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Minimum Energy Control of 3-D Linear Systems

by

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The definition of the state-transition matrix is given and the general solution formula for 3-D linear systems is derived. The Cayley-Hamilton theorem is extended for 3-D linear systems. Necessary and sufficient conditions for local controllability of 3-D linear systems are formulated and proved. The minimum energy control problem for 3-D linear systems is solved and illustrated by a simple example.

1. Introduction

Two-dimensional (2-D) linear systems theory has received great attention during the last ten years [1—9]. Recently there has been some interest shown in three-dimensional (3-D) linear systems theory [10] which finds applications in 3-D computer memory design, 3-D sequential machines and 3-D distributed-parameter process design.

The main purpose of this paper is to formulate and solve the minimum energy control problem for 3-D linear systems. The minimum energy control problem for 2-D linear systems has been considered by Klamka in [5].

To solve the minimum energy control problem of 3-D systems first the definition of the state-transition matrix will be given and the general solution formula for 3-D linear systems will be derived. Next necessary and sufficient conditions for local controllability of 3-D linear systems will be presented.

2. State-transition matrix and general solution formula

Consider a 3-D linear system described by the equations [10]

$$x' = Ax + Bu \tag{1a}$$

$$y = Cx + Du \tag{1b}$$

where

$$x'=\begin{bmatrix} x^h(i+1,j,k) \\ x^v(i,j+1,k) \\ x^d(i,j,k+1) \end{bmatrix}, \quad x=\begin{bmatrix} x^h(i,j,k) \\ x^v(i,j,k) \\ x^d(i,j,k) \end{bmatrix}$$

$$A=\begin{bmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{bmatrix}, \quad B=\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}, \quad C=[C_1\ C_2\ C_3]$$

$x^h(i,j,k)=x^h \in R_{n_1}$ is the horizontal state vector,
$x^v(i,j,k)=x^v \in R_{n_2}$ is the vertical state vector,
$x^d(i,j,k)=x^d \in R_{n_3}$ is the depth state vector,
$u(i,j,k)=u \in R_m$ is the input vector,
$y(i,j,k)=y \in R_l$ is the output vector,
A_i, B_i, C_i, D are constant matrices of appropriate dimensions and $(i,j,k) \in Z\times Z\times Z=Z^3$; Z is the set of integer numbers. It is assumed that the boundary conditions

$$x^h(0,p,q), \quad x^v(r,0,q), \quad x^d(r,p,0) \tag{1c}$$

for

$$r,p,q=0,1,2,\ldots \text{ are given}$$

DEFINITION 1. The following partial ordering is used for integer triple
$(r,p,q)\leqslant(i,j,k)$ if and only if $r\leqslant i$, $p\leqslant j$ and $q\leqslant k$
$(r,p,q)=(i,j,k)$ if and only if $r=i$, $p=j$ and $q=k$
$(r,p,q)<(i,j,k)$ if and only if $(r,p,q)\leqslant(i,j,k)$ and $(r,p,q)\neq(i,j,k)$.

DEFINITION 2. For

$$A=\begin{bmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{bmatrix} \tag{2}$$

the matrix $A^{i,j,k}$ which has the following properties
1) $A^{0,0,0}=I$ (the identity matrix)

2)
$$A^{1,0,0}=\begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{0,1,0}=\begin{bmatrix} 0 & 0 & 0 \\ A_4 & A_5 & A_6 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^{0,0,1}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_7 & A_8 & A_9 \end{bmatrix}$$

3)
$$A^{i,j,k}=A^{1,0,0}A^{i-1,j,k}+A^{0,1,0}A^{i,j-1,k}+A^{0,0,1}A^{i,j,k-1} \tag{2a}$$

for $(i,j,k)>(0,0,0)$

4) $A^{-i,j,k}=A^{i,-j,k}=A^{i,j,-k}=0$ (the zero matrix)

for $i,j,k=1,2,\ldots$
is called the state — transition matrix.

From the definition 2 the following properties of $A^{i,j,k}$ follow:

1) $A=A^{1,0,0}+A^{0,1,0}+A^{0,0,1}$

2) $A^{i,0,0}=(A^{1,0,0})^i, \quad A^{0,j,0}=(A^{0,1,0})^j, \quad A^{0,0,k}=(A^{0,0,1})^k$

for $i,j,k=1,2,\dots$

3) $A^{1,0,0}=I^{1,0,0}A, \quad A^{0,1,0}=I^{0,1,0}A, \quad A^{0,0,1}=I^{0,0,1}A$

where

$$I^{1,0,0}=\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I^{0,1,0}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I^{0,0,1}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{n_3} \end{bmatrix}$$

4) $I^{1,0,0}A^{0,j,k}=0$ for $(j,k)>(0,0)$

$I^{0,1,0}A^{i,0,k}=0$ for $(i,k)>(0,0)$

$I^{0,0,1}A^{i,j,0}=0$ for $(i,j)>(0,0)$

THEOREM 1. *The solution to the equation* (1a) *satisfying the boundary conditions* (1c) *has the form*

$$\begin{aligned} x(i,j,k)= & \sum_{(0,0)\leqslant(p,q)\leqslant(j,k)}\sum A^{i,j-p,k-q}x^{1,0,0}(0,p,q)+ \\ & +\sum_{(0,0)\leqslant(r,q)\leqslant(i,k)}\sum A^{i-r,j,k-q}x^{0,1,0}(r,0,q)+ \\ & +\sum_{(0,0)\leqslant(r,p)\leqslant(i,j)}\sum A^{i-r,j-p,k}x^{0,0,1}(r,p,0)+ \\ & +\sum_{(0,0,0)\leqslant(r,p,q)<(i,j,k)}\sum\sum M(i-r,j-p,k-q)\,u(r,p,q) \end{aligned} \tag{3}$$

where

$$x^{1,0,0}(0,p,q)=\begin{bmatrix} x^h(0,p,q) \\ 0 \\ 0 \end{bmatrix}, \quad x^{0,1,0}(r,0,q)=\begin{bmatrix} 0 \\ x^v(r,0,q) \\ 0 \end{bmatrix},$$

$$x^{0,0,1}(r,p,0)=\begin{bmatrix} 0 \\ 0 \\ x^d(r,p,0) \end{bmatrix}$$

$$M(i,j,k)=A^{i-1,j,k}\begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix}+A^{i,j-1,k}\begin{bmatrix} 0 \\ B_2 \\ 0 \end{bmatrix}+A^{i,j,k-1}\begin{bmatrix} 0 \\ 0 \\ B_3 \end{bmatrix} \tag{4}$$

P r o o f. The proof is accomplished by induction. Substituting $i=j=k=0$ into (3) we obtain

$$\begin{aligned} x(0,0,0)= & A^{0,0,0}x^{1,0,0}(0,0,0)+A^{0,0,0}x^{0,1,0}(0,0,0)+ \\ & +A^{0,0,0}x^{0,0,1}(0,0,0)+M(0,0,0)\,u(0,0,0)=A^{0,0,0}x(0,0,0) \end{aligned}$$

Thus, the hypothesis is true for $(i,j,k)=(0,0,0)$.

Assuming that the hypothesis is true for all (r, p, q) such that $(0, 0, 0) \leqslant (r, p, q) < (i, j, k)$ we shall show that it is true for (i, j, k).

From (1a) and (3) we have

$$x(i,j,k) = A^{1,0,0}\, x(i-1,j,k) + A^{0,1,0}\, x(i,j-1,k) + g^{0,0,1}\, x(i,j,k-1) =$$

$$= A^{1,0,0}\Big[\sum_{(0,0)\leqslant(p,q)\leqslant(j,k)}\sum A^{i-1,j-p,k-q}\, x^{1,0,0}(0,p,q) + \sum_{(0,0)\leqslant(r,q)\leqslant(i-1,k)}\sum A^{i-r-1,j,k-q} x^{0,1,0}(r,0,q) + \sum_{(0,0)\leqslant(r,p)(i-1,j)}\sum A^{i-r-1,j-p,k} \times$$

$$\times x^{0,0,1}(r,p,0) + \sum\sum_{(0,0,0)\leqslant(r,p,q)<(i-1,j,k)}\sum M(i-r-1, j-p, k-q)\, u(r,p,q)\Big] +$$

$$+ A^{0,1,0}\Big[\sum_{(0,0)\leqslant(p,q)\leqslant(j-1,k)}\sum A^{i,j-p-1,k-q}\, x^{1,0,0}(0,p,q) + \sum_{(0,0)\leqslant(r,q)\leqslant(i,k)}\sum A^{i-r,j-1,k-q}\, x^{0,1,0}(r,0,q) + \sum_{(0,0)\leqslant(r,p)\leqslant(i,j-1)}\sum A^{i-r,j-p-1,k} \times$$

$$\times x^{0,0,1}(r,p,0) + \sum\sum_{(0,0,0)\leqslant(r,p,q)<(i,j-1,k)}\sum M(i-r, j-p-1, k-q)\, u(r,p,q)\Big] +$$

$$+ A^{0,0,1}\Big[\sum_{(0,0)\leqslant(p,q)\leqslant(j,k-1)}\sum A^{i,j-p,k-q-1}\, x^{1,0,0}(0,p,q) + \sum_{(0,0)\leqslant(r,q)\leqslant(i,k-1)}\sum A^{i-r,j,k-q-1}\, x^{0,1,0}(r,p,0) + \sum_{(0,0)\leqslant(r,p)\leqslant(i,j)}\sum A^{i-r,j-p,k-1} \times$$

$$\times x^{0,0,1}(r,p,0) + \sum\sum_{(0,0,0)\leqslant(r,p,q)<(i,j,k-1)}\sum M(i-r, j-p, k-q-1)\, u(r,p,q)\Big]$$

Taking into account (2a), (4) and performing some simple manipulations we obtain

$$x(i,j,k) = \sum_{(0,0)\leqslant(p,q)\leqslant(j,k)}\sum A^{i,j-p,k-q}\, x^{1,0,0}(0,p,q) + \sum_{(0,0)\leqslant(r,q)\leqslant(i,k)}\sum A^{i-r,j,k-q} \times$$

$$\times x^{0,1,0}(r,0,q) + \sum_{(0,0)\leqslant(r,p)\leqslant(i,j)}\sum A^{i-r,j-p,k}\, x^{0,0,1}(r,p,0) +$$

$$+ \sum\sum_{(0,0,0)\leqslant(r,p,q)<(i,j,k)}\sum M(i-r, j-p, k-q)\, u(r,p,q) \qquad \blacksquare$$

If the input vector $u(i,j,k)=0$ for all (i,j,k) and the boundary conditions (1c) are zero for $(i,j,k) \neq (0,0,0)$ then from (3) it follows that

$$x(i,j,k) = A^{i,j,k}\, x(0,0,0) \tag{5}$$

3. 3-D Cayley-Hamilton theorem

DEFINITION 3. The determinant of the matrix $[I^{1,0,0}\, x + I^{0,1,0}\, y + I^{0,0,1}\, z - A]$ for the matrix (2) is called the 3-D characteristic function of the matrix, i.e.

$$f(x,y,z)=det\,[I^{1,0,0}x+I^{0,1,0}y+I^{0,0,1}z-A]=$$

$$=\begin{vmatrix} I_{n_1}x-A_1, & -A_2, & -A_3 \\ -A_4, & I_{n_2}y-A_5, & -A_6 \\ -A_7, & -A_8, & I_{n_3}z-A_9 \end{vmatrix}=\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3} a_{ijk}x^i y^j z^k \tag{6}$$

where $a_{n_1 n_2 n_3}=1$

The equation $f(x, y, z)=0$ is called the characteristic equation of the matrix (2).

3-D Cayley-Hamilton theorem

Every matrix (2) satisfies its own characteristic equation, i.e.

$$f(A)=f(x,y,z)\,\Big|_{x^i y^j z^k=A^{i,j,k}}=\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3} a_{ijk}A^{i,j,k}=0 \tag{7}$$

P r o o f. From definition of the inverse matrix we have the relation

$$If(x,y,z)=\left[\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3} A_{ijk}x^i y^j z^k\right][I^{1,0,0}x+I^{0,1,0}y+I^{0,0,1}z-A]=$$

$$=\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3} A_{ijk}[I^{1,0,0}x^{i+1}y^j z^k+I^{0,1,0}x^i y^{j+1}z^k+I^{0,0,1}x^i\times$$

$$y^j z^{k+1}-Ax^i y^j z^k] \tag{8}$$

where

$$\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3} A_{ijk}x^i y^j z^k=Adj\,[I^{1,0,0}x+I^{0,1,0}y+I^{0,0,1}z-A]$$

Substituting $A^{i,j,k}$ for the term $x^i y^j z^k$ and taking into account (2a) and the properties (1) and (3) of $A^{i,j,k}$ we obtain.

$$[I^{1,0,0}A^{i+1,j,k}+I^{0,1,0}A^{i,j+1,k}+I^{0,0,1}A^{i,j,k+1}-AA^{i,j,k}]=0$$

and $f(A)=0$ ■

The 3-D Cayley-Hamilton theorem can be used to find $A^{i,j,k}$ for $(i,j,k)\geqslant(n_1,n_2,n_3)$.

4. Local controllability

For $(h,f,g)<(r,p,q)$ we can define the cube $[(h,f,g),(r,p,q)]\subset Z^3$ as follows

$$[(h,f,g),(r,p,q)]=\{(i,j,k)\in Z^3:(h,f,g)\leqslant(i,j,k)\leqslant(r,p,q)\} \tag{9}$$

DEFINITION 4. The 3-D system (1) is said to be locally controllable in the cube $[(0,0,0),(r,p,q)]$, if for every set of boundary conditions:

$$x^h(0,j,k),\ (0,0)\leqslant(j,k)\leqslant(p,q);\quad x^v(i,0,k),\ (0,0)\leqslant(i,k)\leqslant(r,q);$$
$$x^d(i,j,0),\ (0,0)\leqslant(i,j)\leqslant(r,p),$$

and every vector $S \in R^{n_1+n_2+n_3}$, there exists a sequence of vector inputs $u(i,j,k)$, $(0,0,0) \leqslant (i,j,k) < (r,p,q)$ such that

$$x(r,p,q) = S$$

THEOREM 2. *The 3-D system (1) is locally controllable in the cube* $[(0,0,0),(r,p,q)]$ *if and only if*

$$\text{rank}\,[M(1,0,0), M(0,1,0), M(0,0,1), \ldots, M(i,j,k), \ldots, M(r,p,q)] = n_1+n_2+n_3 \tag{10}$$

where $M(i,j,k)$ *is defined by* (4).

Proof. For the given boundary conditions and the vector $S = x(r,p,q)$ from (3) we have

$$\bar{S}(r,p,q) = S - \sum_{(0,0)\leqslant(j,k)\leqslant(p,q)}\sum A^{r,p-j,q-k}x^{1,0,0}(0,j,k) - \sum_{(0,0)\leqslant(i,k)\leqslant(r,q)}\sum A^{r-i,p,q-k}x^{0,1,0}(i,0,k) - \sum_{(0,0)\leqslant(i,j)\leqslant(r,p)}\sum A^{r-i,p-j,q}\times$$
$$\times x^{0,0,1}(i,j,0) = \sum\sum_{(0,0,0)\leqslant(i,j,k)\leqslant(r,p,q)}\sum M(r-i,p-j,q-k)\,u(i,j,k) = C(r,p,q)\begin{bmatrix} u(r-1,p,q) \\ u(r,p-1,q) \\ u(r,p,q-1) \\ \cdots\cdots\cdots \\ u(r-i,p-j,q-k) \\ \cdots\cdots\cdots\cdots \\ u(0,0,0) \end{bmatrix} \tag{11}$$

where

$$C(r,p,q) = [M(1,0,0), M(0,1,0), M(0,0,1), \ldots, M(i,j,k), \ldots, M(r,p,q)]$$

Note that for every boundary conditions and every vector S a sequence of vector inputs $u(i,j,k)$, $(0,0,0) \leqslant (i,j,k) < (r,p,q)$ can be found if and only if the condition (10) is satisfied.

REMARK. From 3-D Cayley-Hamilton theorem it follows that in the condition (10) it can be assumed $(r,p,q) < (n_1,n_2,n_3)$.

Let us define the local controllability matrix $W(r,p,q)$ of the system (1) in the following way

$$W(r,p,q) = \sum\sum_{(0,0,0)\leqslant(i,j,k)<(r,p,q)}\sum M(r-i,p-j,q-k)\,M^T(r-i,p-j,q-k) = \sum\sum_{(0,0,0)<(i,j,k)\leqslant(r,p,q)}\sum M(i,j,k)\,M^T(i,j,k) \tag{12}$$

where "T" denotes the transposition

THEOREM 3. *The 3-D system* (1) *is locally controllable in the cube* $[(0, 0, 0), (r, p, q)]$ *if and only if rank* $W(r,p,q)=n_1+n_2+n_3$ (13)

P r o o f. From the relation $W(r, p, q)=C(r, p, q)\, C^T(r, p, q)$ it follows that the condition (10) is satisfied if and only if the condition (13) holds. ∎

5. Minimum energy control

Note that for the system (1) locally controllable in the cube $[(0, 0, 0), (r, p, q)]$ there generally exist many different sequences of vector inputs $u(i,j,k), (0,0,0)\leqslant \leqslant(i,j,k)<(r,p,q)$ which transfer the system state from initial local state $x(0, 0, 0)$ (with given boundary conditions) to the desired final local state $x(r, p, q)=S$. Among the admissible sequences of vector inputs we shall look for such one which minimizes the performance index

$$J(u)=\sum_{(0,0,0)\leqslant(i,j,k)<(r,p,q)}\sum\sum u^T(i,j,k)\,Qu(i,j,k). \tag{14}$$

where Q is the given $(m\times m)$ dimensional, symmetric and nonsingular weighting matrix.

The minimum energy control problem for 3-D linear system (1) can be stated as follows.

Given the matrices A, B, Q and the boundary conditions (1c) in the cube $[(0, 0, 0), (r,p,q)]$, find the sequence of vector inputs $u^*(i,j,k)$ for $(0,0,0)\leqslant(i,j,k)<(r,p,q)$ which transfer the system state from initial local state $x(0, 0, 0)$ to the final local state $S=x(r,p,q)$ and minimizes the performance index (14).

To solve the problem let us define the matrix

$$W_Q(r,p,q)=\sum_{(0,0,0)\leqslant(i,j,k)<(r,p,q)}\sum\sum M(r-i,p-j,q-k)\,Q^{-1}\,M^T(r-i,p-j,q-k) \tag{15}$$

It is easy to prove that the nonsingularity of the matrix (14) is equivalent to the nonsingularity of the matrix (12). Thus, we can define the sequence of vector inputs

$$u^*(i,j,k)=Q^{-1}\,M^T(r-i,p-j,q-k)\,W_Q^{-1}(r,p,q)\,\bar{S}(r,p,q) \tag{16}$$

where the vector $\bar{S}(r,p,q)$ is defined by (11).

THEOREM 4. *Let us assume that*

i) *the system* (1) *is locally controllable in the cube* $[(0, 0, 0), (r, p, q)]$

ii) $\bar{u}(i,j,k)$ *is any sequence of vector inputs defined for* $(0,0,0)\leqslant(i,j,k)<(r,p,q)$ *which transfers the initial local state* $x(0, 0, 0)$ *to the desired final local state* S. *Then the sequence of vector inputs* (16) *accomplishes the same transfer and*

$$J(u^*)\leqslant J(\bar{u}) \tag{17}$$

Moreover, the minimum value of (14) *is given by*

$$J(u^*)=\bar{S}^T(r,p,q)\,W_Q^{-1}(r,p,q)\,\bar{S}(r,p,q) \tag{18}$$

Proof. First we shall show that the sequence of vector inputs $u^*(i,j,k)$ defined by (16) transfers the initial local state $x(0,0,0)$ to the final local state S. Substituting (16) into (3) for $i=r$, $j=p$, $q=k$ and using (11) and (15) we obtain

$$x(r,p,q)=\sum\sum_{(0,0)\leq(j,k)\leq(p,q)} A^{r,p-j,q-k}x^{1,0,0}(0,j,k)+$$
$$+\sum\sum_{(0,0)\leq(i,k)\leq(r,q)} A^{r-i,p,q-k}x^{0,1,0}(i,0,k)+\sum\sum_{(0,0)\leq(i,j)\leq(r,p)} A^{r-i,p-j,q}x^{0,0,1}(i,j,0)+$$
$$+\sum\sum\sum_{(0,0,0)\leq(i,j,k)<(r,p,q)} M(r-i,p-j,q-k)Q^{-1}M^T(r-i,p-j,q-k)\times$$
$$\times W_Q^{-1}(r,p,q)\bar{S}(r,p,q)=S(r,p,q)$$

Since $\bar{u}(i,j,k)$ and $u^*(i,j,k)$ transfer the initial local state $x(0,0,0)$ to the final local state S we can write

$$\sum\sum\sum_{(0,0,0)\leq(i,j,k)<(r,p,q)} M(r-i,p-j,q-k)\bar{u}(i,j,k)=$$
$$=\sum\sum\sum_{(0,0,0)\leq(i,j,k)<(r,p,q)} M(r-i,p-j,q-k)u^*(i,j,k)$$

and

$$\sum\sum\sum_{(0,0,0)\leq(i,j,k)<(r,p,q)} M(r-i,p-j,q-k)[\bar{u}(i,j,k)-u^*(i,j,k)]=0 \tag{19}$$

From (19) and (16) it follows that

$$\sum\sum\sum_{(0,0,0)\leq(i,j,k)<(r,p,q)} [\bar{u}(i,j,k)-u^*(i,j,k)]^T M^T(r-i,p-j,q-k)W_Q^{-1}\times$$
$$\times(r,p,q)\bar{S}(r,p,q)=\sum\sum\sum_{(0,0,0)\leq(i,j,k)<(r,p,q)} [\bar{u}(i,j,k)-u^*(i,j,k)]^T\times$$
$$\times Qu^*(i,j,k)=0 \tag{20}$$

Using (20) it is easy to show that

$$\sum\sum\sum_{(0,0,0)\leq(i,j,k)<(r,p,q)} \bar{u}^T(i,j,k)Q\bar{u}(i,j,k)=\sum\sum\sum_{(0,0,0)\leq(i,j,k)<(r,p,q)} u^{*T}(i,j,k)\times$$
$$Qu^*(i,j,k)+\sum\sum\sum_{(0,0,0)\leq(i,j,k)<(r,p,q)} [\bar{u}(i,j,k)-u^*(i,j,k)]^T Q\times$$
$$[\bar{u}(i,j,k)-u^*(i,j,k)] \tag{21}$$

Since the last term in (21) is always nonnegative, then the inequality (17) holds.

To obtain the minimum value of the performance index (14) we substitute (16) into (14)

$$J(u^*)=\sum\sum\sum_{(0,0,0)\leq(i,j,k)<(r,p,q)} u^{*T}(i,j,k)Qu^*(i,j,k)=$$
$$=\sum\sum\sum_{(0,0,0)\leq(i,j,k)<(r,p,q)} [Q^{-1}M^T(r-i,p-j,q-k)W_Q^{-1}(r,p,q)\bar{S}(r,p,q)]^T Q\times$$
$$\times[Q^{-1}M^T(r-i,p-j,q-k)W_Q^{-1}(r,p,q)\bar{S}(r,p,b)]$$

Taking into account (15) and that Q^{-1}, $W_Q^{-1}(r,p,q)$ are symmetric we obtain

$$J(u^*)=\bar{S}^T(r,p,q)\,W_Q^{-1}(r,p,q)\Big[\sum_{(0,0,0)\leqslant(i,j,k)<(r,p,q)}\sum\sum M(r-i,p-j,q-k)\times$$

$$\times Q^{-1}M^T(r-i,\,p-j,q-k)\Big]\times[W_Q^{-1}(r,p,q)\,\bar{S}^T(r,p,q)]=$$

$$=\bar{S}^T(r,p,q)\,W_Q^{-1}(r,p,q)\,\bar{S}(r,p,q) \qquad \blacksquare$$

6. Example

Consider the system (1) with the matrices

$$A=\begin{bmatrix}A_1 & A_2 & A_3\\ A_4 & A_5 & A_6\\ A_7 & A_8 & A_9\end{bmatrix}=\begin{bmatrix}1 & 0 & -1\\ 0 & 2 & 1\\ -1 & 0 & -1\end{bmatrix}\qquad B=\begin{bmatrix}B_1\\ B_2\\ B_3\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} \tag{22}$$

and zero boundary conditions.

Let us assume that $r=p=q=1$,

$$x(0,0,0)=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix},\qquad S=x(1,1,1)=\begin{bmatrix}1\\ 2\\ -1\end{bmatrix}$$

and $Q=[1]$. Find $u^*(i,j,k)$ and $J(u^*)$. In this case $n_1=n_2=n_3=1$ and $m=1$. First we check the local controllability of the pair (22). Using (4) we obtain

$$M(1,0,0)=A^{0,0,0}\begin{bmatrix}B_1\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix};\qquad M(0,1,0)=A^{0,0,0}\begin{bmatrix}0\\ B_2\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$$

$$M(0,0,1)=A^{0,0,0}\begin{bmatrix}0\\ 0\\ B_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

$$M(1,1,0)=A^{0,1,0}\begin{bmatrix}B_1\\ 0\\ 0\end{bmatrix}+A^{1,0,0}\begin{bmatrix}0\\ B_2\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$M(0,1,1)=A^{0,0,1}\begin{bmatrix}0\\ B_2\\ 0\end{bmatrix}+A^{0,1,0}\begin{bmatrix}0\\ 0\\ B_3\end{bmatrix}=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$$

$$M(1,0,1)=A^{0,0,1}\begin{bmatrix}B_1\\ 0\\ 0\end{bmatrix}+A^{1,0,0}\begin{bmatrix}0\\ 0\\ B_3\end{bmatrix}=\begin{bmatrix}-1\\ 0\\ -1\end{bmatrix}$$

$$M(1,1,1)=A^{0,1,1}\begin{bmatrix}B_1\\ 0\\ 0\end{bmatrix}+A^{1,0,1}\begin{bmatrix}0\\ B_2\\ 0\end{bmatrix}+A^{1,1,0}\begin{bmatrix}0\\ 0\\ B_3\end{bmatrix}=\begin{bmatrix}0\\ -1\\ 0\end{bmatrix}$$

and

$$[M(1,0,0), M(0,1,0), M(0,0,1), \ldots, M(1,1,1)] = \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \end{bmatrix}$$

Thus, by the Theorem 2 the pair (22) is locally controllable in the cube [(0, 0, 0), (1, 1, 1)].

To find $u^*(i,j,k)$ we calculate

$$W_Q(1,1,1) = \sum_{(0,0,0)\leqslant(i,j,k)<(1,1,1)} \sum \sum M(r-i, p-j, q-k) Q^{-1} M^T(r-i, p-j, q-k) =$$

$$= M(1,1,1) M^T(1,1,1) + M(0,1,1) M^T(0,1,1) + M(1,0,1) M^T(1,0,1) +$$

$$+ M(1,1,0) M^T(1,1,0) + M(1,0,0) M^T(1,0,0) + M(0,1,0) M^T(0,1,0) +$$

$$+ M(0,0,1) M^T(0,0,1) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

and

$$W_Q^{-1}(1,1,1) = \frac{1}{3} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Using (16) and taking into account that $\bar{S} = S$ we obtain

$$u^*(0,0,0) = Q^{-1} M^T(1,1,1) W_Q^{-1}(1,1,1) S(1,1,1) = -\frac{2}{3}$$

$$u^*(1,0,0) = Q^{-1} M^T(0,1,1) W_Q^{-1}(1,1,1) S(1,1,1) = \frac{2}{3}$$

$$u^*(0,1,0) = Q^{-1} M^T(1,0,1) W_Q^{-1}(1,1,1) S(1,1,1) = 0$$

$$u^*(0,0,1) = Q^{-1} M^T(1,1,0) W_Q^{-1}(1,1,1) S(1,1,1) = 0$$

$$u^*(1,1,0) = Q^{-1} M^T(0,0,1) W_Q^{-1}(1,1,1) S(1,1,1) = -1$$

$$u^*(1,0,1) = Q^{-1} M^T(0,1,0) W_Q^{-1}(1,1,1) S(1,1,1) = \frac{2}{3}$$

$$u^*(0,1,1) = Q^{-1} M^T(1,0,0) W_Q^{-1}(1,1,1) S(1,1,1) = 1$$

and using (18)

$$J(u^*) = S^T(1,1,1) W_Q^{-1}(1,1,1) S(1,1,1) = \frac{10}{3}$$

7. Conclusions

For 3-D linear systems the definition of state-transition matrix is given and the general solution to state-space equation (1a) with boundary conditions (1c) is derived. The well-known Cayley-Hamilton theorem is extended for 3-D linear systems. Necessary and sufficient conditions for local controllability of 3-D linear systems are formulated and proved. Finally the minimum energy control problem for the system (1) is formulated and solved. The presented method is an extension of Klamka's method presented in [5] for 2-D linear systems. The above considerations can be extended for n-D linear systems.

References

[1] Bose N. K. Multidimensional Systems: Theory and Applications, IEEE Press, 1979.
[2] Eising R. Realization and stabilization of 2-D systems. *IEEE Trans. Autom. Control*, vol AC-23, 1978, 5, 793-799.
[3] Eising R. Controllability and observability of 2-D systems. *IEEE Trans. Autom. Cotrol*, AC-24, 1979, 3, 508-510
[4] Fornasini M. and Marchesini G State space realization theory of two-dimemsional filters. *IEEE Ttans. Autom. Control*, AC-21, 1976, 4, 484—492.
[5] Klamka J. Minimum energy control for 2-D systems. *Systems Science*, 1983 (in press).
[6] Kung S., Levy B., Morf M., Kailath T. New results in 2-D systems theory, part II: 2-D state-space models-realisation and the notions of controllability, observability and minimality. *Proceedings of the IEEE*, 65, 1977, 4, 861—872.
[7] Paraskevopoulos P. N. Characteristic polynomial assignment and determination of the residual polynomial in 2-D systems. *IEEE Trans. Autom. Control*, AC-26, 1981, 2, 541—543.
[8] Paraskevopoulos P. N. and Mertzios B. G. Transfer function factorization of 2-D systems using state feedback. *Int. J. Systems Science*, 12, 1981, 9, 1135—1147.
[9] Roesser R. A discrete state space model for linear image processing. *IEEE Trans. Autom. Control*, 10—20, 1975, 1, 1—10.
[10] Tzafestas S. G. and Pimenides T. G. Exact model-matching control of three-dimensional systems using state and output feedback. *Int. J. Systems Science*, 13, 1982, 11, 1171—1187.

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Sterowanie o minimalnej energii dla układów liniowych typu 3-D

Wprowadza się definicje macierzy tranzycyjnej stanu i wyprowadza ogólną postać rozwiązania dla układów liniowych typu 3-D. Uogólnia się twierdzenie Cayley'a-Hamiltona na przypadek liniowych układów typu 3-D. Formułuje się i udowadnia warunki konieczne i dostateczne lokalnej sterowalności liniowych układów typu 3-D. Rozwiązuje się i ilustruje prostym przykładem problem wyznaczania sterowania minimalizującego energię dla takich układów.

Управление с минимальной энергией для линейных систем типа 3-D

Вводится определение транзитивной матрицы состояния и выводится общий вид решения для линейных систем типа 3-D. Обобщается теорема Кали-Гамильтона для случая линейных систем типа 3-D. Формулируются и обобщаются необходимые и достаточные условия локальной управляемости линейных систем типа 3-D. Решается и иллюстрируется простым примером проблема определения управления с минимальной энергией для таких систем.