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On the continuity of the implicit mapping

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A theorem on the continuity of implicit mapping related to Clarke's generalized Jacobian is formulated and proved.

1. Introduction

We say that a mapping f: $R^n \to R^n$ is locally Lipschitzian in an open subset $G \subset R^n$ if for every $x_0 \in G$ there exists a neighbourhood $Q(x_0) \subset G$ and some constant K, such that

$$|f(x)-f(y)| \leqslant K|x-y|, \quad \forall x, y \in Q(x_0), \tag{1}$$

where $|\cdot|$ denotes the usual Euclidean norm.

The usual $n \times n$ Jacobian matrix of partial derivatives, when it exists, is denoted $Jf(x)$. We topologize the vector space $\mathcal{M}$ of $n \times n$ matrices by the norm

$$\|M\| = \max |m_{ij}|, \text{ where } M=(m_{ij}), \quad 1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant n.$$

DEFINTION 1. The generalized Jacobian of f at $x_0 \in G$, denoted by $\partial f(x)_0$, is the convex hull of all matrices M of the form

$$M = \lim_{i \to \infty} Jf(x_i), \tag{2}$$

where the sequence $\{x_i\}$ converges to x_0 and f is differentiable at x_i for each i.

PROPOSITION 1. *$\partial f(x_0)$ is a non empty compact cvnvex subset vf $\mathcal{M}$, $\partial f(x_0) \subset \mathcal{M}$* (see [1]).

DEFINTION 2. $\partial f(x_0)$ is said to be of maximal rank if every M in $\partial f(x_0)$ is of maximal rank.

The following theorem is proved in [1].

THEOREM 1. *If $\partial f(x_0)$ is of maximal rank, then there exist neighbourhoods U and V of x_0 and $f(x_0)$ respectively, and a Lipschitzian function $g\colon V\to R^n$ such that*

a) $g(f(x))=x$ *for every* $x\in U$,

b) $f(g(y))=y$ *for every* $y\in V$.

2. Theorem on the continuity of implicit Lipschitzian mappings

We consider a topological space E and a mapping f of an open subset $W\subset E\times R^n$ into R^n. Let $f(a,b)=c$ for the fixed points $a\in E$, $b\in R^n$, $(a,b)\in W$ and $c\in R^n$.

For fixed point x the generalized Jacobian of $y\to f(x,y)$ at a point b will be denoted by $\partial_y f(x,b)$.

DEFINTION 3. We say that a function $f(x,y)$ is locally Lipschitzian with respect to y in W, if for arbitrary point $(a,b)\in W$ there exists a neighbourhood $Q(a,b)$ in W and the function $y\to f(x,y)$ is Lipschitzian when $(x,y)\in Q(a,b)$ i.e. there exists some constant $\delta>0$ such that

$$|f(x,y_1)-f(x,y_2)|\leqslant\delta\,|y_1-y_2|$$

for every (x,y_1), $(x,y_2)\in Q(a,b)$.

THEOREM 2. *Let a mapping $f\colon W\to R^n$ satisfy the following conditions:*

(α) *$f(x,y)$ is locally Lipschitzian in W with respect to y.*

(β) *$\partial_y f(x,b)$ is of maximal rank, for $(x,b)\in W$,*

(γ) *the mapping $x\to f(x,y)$ is continuous, for $(x,y)\in W$,*

then there exist neighbourhoods U and V of (a,b) and (a,c) respectively, and a mapping $g\colon V\to R^n$ such that

(i) *$g(x,z)$ is Lipschitzian with respect to z in V,*

(ii) *$g(x,f(x,y))=y$ for every $(x,y)\in U$ and $f(x,g(x,z))=z$ for every $(x,z)\in V$,*

(iii) *the mapping $x\to g(x,z)$ is continuous for $(x,z)\in V$.*

Proof. By Theorem 1 for a fixed $(a,b)\in W$ there exist neighbourhoods $U_a(b)$ and $V_a(c)$ in R^n of b and c respectively, and a Lipschitzian function $g(a,z)$: $V_a(c)\to R^n$ with Lipschitz-constant $\delta>0$, which has the following properties:

$$g(a,f(a,y))=y \qquad \forall\, y\in U_a(b)$$

$$f(a,g(a,z))=z \qquad \forall\, z\in V_a(c).$$

The mapping $x\to f(x,b)$ is continuous so that there exists also a neighbourhood $A(a)$ such that

$$f(x,b)\in V_a(c) \quad \text{for arbitrary} \quad x\in A(a).$$

From the above considerations it follows that we can apply Theorem 1 at the points (x,b) and z, where $z=f(x,b)$ and $x\in A(a)$.

Hence there exist neighbourhoods $U_x(b)$ and $V_x(z)$, which depend on $x \in A(a)$, and a Lipschitzian function $g(x, \xi)$ such that

$$g(x, f(x, y)) = y \quad \forall y \in U_x(b), \quad x \in A(a),$$
$$f(x, g(x, \xi)) = \xi \quad \forall \xi \in V_x(z), \quad x \in A(a),$$

Let $\bar{U}_x(b) = U_a(b) \cap U_x(b)$, $\bar{V}_x(z) = V_a(c) \cap V_x(z)$ and

$$U(b) = \bigcup_{x \in A(a)} \bar{U}_x(b); \quad U(b) \subset U_a(b)$$

$$V(c) = \bigcup_{x \in A(a)} \bar{V}_x(z); \quad V(c) \subset V_a(c).$$

At the begining we shall prove the theorem in the neighbourhoods $U \equiv A(a) \times \times U(b)$ and $V \equiv A(a) \times V(c)$ in W of points (a, b) and (a, c) respectively, and mapping $g \colon V \to R^n$.

Obviously, by Theorem 1 the mapping $g(x, z)$ is Lipschitzian in V with respect to ϑz with Lipschitz-constant $\delta > 0$, since $V(c) \subset V_a(c)$, i.e.

$$\forall (x, z'), (x, z'') \in V \equiv A(a) \times V(c) \Rightarrow z', z'' \in V(c) \subset V_a(c)$$

and

$$|g(x, z'') - g(x, z'')| \leqslant \delta |z' - z''|.$$

Now we shall prove (ii). Let $(x, y) \in U \equiv A(a) \times U(b)$, then $U(b) = \bigcup_{x \in A(a)} \bar{U}_x(b)$ and there exists

$$\bar{U}_x(b) \ni y, \quad \bar{U}_x(b) = U_a(b) \cap U_x(b).$$

Hence we get

$$z = f(x, y) \in \bar{V}_x(z) \quad \text{and } g(x, f(x, y)) = y.$$

On the other hand for arbitrary $(x, \xi) \in V$, $V \equiv A(a) \times V(c)$, there exists $\bar{V}_x(z) = V_a(c) \cap V_x(z)$, $(x, \xi) \in \bar{V}_a(z)$. Hence we obtain also

$$y = g(x, \xi), \quad y \in U_x(b) \quad \text{and} \quad f(x, g(x, \xi)) = \xi.$$

Now we shall prove (iii). Let (x_0, z), $(x, z) \in V$. We put $y_0 = g(x_0, z)$, $y = g(x, z)$, then we have

$$g(x_0, f(x_0, y)) = y, \quad g(x; f(x, y)) = y. \tag{3}$$

It follows that

$$g(x, z) - g(x_0, z) = g(x, f(x, y)) - g(x_0, f(x_0, y)) +$$
$$+ g(x_0, f(x_0, y)) - g(x_0, f(x, y)) \quad \text{and}$$
$$g(x, z) - g(x_0, z) = g(x_0, f(x_0, y)) - g(x_0, f(x, y)). \tag{4}$$

Taking advantage of the fact that $g(x, z)$ is Lipschitzian with respect to z in V we can obtain from (4) the following inequality:

$$|g(x, z) - g(x_0, z)| \leqslant \delta |f(x_0, y) - f(x, y)|. \tag{5}$$

Since the mapping $f(x, y)$ is continuous with respect to x in W, inequality (5) implies the continuity of the mapping $g(x, z)$ with respect to x in V, and completes our proof. ■

References

[1] CLARKE F. H. On the inverse function theorem. *Pacific Journal of Mathematics* **64** (1976) 1.

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O ciągłości odwzorowania uwikłanego

Formułuje się i dowodzi twierdzenia o ciągłości odwzorowania uwikłanego, związanego z uogólnianym jakobianem Clarke'a.

О непрерывности неявного отображения

Формулируется и доказывается теорема о непрерывности неявного отображения, связанного с обобщаемым якобианом Кларка.