] \text{ minimizing } I_i(\lambda v, q^j); \tag{4.5d'}$$

$$q^{j+1} := q^j + \lambda^* K_i \bar{v}. \tag{4.5e'}$$

Under the assumptions of Theorem 3.5 the modified iteration converges.

P r o o f. Denote the function performed via the iteration by $\varphi(v^j) := v^{j+1}$. φ is not necessarily continuous but $I_i(\varphi(\cdot), q^0)$ is. (4.5d′, e′) ensures $I_i(\varphi(v), q^0) \leqslant \leqslant I_i(v, q^0)$ since $\varphi(v) - v$ is a descent direction. There is a subsequence $\{j_k\}$ such that $\lim v^{j_k} = v^*$, $\lim I_i(v^{j_k}, q^0) =: I^*$. If v^* would not be optimal, $v^{**} := \varphi(v^*)$ would differ from v^* and yield $I_i(v^{**}, q^0) < I^*$ (cf. Theorem 3.5). However, by monotonous convergence $I_i(v^j, q^0) \downarrow I^*$

$$I_i(v^{**}, q^0) = \lim_{k\to\infty} I_i(\varphi(v^{j_k}), q^0) = \lim_{k\to\infty} I_i(v^{j_k+1}, q^0) = I^*.$$

Hence, v^* is optimal. The uniqueness of the optimal control implies $\lim v^j = v^*$.

An equivalent representation of the two-grid iteration is the following iteration minimizing $I_i(v, q^0)$:

$$\hat{v}^j = \arg\min_{v\in V(u)} \{(I_i'(v^j, q^0), v - v^j)_{n_i} + (K_{i-1}(v - v^j), v - v^j)_{n_i}\}, \qquad v^{j+1} := \hat{v}^j.$$

The modification (4.5d′, e′) corresponds to

$$v^{j+1} := v^j + \lambda^*(\hat{v}^j - v^j), \qquad \lambda^* = \arg\min_{0<\lambda<1} I_i(v^j + \lambda(\hat{v}^j - v^j), q^0).$$

Convergence rates can be shown under stronger assumption on K_{i-1}. Hughes [5, Theorem 3.1] proved that $I_i(v^j, q^0) - I_i(v^*, q^0) = 0(1/j)$ if either

$$0 \leqslant (K_{i-1} v, v)_{n_i} \leqslant (K_i v, v)_{n_i} \quad \text{for all} \quad v \in R^{n_i}$$

or

$$\underline{c}\, \|v\|^2 \leqslant (K_{i-1} v, v)_{n_i} \leqslant \bar{c}\, \|v\|^2, \quad 0 < \underline{c} \leqslant \bar{c} < \infty \quad \text{for all} \quad v \in R^{n_i}.$$

The first condition is equivalent to $\|S_{n_i} \rho_i v\| \leqslant \|S_{n_i} v\|$ if $\pi_i = \rho_i^*$ and is hard to prove. The second condition can be enforced by replacing K_{i-1} with

$$K_{i-1} + \varepsilon_{i-1} I, \quad \varepsilon_{i-1} > 0.$$

Thus, $\varepsilon_k = \text{const}/n_k$ might for instance be a reasonable choice, since K_i $(i < l)$ are then still consistent with K_l.

4.4. Multi-Grid Iteration

So far we considered a two-grid iteration for solving problem (4.4). Note that the auxiliary problem (4.6) at level $i-1$ is of the same form as (4.4). Therefore, the same method can be applied for solving approximately (4.6) as long as $i-1>0$.

The multi-grid iteration at level $i=0, \ldots, l$ is defined recursively. For $i=0$ (i.e. for the coarsest grid existing) solve the problem (4.4) by any (convergent) iterative method[1]. Assume that the multi-grid iteration at level $i-1\geqslant 0$ is already defined. Then one iteration step at level i reads as follows:

v^j, q^j known; (4.9b)

$\tilde{v}$: solution obtained by μ iterations of the multi-grid method at level $i-1$ applied to (4.6) with zero as starting value; (4.9c)

$$v^{j+1}:=v^j+\tilde{v}; \tag{4.9d}$$

$$q^{j+1}:=q^j+K_i\tilde{v}. \tag{4.9e}$$

The choice of the number μ will be discussed below. Practical values seem to be $\mu=2$ or $\mu=3$.

As in the case of the two-grid iteration (4.5), the convergence of the multi-grid iteration (4.9) cannot be guaranteed. However, the modification mentioned in Remark 4.3 helps again:

REMARK 4.4. Replace (4.9d, e) by (4.5d′, e′) and assume that the iteration at level $i=0$ yields $\tilde{v}$ with $(q^j, \tilde{v})<0$, i.e. $\tilde{v}$ is a descent diiection. Then, the multi-grid iteration with $\mu\geqslant 1$ converges for every $i\in\{0, 1, ..., 1\}$.

As in (1.2) the set $V(u)$ can be characterized by the lower and upper bounds $u_1=-\mathbf{1}-u$, $u_2=\mathbf{1}-u$, where $\mathbf{1}$ is the n_i-vector with all components being equal to one. The following quasi-ALGOL program performs ν steps of the multi-grid iteration (4.9) starting with $v^0=0$.

multi-grid iteration for solving $I_i(v, q)=\mathrm{Min},\ u1\leqslant v\leqslant u2$
description of parameters: i: level number $\in\{0, 1, ..., l\}$, v: desired optimal control (approximate value), $q, u1, u2$: given by the problem $I_i(v,q)=\mathrm{Min},\ u1\leqslant v\leqslant u2$, ν: number of iterations.

procedure mgm $(i, v, q, u1, u2, \nu)$;
if $i=0$ **then**
$v:=$approximate minimizer of "$I_0(w,q)=\mathrm{Min},\ u1\leqslant w\leqslant u2$"
else
begin $v:=0$; $qq:=q$;
    for $k:=1$ **step** 1 **until** ν do
    begin $uu1:=u1-v$; $uu2:=u2-v$;
        $mgm(i-1, vv, qq, uu1, uu2, \mu)$;
        $v:=v+vv$;
        if $k<\nu$ **then** $qq:=qq+K_i*vv$
    end
end;

(4.10)

[1] For the numerical examples of Section 5 we minimized $I_0(v, q)$ successively with respect to the components $v(0), v(1), ..., v(n_0-1)$.

For $i=l$, $ul=-1$, $u2=1$ (i.e. $V(u)=\bar{U}^n_{ad}$), $q=q(0)$ the problem (4.4) is equivalent to the original problem (3.4), $J_{n_l m_l}(u)=\text{Min}$, $u \in U^n_{ad}$. Therefore, the latter problem can be solved by

```
program for solving J_{n_l m_l}(u) = Min, u ∈ U^n_{ad}:
begin array q, u, ul, u2 [0: n_l - 1];
      q := - S*_{n_l m_l} * z_{m_l};
      ul := -1; u2 := +1;
      mgm (l, u, q, ul, u2, v);
      comment u is approximate solution
end;
```

(4.11)

4.5. Computational Effort

The subtractions and additions in (4.10) can be neglected compared with the performance of *mgm* and the multiplication $K_i \cdot vv$. Define W_i: computational effort related to multiplication $K_i \cdot v$ as *a* unit. According to Remark 3.6, one has

$$W_i=0\,(n_i m_i^d), \tag{4.12}$$

where d is the number of spatial coordinates (for our example: $d=1$). Assuming $n_i \simeq 2n_{i-1}$ and $m_i \simeq \sqrt{2}\, m_{i-1}$ (cf. (4.1)) we conclude that

$$W_i=2^{1+d/2}\, W_{i-1} \quad (1 \leqslant i \leqslant l).$$

Let $M_{i,\nu}$ be the effort required by ν multi-grid iterations at level i, i.e. the expense resulting from calling *mgm* $(i, \ldots, \nu)$. From

$$M_{i,\nu}=\nu M_{i-1,\mu}+(\nu-1)\, W_i \quad (1 \leqslant i \leqslant l,\ \nu \geqslant 1) \tag{4.13}$$

and (4.12) we obtain

$$M_{l,\nu}=\nu\mu^{l-2}\, M_{0,\mu}+\nu C_l\, W_l-\left(1+\frac{\nu}{\mu}\right)(C_l-1)\, W_l, \tag{4.14}$$

where

$$C_l=\sum_{i=0}^{l-1} (\mu/2^{1+d/2})^i.$$

$M_{l,\nu}$ is the computational work required by *mgm* in (4.11). For $\mu=2$ the constant C_l is uniformly bounded by $2^{d/2}/(2^{d/2}-1)$. Although $C_l \to \infty$ for $\mu=3$, $d=1$, the values C_l increase very slowly. The following table shows C_l for different values of μ and l:

μ \ l	1	2	3	4	5
2	1.000	1.707	2.207	2.561	2.811
3	1.000	2.061	3.186	4.379	5.645
4	1.000	2.414	4.414	7.243	11.24

Values of C_l for $d=1$

The number $M_{0,\mu}$ depends on the iteration chosen at level $i=0$. It should be proportional to n_l; thus it is asymptotically smaller than W_l. This reason justifies the rough estimate

$$M_{l,\nu} \simeq \nu C_l W_l \tag{4.15}$$

Numerical experiments showed that usually $\mu=2$ or $\mu=3$ are the best choice. For these values the number C_l is adequately small, so that the multi-grid iteration is relatively cheap in terms of the computational effort.

5. Numerical Examples

In the following we use grid steps h_i, prolongations π_i, and restrictions ρ_i that are slightly different from those proposed in Section 4.1 (but not necessarily better). Instead of (4.1) we set $n_i=2^i(n_0-1)+1$. π_l is the piecewise linear interpolation, e.g. π_{l-1} is defined by

$$(\pi_{l-1} v)(2j)=v(j) \qquad (0 \leqslant j \leqslant n_{l-1}-1),$$

$$(\pi_{l-1} v)(2j+1)=\frac{1}{2}(v(j)+v(j+1)) \qquad (0 \leqslant j \leqslant n_{l-1}-2).$$

The restriction ρ_i equals almost the adjoint π_i^* of π_i, the difference residing in the fact that the components $(\rho_i v)(0)$ and $(\rho_i v)(n_i-1)$ are scaled in such a way that the constant n_l-vector $\mathbf{1}$ is mapped into the n_i-vector $\rho_i \mathbf{1}=\mathbf{1}$.

In the first example the differential operator $L=-\beta \frac{\partial}{\partial x} \alpha \frac{\partial}{\partial x}$ of (2.1) is defined by

$$\alpha(x)=\begin{cases} 1 & \text{if } x \leqslant 1/2, \\ 1/2 & \text{if } x>1/2, \end{cases} \qquad \beta(x)=\frac{1}{2}+\frac{1}{4}\sin(2\pi x). \tag{5.1a}$$

The function z is chosen as

$$z(x)=x-1/2. \tag{5.1b}$$

The time step $h_i=T/n_i$ is defined by

$$T=1, \quad n_0=9, \quad n_1=17, \quad n_2=33, \quad n_3=65. \tag{5.1c}$$

That means we have $l=3$. The resulting solution at level l is

$$u^*(j)=+1 \ (0 \leqslant j \leqslant 28), \quad u^*(29)=0.338\ldots, \quad u^*(j)=-1 \ (30 \leqslant j \leqslant 64).$$

Functions u^* together with $q(u^*)$ are shown in Fig. 5.1.

The multi-grid iteration with $\mu=2$ resulted in $u^j \to u^*$. The difference $\delta = J_{n_l m_l}(u^j)-J_{n_l m_l}(u^*)$ is decreasing by a factor 1/100 per iteration step:

j	1	2	3	4	5	6	7
δ	19E-2	17E-4	17E-6	17E-8	17E-10	17E-12	<1E-14

Thus, u^7 equals u^* up to rounding errors. According to (4.15) the computation of u^7 requires the same effort as $7C_3=15.4$ initial-boundary values problems on the finest grid. The total CPU time[2] for computing u^7 is less than 0.4s. If one would solve the discrete control problem (3.4) by some other method that requires the entries of the matrix $K=K_l$ explicitly, one would consume 1.1s only for computing the matrix K.

The second example is given by

$$\alpha(x)=\beta(x)=1, \quad z(x)=8(x-1/2)^2-1, \quad T=1. \tag{5.2}$$

For the same discretization parameter as above and by the same computational work one obtains u^* and $q(u^*)$ as given in Fig. 5.2. It is a typical phenomenon

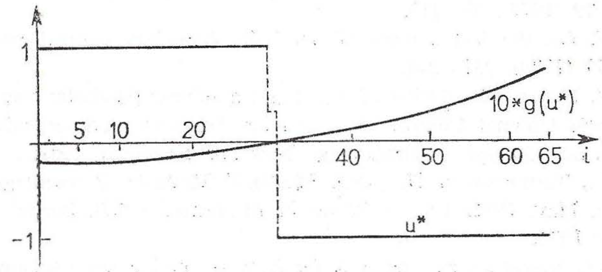

Fig. 5.1. Optimal u^* and $q(u^*)$ for first example

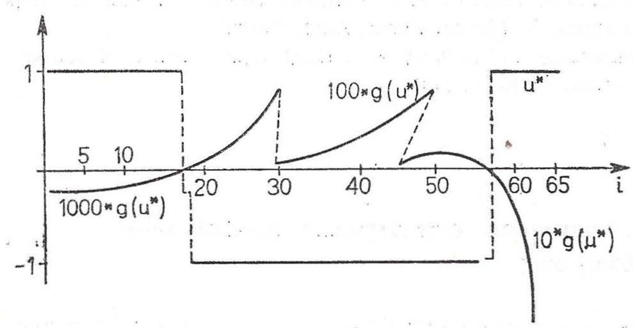

Fig. 5.2. Optimal u^* and $q(u^*)$ for second example
Attention: $q(u^*)$ erroneously printed $g(u^*)$ in the figure

that the values of $q(u^*)$ are close to zero. As a consequence the position of sign change is badly conditioned. By (3.5) the optimal control is badly conditioned, too.

There may be several reasons for $q(u_n^*)$ being very close to zero:

(i) z_m is almost in the image $\{S_{nm}u_n\colon\ u_n \in U_{ad}^n\}$,
(ii) $z_m - S_{nm}u_n^*$ is oscillating,
(iii) z_m is oscillating.

In case (i) the smallness of $z_m - S_{nm}u_n^*$ implies the smallness of $q(u_n^*)=S_{nm}^*(S_{nm}u_n^*-z_m)$. $w_m=S_{nm}u_n^*-z_m$ is the final value of the adjoint problem giving $q(u_n^*)=S_{nm}^*w_m$. Since parabolic equation damps oscillations, $q(u_n^*)$ is small in case (ii), too. By

[2] The computations were performed on Cyber 76 of the Rechenzentrum der Universität zu Köln.

the same argument the image $S_{nm} u_n$ consists of smooth functions so that case (iii) leads to (ii).

The numerical experiments showed that the proposed multi-grid iteration (without modification (4.5d′, e′)) is increasingly slow or even divergent if $q(u_n^*)$ becomes increasingly smaller.

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Pewna metoda numeryczna rozwiązywania parabolicznego zadania typu bang-bang

Dla równania parabolicznego rozważa się zadanie sterowania brzegowego z funkcjonałem jakości zależnym od stanu końcowego.

Rozwiązanie zadania jest typu bang-bang. Aproksymując równanie paraboliczne schematem różnicowym otrzymuje się dyskretny problem sterowania. Dla numerycznego rozwiązania tego problemu opracowano iteracyjną metodę wielosiatkową. Przykłady numeryczne wskazują na efektywność metody.

Некоторый численный метод решения параболической задачи типа банг-банг

Для параболического уравнения решается краевая задача управления с функционалом качества, зависящим от конечного состояния. Решение задачи является типа банг-банг. Аппроксимируя параболическое уравнение по разностной схеме, получаем дискретную задачу управления. Для численного решения этой задачи разработан итерационный многосеточный метод. Численные примеры иллюстрируют эффективность метода.