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Duality and stability theorems for convex multivalued constrained minimization problems

by

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In this paper we investigate duality and stability relations for abstract convex minimization problems with constraints set described by a convex multifunction. The proposed duality generalize some of the existing dualization schemes by admitting various dualization parameters. The obtained form of a dual problem allows to derive easily some stability results for dual solutions.

Introduction

In this paper we present duality theorems for minimization problems of the form

$$P \qquad \inf \{f(x) : x \in \Gamma y_0\}$$

where $f: X \to R$ is a real-valued function defined on a topological space X and $\Gamma: Y \to X$ is a multifunction defined on a topological space Y, i.e. for every $y \in Y$ $\Gamma y \subset 2^X$. Such formulation allows to consider in a unified way minimization problems with different types of constraints (e.g. equalities, inequalities).

Parameter $y_0 \in Y$ may be considered as representing certain quantities defining the problem whose effect on the solutions is of interest. For this reason we may refer to y_0 as a data vector. One of the most frequently exploited feasible set multifunctions $\Gamma: Y \to X$ is of the form

$$\Gamma y_0 = \{x : g(x) \leqslant y_0\}$$

where $g: X \to Y$ is a certain mapping. The formulation of P admits also other, more general, choices of data, e.g.

$$\Gamma y_0 = \{x : g(x, y_0) \leqslant 0\}$$

which stresses the fact that we consider as the data of the problem not only right-hand-side vector but also a left-hand-side mapping.

The problem P may be embedded, by a multifunction Γ in a family of perturbed problems

$$P_y \qquad \inf\{f(x) : x \in \Gamma y\}$$

where y reflects uncertainity in data due to errors of measurement or numerical representation. In the present paper we formulate the dual problem of P with y as a dualization parameter. The proposed duality generalize the schemes of Tind and Wolsey [16] and the earlier one of Gould [7] by assuming more general description of the constraints set and by admitting dualization parameters other than right-hand-side vectors. This last fact is of particular importance in stability problems.

The first section contains necessary definitions and facts about convex multifunctions. In the second section a dual program is formulated and its basic properties are examined. In the third section we compare the dual formulated in this paper with Lagrangean dual of Kurcyusz [18] and Rockafellar [19]. The last part contains sensitivity analysis of dual solutions. The weak* compactness of dual solutions multifunction is also investigated.

1. Convex multifunctions

Let X and Y be any real linear spaces. In the sequel we consider a convex minimization problem

$$P \qquad \inf\{f(x) : x \in \Gamma y_0\}$$

where $f: X \to R$ is a convex function and $\Gamma: Y \to X$ is a convex multifunction, i.e.

$$\lambda \Gamma y_1 + (1-\lambda)\Gamma y_2 \subset \Gamma(\lambda y_1 + (1-\lambda) y_2)$$

for every $y_1, y_2 \in Y$ and $\lambda \in \langle 0, 1 \rangle$.

If we denote by $G(\Gamma) \subset Y \times X$ the graph of a multifunction Γ

$$G(\Gamma) = \{(y, x) \in Y \times X : x \in \Gamma y\}$$

then convexity of Γ is equivalent to convexity of the set $G(\Gamma)$. Namely, if $(y_1, x_1), (y_2, x_2) \in G(\Gamma)$ then $(\lambda x_1 + (1-\lambda) x_2, \lambda y_1 + (1-\lambda) y_2) \in G(\Gamma)$ for every $\lambda \in \langle 0, 1 \rangle$ if and only if $\lambda x_1 + (1-\lambda) x_2 \in \Gamma(\lambda y_1 + (1-\lambda) y_2)$ what amounts to convexity of Γ. The set $\operatorname{dom} \Gamma = \{y \in Y : \Gamma y \neq \emptyset\}$ is called the *dvmain* of Γ.

Convexity of Γ implies also convexity of the set Γy for every $y \in Y$. Multifunctions of this kind were considered by many authors, e.g. in the context of optimization by Borwein [2], Rockafellar [13], in application to stability problems by Robinson [11, 12].

An extended real-valued function $f\Gamma: Y \to \bar{R} = R \cup \{\pm\infty\}$ defined as

$$f\Gamma(y) = \inf\{f(x) : x \in \Gamma y\}$$

with the convention $f\Gamma(y) = +\infty$ whenever $\Gamma y = \emptyset$ will be called the *perturbation function* of the problem P.

PROPOSITION 1. Let X, Y be any real linear spaces. If f and Γ are convex then the perturbation function $f\Gamma$ of the problem P is convex.

Proof. It is enough to observe that $F: X\times Y \to \bar{R}$ defined as

$$F(x,y)=\begin{cases} f(x) & x\in\Gamma y \\ +\infty & x\notin\Gamma y \end{cases}$$

is convex in x and y. Now the proof follows immediately from Theorem 1 of Rockafellar [19]. ∎

1a. Directions of recession. Recession multifunction

Analogously to the recession cone of a convex function we can define, for arbitrary convex multifunction $\Gamma: Y\to X$ its *recession cone* $\Gamma 0^+$ as

$$\Gamma 0^+ = \{z\in Y\colon \Gamma y\subset \Gamma(y+\lambda z) \text{ for every } y\in \operatorname{dom}\Gamma,\ \lambda\in R,\ \lambda\geqslant 0\}.$$

PROPOSITION 2. Let X, Y be any real linear spaces. If $\Gamma: Y\to X$ is a convex multifunction then

(i) $\Gamma 0^+ = 0^+G(\Gamma)\cap\{(y,x)\in Y\times X\colon x=0\}$
where $0^+G(\Gamma)$ denotes the recession cone of $G(\Gamma)$

(ii) $\Gamma 0^+=\{z\colon \Gamma y\subset\Gamma(y+z) \text{ for } y\in\operatorname{dom}\Gamma\}$

(iii) $\Gamma 0^+$ is convex.

Proof. By definition $(z,0)\in 0^+G(\Gamma)\cap\{(y,x)\in Y\times X\colon x=0\}$ if for every $(y,x)\in G(\Gamma)$ and every $\lambda\geqslant 0$ $(y,x)+\lambda(z,0)\in G(\Gamma)$. This is equivalent to the condition $\Gamma y\subset\Gamma(y+\lambda z)$ for every $y\in\operatorname{dom}\Gamma$ and every $\lambda\geqslant 0$ what proves the part (i) Part (ii) follows directly from Rockafellar [13, sec. 8]. Part (iii) is an immediate consequence of the fact that $\Gamma 0^+$ can be expressed as an intersection of two convex sets according to (i). ∎

REMARK. It is worth noting that the definition of the recession cone $f0^+$ of a convex function $f\colon Y\to R$, given by Rockafellar [13] is based on the epigraph of f and not on graph. So, the definition introduced above is more restrictive, when applied to function and give the set of all directions of constancy.

Together with the recession cone $\Gamma 0^+$ we can consider also, for a given convex multifunction Γ its *multifunction of recession* $0^+\Gamma$, $0^+\Gamma\colon Y\to X$, defined as

$$G(0^+\Gamma)=0^+G(\Gamma).$$

If $(z,t)\in 0^+G(\Gamma)$ then $(y,x)+\lambda(z,t)\in G(\Gamma)$ for every $(y,x)\in G(\Gamma)$ and $\lambda\geqslant 0$. Equivalently $\Gamma y+t\subset\Gamma(y+z)$ for every $y\in\operatorname{dom}\Gamma$, and consequently $t\in\bigcap_{y\in\operatorname{dom}\Gamma}\Gamma(y+z)\setminus\Gamma y$. By the last formula we obtain the inclusion

$$0^+\Gamma z\subset\{t\in X\colon t\in\bigcap_{y\in\operatorname{dom}\Gamma}\Gamma(y+z)\setminus\Gamma y\}.$$

The following proposition is a straightforward consequence of the notions introduced above.

Proposition 3. Let X, Y be any real linear spaces. Then $z \in \Gamma 0^+$ if and only if $0 \in 0^+ \Gamma z$.

The following fact relates a feasible set multifunction to the recession cone of the perturbation function.

Proposition 4. Let X, Y be any real spaces. If $\Gamma: Y \to X$ is a feasible set multifunction of the problem P then $\Gamma 0^+ \subset 0^+ f\Gamma$ where $0^+ f\Gamma$ is the recession cone of the perturbation function.

Proof. If $z \in \Gamma 0^+$ then $\Gamma y \subset \Gamma(y+\lambda z)$ for every $y \in \operatorname{dom} \Gamma$ and $\lambda \geqslant 0$. It implies that $\inf\{f(x): x \in \Gamma y\} \geqslant \inf\{f(x): x \in \Gamma(y+\lambda z)\}$ for every function f defined on X, $y \in \operatorname{dom} \Gamma$ and $\lambda \geqslant 0$. This is equivalent to $f\Gamma(y+\lambda z) \leqslant f\Gamma(y)$ for every $y \in \operatorname{dom} \Gamma$ and $\lambda \geqslant 0$. So, z belongs to the recession cone of $f\Gamma$. ∎

1b. Dual relations

Let us assume now that Y and U are two locally convex Hausdorff linear topological spaces in duality with respect to the bilinear form $\langle y, u\rangle$, $y \in Y$, $u \in U$ with topologies compatible with this duality (i.e. functionals $\varphi(y)=\langle y, u\rangle$ and $\psi(u)=\langle y, u\rangle$ are continuous in these topologies).

For arbitrary functional $f: Y \to R$ by its conjugate $f^*: U \to R$ we mean

$$f^*(u)=\sup_{y \in Y}\{\langle y, u\rangle - f(y)\}.$$

Moreover, let us denote by $\operatorname{dom} f=\{y: f(y)<+\infty\}$ the effective domain of f, by $Q^0=\{u: \langle u, z\rangle \leqslant 0 \text{ for every } z \in Q\}$ the polar cone of arbitrary cone $Q \subset Y$ and by $\operatorname{cone} C=\{\lambda x: \lambda \geqslant 0, x \in C\}$ the conic hull of arbitrary set $C \subset X$.

By Proposition 4 $\Gamma 0^+$ is contained in the recession cone of the perturbation function. The following proposition gives the dual relations.

Proposition 5. Let Y, U be a dual pair of spaces. Then

$$\operatorname{dom} f\Gamma^* \subset \operatorname{cone} \operatorname{dom} f\Gamma^* \subset (\Gamma 0^+)^0.$$

Proof. Let us consider $\sup_{y \in Y}\{\langle y, u\rangle - f\Gamma(y)\}$ for arbitrary $u \in U$ and let $z=t+k$ for arbitrary but fixed $t \in Y$ and $k \in \Gamma 0^+$. Then $\langle z, u\rangle - f\Gamma(z)= =\langle t, u\rangle + \langle k, u\rangle - f\Gamma(t+k) \geqslant \langle t, u\rangle + \langle k, u\rangle - f\Gamma(t)$. Thus $\sup_{y \in Y}\{\langle y, u\rangle - f\Gamma(y)\} \geqslant \geqslant \sup_z\{\langle z, u\rangle - f\Gamma(z)\} \geqslant \sup_{k \in \Gamma 0^+}\langle k, u\rangle + \langle t, u\rangle - f\Gamma(t)$.

If there exists $k \in \Gamma 0^+$ such that $\langle k, u\rangle > 0$ then $\sup_{k \in \Gamma 0^+}\langle k, u\rangle = +\infty$ what implies $f\Gamma^*(u)=+\infty$. So, if $u \notin (\Gamma 0^+)^0$ then $u \notin \operatorname{dom} f\Gamma^*$. ∎

2. Dual problem

Denoting the class of all affine functionals defined on Y by $\operatorname{Aff}(Y)$ we introduce

$$\mathscr{F}=\{F \in \operatorname{Aff}(Y): Fy=\langle y, u\rangle + u_0,\ u_0 \in R,\ u \in (\Gamma 0^+)^0\}$$

Now we can define the dual to the problem P as

supremize Fy_0

D subject to: $f(x) \geqslant \sup_{y: x \in \Gamma y} Fy$ for every $x \in X$ $F \in \mathscr{F}$

Example

Let us consider the problem

$P \quad \inf\{f(x): b_0 - g(x) \in D\}$

where $f: X \to R$, $g: X \to Y$ is a mapping, $b_0 \in Y$, $D \subset Y$ is a closed convex cone. If we define multifunction $\Gamma: Y \to X$ as $\Gamma b = \{x: b - g(x) \in D\}$ then $\Gamma 0^+ = D$ and $\sup_{b: x \in \Gamma b} Fb = Fg(x)$ for every F belonging to $\mathscr{F}$ and the dual D takes the form of the dual proposed by Tind and Wolsey [16]. Namely, we obtain

$$\begin{array}{ll} & \text{supremize}_F\ Fb_0 \\ D & \text{subject to: } f(x) \geqslant Fg(x) \text{ for every } x \in X\ F \in \mathscr{F}. \end{array}$$

Further specifications of functions f and g as linear gives usual dual pair of linear problems. Detailed calculations of this fact are contained in the paper of Gould [7, Th. 11].

REMARK. The condition $u \in (\Gamma 0^+)^0$ corresponds to the requirement on sign of dual variables. It is introduced in the definition of price functions F to indicate the analogy to the existing duals and play the same role as in [16] the requirement for price functions to be nondecreasing. This condition excludes from considerations certain points where $f\Gamma^*$ takes $+\infty$. In fact, such condition is not essential for the maximization in the dual problem and might be omitted. Similar observation was made by Gould [7] and also by Tind and Wolsey [16] and is motivated by

PROPOSITION 6. The conjugate function of the perturbation function $f\Gamma$ can be expressed by the following formula

$$-f\Gamma^*(u) = \inf_{x \in X}\{f(x) - \sup_{y: x \in \Gamma y}\langle y, u\rangle\}$$

for arbitrary $u \in U$.

P r o o f. For every $x \in X$ and every $u \in U$

$$f\Gamma^*(u) \geqslant \sup_{y: x \in \Gamma y}\{\langle y, u\rangle - f\Gamma(y)\} \geqslant \sup_{y: x \in \Gamma y}\langle y, u\rangle - f(x).$$

So, for every $x \in X$ and every $u \in U$

$$-f\Gamma^*(u) \leqslant f(x) - \sup_{y: x \in \Gamma y}\langle y, u\rangle$$

and finally $-f\Gamma^*(u) \leqslant \inf_{x \in X}\{f(x) - \sup_{y: x \in \Gamma y}\langle y, u\rangle\}$.

On the other hand, for every $y \in Y$

$$\begin{aligned} &\inf_{x \in X}\{f(x) - \sup_{y: x \in \Gamma y}\langle y, u\rangle\} \leqslant \\ &\leqslant \inf_{x \in \Gamma y}\{f(x) - \sup_{y: x \in \Gamma y}\langle y, u\rangle\} \leqslant f\Gamma(y) - \langle y, u\rangle. \end{aligned}$$

So, according to the last inequality

$$\inf_{x \in X}\{f(x) - \sup_{y: x \in \Gamma y}\langle y, u\rangle\} \leqslant \inf_{y \in Y}\{f\Gamma(y) - \langle y, u\rangle\} = -f\Gamma^*(u). \quad \blacksquare$$

Proposition 6 allows to formulate the dual problem D in the equivalent form D' as

$$\begin{array}{ll} & \text{supremize } \langle y_0, u\rangle + u_0 \\ D' & \text{subject to: } -f\Gamma^*(u) \geqslant u_0 \\ & \qquad u \in (\Gamma 0^+)^0,\ u_0 \in R \end{array}$$

By proposition 5, if $u \notin (\Gamma 0^+)^0$ then $f\Gamma^*(u) = +\infty$. So, in other way, if there exists (u, u_0) such that $-f\Gamma^*(u) \geqslant u_0$ then $u \in (\Gamma 0^+)^0$. The formulation D' of the dual problem D admits a simple geometric interpretation. Namely, the problem D might be viewed as a problem of finding among all supporting hyperplanes of $f\Gamma$ a such one which has the maximal value at y_0. In fact, the condition $-f\Gamma^*(u) \geqslant u_0$ can be rewritten as

$$-\sup_{y \in Y} \{\langle y, u\rangle - f\Gamma(y)\} \geqslant u_0$$

and the dual takes the form

$$D'' \quad \begin{array}{l} \text{supremize } \langle y_0, u\rangle + u_0 \\ \text{subject to: } f\Gamma(y) \geqslant \langle y, u\rangle + u_0 \text{ for every } y \in Y \\ \qquad u \in (\Gamma 0^+)^0, \quad u_0 \in R \end{array}$$

For the sake of completeness we should notice that for a primal minimization problem of the form

$$\inf\{f(x) : x \in \Gamma y_0,\ x \in C\}$$

where C is a convex subset of X (C may represent the constraints of particularly simple form, e.g. $x \geqslant 0$ or the constraints which are not subject to perturbations) the dual D takes the form

$$D \quad \begin{array}{l} \text{supremize } Fy_0 \\ \text{subject to: } f(x) \geqslant \sup_{y\,:\,x \in \Gamma y} Fy \text{ for every } x \in C \\ \qquad F \in \mathcal{F} \end{array}$$

and the formula for $f\Gamma^*(u)$ from Proposition 6 may be restated as

$$-f\Gamma^*(u) = \inf_{x \in C} \{f(x) - \sup_{y\,:\,x \in \Gamma y} \langle y, u\rangle\}.$$

Denoting by $\gamma(0)$ the optimal value of the dual problem D we may formulate

THEOREM 2.1 (*weak duality*)

If x is feasible for P and F is feasible for D then $f\Gamma(y_0) \geqslant \gamma(0)$.

Proof. $f(x) \geqslant \sup_{y\,:\,x \in \Gamma y} Fy \geqslant Fy_0$. The first inequality follows from feasibility of F, the second one follows from feasibility of x. These inequalities imply immediately $f\Gamma(y_0) \geqslant \gamma(0)$. ■

Now we can formulate the two following properties.

Property 1. D is feasible if and only if $f\Gamma(y) > -\infty$ for every $y \in Y$.

Property 2. If y_0 belongs to the interior of $\operatorname{dom}\Gamma$ (interior in any admissible topology) and $f\Gamma(y_0)$ is finite then the problem D has a solution.

These properties follow immediately from the formulation D of the dual and the conditions assuring existence of nonvertical support for convex functions.

The question of equality between optimal values of P and D can be reduced to the question whether the equality $f\Gamma(y_0) = f\Gamma^{**}(y_0)$ holds since

$$\sup\{\langle y_0, u\rangle + u_0 : -f\Gamma^*(u) \geqslant u_0,\ u \in (\Gamma 0^+)^0,\ u_0 \in R\} =$$
$$\sup\{\langle y_0, u\rangle - f\Gamma^*(u) : u \in (\Gamma 0^+)^0\} = \sup_{u \in U}\{\langle y_0, u\rangle - f\Gamma^*(u)\} = f\Gamma^{**}(y_0).$$

Corresponding theorems are contained in Rockafellar [19], Dolecki [5], Joly and Laurent [17].

2a. Symmetry of the dual

There exist several ways of introducing parameters in D. Here we perturb D by adding linear functional to the objective function f.

Let X and V be in duality with respect to the bilinear form $\langle x, v\rangle$ where $x \in X$ and $v \in V$.

A multifunction Γ^*: $V \to \mathrm{Aff}(Y)$ defined as

$$\Gamma^* v = \{F \in \mathscr{F}: f(x) + \langle x, v\rangle \geqslant \sup_{y:\, x \in \Gamma y} Fy \text{ for every } x \in X\}$$

will be called the dual multifunction of the problem P.

Now we may introduce the perturbed dual problem D_v for $v \in V$ as

$$D_v \sup \{Fy_0 : F \in \Gamma^* v\}.$$

In these terms the problem D corresponds to the parameter $v=0$. Since f and Γ are assumed to be convex throughout this paper the dual multifunction Γ^* is also convex by direct calculations. So, the dual perturbation function $\gamma: V \to \bar{R}$ defined as

$$\gamma(v) = \sup \{Fy_0:\ F \in \Gamma^* v\}$$

is concave with $\gamma(0)$ equal to the optimal value of D.

If we recall that the recession cone of Γ^* is equal to

$$\Gamma^* 0^+ = \{t \in V:\ \Gamma^* v \subset \Gamma^*(v + \lambda t) \text{ for every } v \in \operatorname{dom} \Gamma^*,\ \lambda \in R,\ \lambda \geqslant 0\}$$

then by similar arguments as in Proposition 4 $\Gamma^* 0^+ \subset 0^- \gamma$ where $0^- \gamma$ is the progression cone of γ in the sense that $z \in 0^- \gamma$ if and only if $\gamma(v) \leqslant \gamma(v + \lambda z)$ for every $v \in V$ and $\lambda \geqslant 0$. Basing on the similar arguments as in Proposition 5 we obtain

$$\operatorname{dom} \gamma^* \subset (\Gamma^* 0^+)^*$$

where $(\Gamma^* 0^+)^*$ is the dual cone of $\Gamma^* 0^+$, i.e.

$$(\Gamma^* 0^+)^* = \{x \in X:\ \langle x, t\rangle \geqslant 0 \text{ for every } t \in \Gamma^* 0^+\}.$$

If we denote by $\mathrm{Aff}(V)$ the class of all affine functionals defined on V then we may introduce

$$\mathscr{X} = \{\hat{x} \in \mathrm{Aff}(V):\ \hat{x}(v) = \langle x, v\rangle + x_0,\ x \in (\Gamma^* 0^+)^*,\ x_0 \in R\}.$$

So, the dual to the problem D may be formulated as

$$DD \quad \begin{array}{l} \text{infimize } \hat{x}(0) \\ \text{subject to: } Fy_0 \leqslant \inf_{v:\, F \in \Gamma^* v} \hat{x}(v) \text{ for every } F \in \mathscr{F} \\ \qquad \hat{x} \in \mathscr{X} \end{array}$$

By calculations, similar to those of the Proposition 6, we obtain the formula

$$-\gamma^*(x) = \sup_{F \in \mathscr{F}} \{Fy_0 - \inf_{v:\, F \in \Gamma^* v} \langle x, v\rangle\}.$$

Thus the problem DD may be restated as

$$DD' \quad \begin{array}{l} \text{infimize } x_0 \\ \text{subject to: } x_0 \geqslant -\gamma^*(x) \\ \qquad x \in (\Gamma^* 0^+)^*, \; x_0 \in R \end{array}$$

REMARK. The necessity of introducing the progression cone is motivated by the fact that the dual perturbation function is concave. This necessity might be omitted by passing to the minimization in the dual problem D. However the construction of the problem DD presents a straightforward way of constructing the dual for maximization problems.

The following theorem establishes the symmetry of the proposed dualization scheme.

THEOREM 2.2. *The duality induced by D is symmetric in the sense that the optimal values of DD and P are equal under the assumption that the equality between optimal values of P and D holds.*

Proof. The optimal value of the problem DD is equal to $\inf\{-\gamma^*(x): x \in (\Gamma^* 0^+)^*\} = \gamma^{**}(0)$. On the other hand, according to the formula cited above

$$-\gamma^*(x) = \sup_{F\in\mathcal{F}} \{Fy_0 - \inf_{v:\, F\in\Gamma^* v} \langle x, v\rangle\} \geqslant$$
$$\geqslant Fy_0 - \inf_{v:\, F\in\Gamma^* v} \langle x, v\rangle \quad \text{for every } F \in \mathcal{F}.$$

So $\gamma^*(x) \leqslant -Fy_0 - \sup_{v:\, F\in\Gamma^* v} \langle x, -v\rangle$. Thus, for the function conjugate to γ^* we obtain

$$\gamma^{**}(v) \geqslant Fy_0 - \delta(-v: \Gamma^{*-1} F)$$

for every $F \in \mathcal{F}$ where $\Gamma^{*-1} F = \{v \in V: F \in \Gamma^* v\}$. From this formula we obtain for $v = 0$

$$\gamma^{**}(0) \geqslant \sup_{F\in\mathcal{F}} \{Fy_0 - \delta(0: \Gamma^{*-1} F)\} = \sup\{Fy_0: F \in \Gamma^* 0\} = \gamma(0).$$

To get a bound from above for $\gamma^{**}(0)$ let us observe that

$$-\gamma^{**}(0) = -\inf_{x\in X} -\gamma^*(x) = -\inf_{x\in X} \sup_{F\in\mathcal{F}} \{Fy_0 - \inf_{v:\, F\in\Gamma^* v} \langle x, v\rangle\} \geqslant$$
$$\geqslant -\inf_{x\in\Gamma y_0} \sup_{F\in\mathcal{F}} \{Fy_0 - \inf_{v:\, F\in\Gamma^* v} \langle x, v\rangle\} =$$
$$= \sup_{x\in\Gamma y_0} \{-\inf_{F\in\mathcal{F}} \{Fy_0 - \inf_{v:\, F\in\Gamma^* v} \langle x, v\rangle\}\} \geqslant$$
$$\geqslant \sup_{x\in\Gamma y_0} \{-\inf_{F\in\Gamma^* v} \{Fy_0 - \inf_{v:\, F\in\Gamma^* v} \langle x, v\rangle\}\}.$$

If $F \in \Gamma^* v$ then $f(x) + \langle x, v\rangle \geqslant Fy_0$ for every $x \in \Gamma y_0$ and consequently

$$-\langle x, v\rangle \leqslant f(x) - Fy_0$$
$$\sup_{v:\, F\in\Gamma^* v} -\langle x, v\rangle \leqslant f(x) - Fy_0$$
$$Fy_0 - \inf_{v:\, F\in\Gamma^* v} \langle x, v\rangle \leqslant f(x)$$
$$-\inf_{F\in\Gamma^* v} \{Fy_0 - \inf_{v:\, F\in\Gamma^* v} \langle x, v\rangle\} \geqslant -f(x).$$

So finally $-\gamma^{**}(0) \geqslant \sup_{x \in \Gamma y_0} -f(x) = -f\Gamma(y_0)$. Under the assumption that $\gamma(0) = f\Gamma(y_0)$ what implies $\gamma^{**}(0) = f\Gamma(y_0)$. ■

2b. Solutions of the dual

A convex function $f: Y \to R$ is said to be subdifferentiable at $y_0 \in Y$ if there exists $u \in U$ such that

$$f(y) \geqslant f(y_0) + \langle y - y_0, u \rangle$$

for every $y \in Y$, i.e. there exists a continuous affine minorant of f which takes the value $f(y_0)$ at y_0. Such an element $u \in U$ is called a *subgradient* of f at y_0.

The following two theorems characterize the solutions of the dual problem D in terms of subgradients of the perturbation function $f\Gamma$ at y_0.

THEOREM 2.3. *If $\hat{u} \in U$ is a subgradient of $f\Gamma$ at y_0 then $(\hat{u}, -f\Gamma^*(\hat{u}))$ is a solution of D.*

Proof. Let us observe firstly that the pair $(\hat{u}, -f\Gamma^*(\hat{u}))$ is feasible. As a subgradient of $f\Gamma$ at y_0, $\hat{u}$ satisfies $f\Gamma^*(\hat{u}) + f\Gamma(y_0) = \langle y_0, \hat{u} \rangle$. So $f\Gamma^*(\hat{u})$ must be finite what implies that $\hat{u} \in (\Gamma 0^+)^0$ and obviously $u_0 = -f\Gamma^*(\hat{u}) \leqslant -f\Gamma^*(\hat{u})$. The optimality follows from the relation $\langle \hat{u}, y_0 \rangle - f\Gamma^*(\hat{u}) = f\Gamma(y_0) + f\Gamma^*(\hat{u}) - f\Gamma^*(\hat{u}) = f\Gamma(y_0)$ what completes the proof. ■

THEOREM 2.4. *If the equality of the optimal values of P and D holds and $(\hat{u}, \hat{u}_0)$ is a solution of the dual problem D then $\hat{u}$ is a subgradient of $f\Gamma$ at y_0 and $\hat{u}_0 = -f\Gamma^*(\hat{u})$.*

Proof. For every feasible pair (u, u_0) we have $\langle u, y_0 \rangle + u_0 \leqslant \langle \hat{u}, y_0 \rangle + \hat{u}_0 = \langle \hat{u}, y_0 \rangle - f\Gamma^*(\hat{u}) = f\Gamma(y_0)$. By the definition of the conjugate function $\sup_{y \in Y} \{\langle y, \hat{u} \rangle - f\Gamma(y)\} = -\Gamma(y_0) - \langle \hat{u}, y_0 \rangle$. So for every $y \in Y$ $\langle y, \hat{u} \rangle - f\Gamma(y) \leqslant -f\Gamma(y_0) - \langle \hat{u}, y_0 \rangle$ what completes the proof. ■

3. Relation to other duals

3a. Lagrangean duals

For the problem P we consider the Lagrangean $L: X \times U \to \bar{R}$ defined as

$$L(x, u) = f(x) - \sup_{y: x \in \Gamma y} \langle y, u \rangle + \langle y_0, u \rangle = L(x, u, y_0).$$

The Lagrangean of this form was introduced by Kurcyusz [18] and was investigated by Kurcyusz and Dolecki [6], Dolecki [5]. The dual pair of problems connected with $L(x, u)$ can be written in the usual form as

LP $\quad \inf_{x \in X} \sup_{u \in U} L(x, u) = LP_{\text{opt}}$

LD $\quad \sup_{u \in U} \inf_{x \in X} L(x, u) = LD_{\text{opt}}$

THEOREM 3.1. *The problem LD is equivalent to D.*

Proof. The problem LD may be rewritten as $\sup_{u\in U}\{\langle y_0, u\rangle+\inf_{x\in X}\{f(x)-$ $-\sup_{y:\,x\in\Gamma y}\langle y, u\rangle\}\}$.

If $\inf_{x\in X}\{f(x)-\sup_{y:x\in\Gamma y}\langle y, u\rangle\}=-\infty$ for all $u\in U$ then $f\Gamma^*(u)=+\infty$ and D is infeasible so $\gamma(0)=-\infty$. Otherwise, there exists $u\in U$ such that $\inf_{x\in X}\times$ $\times\{f(x)-\sup_{y:\,x\in\Gamma y}\langle y, u\rangle\}>-\infty$. According to the Proposition 5 such u satisfies also $u\in(\Gamma 0^+)^0$ so D is feasible and $\gamma(0)=\sup\{\langle y_0, u\rangle-f\Gamma^*(u)\colon u\in(\Gamma 0^+)^0\}=$ $=\sup_{u\in U}\{\langle y_0, u\rangle+\inf_{x\in X}\{f(x)-\sup_{y:\,x\in\Gamma y}\langle y, x\rangle\}\}=LD_{\mathrm{opt}}$. ∎

Abstract minimization problems with perturbations were investigated also by Rockafellar [13, 19]. He considers a representation of the minimization problem P in the form

$$\inf_{x\in X} F(x, 0)$$

and the family of perturbed problems

$$\inf_{x\in X} F(x, y)$$

where $F\colon X\times Y\to\bar{R}$ is an extended-real-valued function with $F(x, 0)=f(x)$. The minimization over all the space X is obtained if we redefine the minimized function f of the constrained problem P so that $f(x)=+\infty$ for $x\notin\Gamma y_0$, $y_0=0$. The Lagrangean function $K\colon X\times U\to\bar{R}$ connected with such family of perturbed problems is defined as

$$K(x, u)=\inf_{y\in Y}\{F(x, y)+\langle y, u\rangle\}.$$

The Rockafellar's way of introducing perturbations in the form of function $F(x, y)$ is rather general. It admits various forms of perturbations in constraints set as well as perturbations in the minimized function.

If we consider the problem P with $y_0=0$ then the function $F(x, y)$ connected with the family of perturbed problems P_y is of the form

$$F(x, y)=f(x)+\delta(y\colon\Gamma^{-1}x)$$

where $\delta(y\colon\Gamma^{-1}x)=0$ if $y\in\Gamma^{-1}x$ and $+\infty$ otherwise, $\Gamma^{-1}x=\{y\colon x\in\Gamma y\}$. For such a function $F(x, y)$ the Lagrangean $K(x, u)$ takes the form

$$\begin{aligned}K(x, u)=\inf_{y\in Y}\{F(x, y)+\langle y, u\rangle\}&=\inf_{y\in Y}\{f(x)+\delta(y\colon\Gamma^{-1}x)+\langle y, u\rangle\}=\\ =f(x)+\inf_{y\in Y}\{\delta(y\colon\Gamma^{-1}x)+\langle y, u\rangle\}&=f(x)-\delta^*(-u\colon\Gamma^{-1}x)=\\ &=f(x)-\sup_{y:\,x\in\Gamma y}\langle -u, y\rangle=L(x, -u, 0).\end{aligned}$$

So, in this sense the dual problems derived from $L(x, u)$ and $K(x, u)$ and the problem D are equivalent.

3b. Surrogate dual

For the problem P a surrogate problem may be formulated as

SP $$\inf\{f(x)\colon Fy\leqslant\sup_{y:x\in\Gamma y}Fy\}=SP_{\mathrm{opt}}$$

where $F \in \mathscr{F}$. It is an immediate observation that if $x \in \Gamma y_0$ then $Fy_0 \leqslant \sup_{y: x \in \Gamma y} Fy$ and consequently $f\Gamma(y_0) \geqslant SP_{\text{opt}}$. If we denote

$$v(F) = \inf\{f(x) \colon Fy_0 \leqslant \sup_{y;\, x \in \Gamma y} Fy\}$$

then the surrogate dual problem may be defined as

$$SD \qquad \sup\{v(F) \colon F \in \mathscr{F}\}.$$

The dual of this form was introduced by Greenberg and Pierskalla [8] for nonlinear inequality type constraints and was investigated by many authors. Recently the basic properties of surrogate dual in infinite dimensional spaces for inequality type constraints were investigated by Sikorski [15]. We show here only the following equivalence result.

THEOREM 3.2. *If the equality $f\Gamma(y_0) = LD_{\text{opt}}$ holds then the surrogate dual problem SD is equivalent to D.*

Proof $SD_{\text{opt}} = \sup_{F \in \mathscr{F}} v(F) = \sup_{F \in \mathscr{F}} \sup_{\lambda \geqslant 0} \inf\{f(x) + \lambda F y_0 - \lambda \sup_{y:\, x \in \Gamma y} Fy\} = \sup_{u \in U} \inf_{x \in X} L(x, u)$. ∎

4. Stability results

In the sequel we assume that the space of parameter Y is a Hausdorff topological space satisfying the first countability axiom at y_0 with W_n as a countable basis of neighborhood at y_0. In [1] the sufficient conditions for feasible set multifunction Γ at y_0 are given which assure for every continuous minimized function f the upper semicontinuity of solutions at y_0. We recall that for arbitrary subset D of a metric space $X = (X, \rho)$ by its measure of noncompactness $\psi(D)$ we mean the infimum of all those $r \geqslant 0$ for which D may be split to a finite number of subsets $\{D_i\}_{i=1,\dots,n}$ $\bigcup_{i=1}^n D_i = D$ such that $\sup_{v, z \in D_i} \rho(v, z) \leqslant r$ for each i. Moreover, if we denote by $\operatorname{Inac} \Gamma y_0 = \bigcap_n \overline{\bigcup_{y \in W_n} \Gamma y_0 \setminus \Gamma y}$ the *inner carrier* of Γ at y_0 then we can state

THEOREM 4.1 [1]. *Let Y be a Hausdorff topological space that fulfills the first countability axiom, let $X = (X, \rho)$ be a complete metric space and let Γ be u.s.c. and l.s.c. at $y_0 \in Y$. Suppose furthermore that*

$$\lim_n \psi(B_n) = 0 \qquad \text{and} \qquad \operatorname{Inac} \Gamma y_0 \subset \Gamma y_0$$

for $B_n = \bigcup_{y \in W_n} \Gamma y_0 \setminus \Gamma y$. Then for every continuous function f defined on X a solution multifunction M of the problem P

$$My = \{x \in \Gamma y \colon f(x) \leqslant f\Gamma(y)\}$$

is u.s.c. at y_0.

The abreviations u.s.c. and l.s.c. are used for upper and lower semicontinuity respectively.

One of the important consequences of the Theorem 4.1 is the following.

THEOREM 4.2. *If* $\Gamma\colon Y \to X$ *satisfies the assumptions of the Theorem* 4.1 *and* $f\Gamma(y_0)$ *is finite then there exists a neighborhood* W *of* y_0 *such that for every* $y \in W f\Gamma(y_0)$ *is finite.*

P r o o f. By contrary, let us assume, that in every neighborhood W_n there exists y_n such that $f\Gamma(y_n)$ is not finite. Only two situations should be considered:

1. if $f\Gamma(y_n)=+\infty$ then it is equivalent to the fact that $\Gamma y_n=\emptyset$ what amounts to the absence of l.s.c. at y_0

2. if $f\Gamma(y_n)=-\infty$ then there exist $\{x_n\}_{n=1}^{\infty}$ $x_n \in \Gamma y_n$ such that $f(x_n)$ tends to $-\infty$; if $\{x_n\}_{n=1}^{\infty}$ contains a convergent subsequence with limit point x_0 and $x_0 \in \Gamma y_0$ then f is not continuous at x_0; if $x_0 \notin \Gamma y_0$ then Γ is not u.s.c. at y_0; and finally, if $\{x_n\}_{n=1}^{\infty}$ does not contain any convergent subsequence then Γ is not u.s.c. at y_0. ■

So, in other words $y_0 \in \operatorname{int} \operatorname{dom} f\Gamma$. Since $f\Gamma$ is convex (according to the proposition 1) the condition $y_0 \in \operatorname{int} \operatorname{dom} f\Gamma$ implies that $f\Gamma$ is continuous at y_0 and there exists a neighborhood W of y_0 such that $f\Gamma$ is bounded from above on W (Brøndsted [3]).

Now the stability of dual solutions follows immediately from the results of Moreau [9, 10] and Rockafellar [14]. Namely, we have

THEOREM 4.3 [9] *If* $f\colon Y \to R$ *is any convex function bounded from above on some neighborhood of* y_0 *(in any admissible topology on* Y*) then* f *is subdifferentiable at* y_0 *and the set of subgradients is weakly* compact subset of* U.

This theorem generalize the well-known fact of boundedness of subgradients in every interior point of the effective domain of convex function in finite dimensional case. As a corollary of Theorem 4.1 and 4.3 we may formulate the final

THEOREM 4.4. *If* $\Gamma\colon Y \to X$ *and* $f\colon X \to R$ *satisfy the assumptions of Theorem* 4.1 *then the primal solution multifunction* M *is u.s.c. at* y_0 *and the set of solutions of the dual problem* D *is weakly* compact.*

5. Generalizations

Throughout the paper we considered convex primal problems assuming convexity of a minimized function f and a feasible set multifunction Γ. The most important consequence of these assumptions was convexity of the perturbation function $f\Gamma$.

Let us observe firstly that the dual problem

D supremize $\langle y_0, u\rangle + u_0$

subject to: $f\Gamma(y) \geqslant \langle y, u\rangle + u_0$ for every $y \in Y$

$u \in (\Gamma 0^+)^0$, $u_0 \in R$

remains well defined if we remove the convexity assumptions. However, having in mind the geometrical interpretation of the dual problem, we cannot expect the existence of the nonvertical support closing the duality gap in nonconvex case. One of the usually treated way to avoid this difficulty is to consider larger classes of dual price functions $\mathscr{F}$. This approach, suggested by Gould in [7] was developed by Dolecki and Kurcyusz in [6] and Dolecki [5]. The analysis of different price function classes $\mathscr{F}$ and the resulting duals was also given by Tind and Wolsey [16] for problems with inequality constraints. These ideas may be used also in the context of duality proposed in the present paper.

There exist several ways of introducing dual multifunction Γ^* to a given multifunction Γ. In finite dimensional spaces dual multifunction Γ^* may be considered as a multifunction having the graph $G(\Gamma^*)$ which is the polar set of the graph $G(\Gamma)$. The properties of such dual multifunction and corresponding pair of linear dual problems were analysed recently by Ruys and Weddepohl [20]. Most recently this idea was considered by Dolecki [21] in general spaces.

The main tool explored in this approach is the notion of pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ between arbitrary sets X and Y understood as an arbitrary function from the product space $X \times Y$ into real numbers R. In particular, if Y is any set of functions, defined on X, then a pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ between X and Y may be defined as $\langle\!\langle x, y \rangle\!\rangle = y(x)$. If ψ and Φ are any class of functions defined on X and Y respectively, and $\langle\!\langle \cdot, \cdot \rangle\!\rangle \colon X \times \psi \to R$ $\langle\!\langle \cdot, \cdot \rangle\!\rangle \colon Y \times \Phi \to R$ are arbitrary pairings then the dual multifunction $\Gamma^* \colon \psi \to \Phi$ may be introduced as

$$\Gamma^* \psi = \{\varphi \in \Phi \colon \langle\!\langle y, \varphi \rangle\!\rangle - \langle\!\langle x, \psi \rangle\!\rangle \leqslant 0 \text{ for every } x \in X \text{ and } y \in \Gamma^{-1} x\}.$$

Taking the class $\Phi = \mathscr{F}$ and the class $\psi = V$ (according to the notations introduced in section 2) and defining $\langle\!\langle y, F \rangle\!\rangle = Fy$, $\langle\!\langle x, v \rangle\!\rangle = f(x) + \langle x, v\rangle$ we obtain

$$\Gamma^* v = \{F \in \mathscr{F} \colon f(x) + \langle x, v\rangle \geqslant \sup_{y: x \in \Gamma y} Fy \text{ for every } x \in X\}$$

which is the same as the dual multifunction introduced in the section 2a.

The notion of the recession cone of multifunction, introduced in the section 1a of the present paper and appearing in the formulation of the dual problem D gives rise to the more detailed description of the set of price functions $\mathscr{F}$. It gives additional information about the structure of the set $\mathscr{F}$ and might be explored in applications and further investigations.

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Dualność i stabilność w wypukłych zadaniach minimalizacji warunkowej przy ograniczeniach zadanych multifunkcyjnie

W pracy formułowany jest problem dualny dla abstrakcyjnych zadań minimalizacji warunkowej, gdzie zbiór ograniczeń zadany jest multifunkcją wypukłą. Proponowany schemat dualny umożliwia dowolny wybór parametru dualizacji, a także uogólnia niektóre istniejące schematy poprzez możliwie ogólną postać zbioru ograniczeń.

Дуальность и устойчивость в выпуклых задачах условной минимизации при ограничениях заданных в виде многозначных отображений

В работе формулируется дуальная проблема для абстрактных задач условной минимизации, когда множество ограничений задано в виде выпуклого многозначного отображения. Предлагаемая дуальная схема позволяет проводить произвольный выбор параметра дуализации, а также обобщает некоторые существующие схемы посредством возможно общего вида ограничений.