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On perturbations of quadratic functionals with constraints

by

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We minimize a perturbed quadratic form $\langle Ax, x\rangle$ subject to the constraint $\|Sx-y\|\leqslant\varepsilon$. We prove convergence criteria for optimal and approximate solutions, values and multipliers, extending some results of [10], where the case $\varepsilon=0$ was considered. This extends the G-convergence type criterion for quadratic forms to the constrained case. We obtain explicit error estimates either with a fixed $\varepsilon<0$ or as $\varepsilon\to0$ (uniform with respect to the perturbations acting on A) for exact solutions and values.

Introduction

We consider tne convergent behavior of approximate and optimal solutions, Lagrange multipliers and optimal values for the following problem. We minimize a perturbed quadratic form $\langle Ax, x\rangle$ on some Banacn space subject to the fixed constraint $\|Sx-y\|\leqslant\varepsilon$. We assume that the perturbations acting on the operator A do not destroy its positive character. When $\varepsilon=0$ we get the problem considered in [10].

When $\varepsilon>0$, the above problem is of interest in its own right. Moreover the following applications motivate this work and are related to this problem: perturbation theory of constrained least squares, parameters evaluation from which the objective functional depends in quadratic constrained optimization, stability analysis of the linear regulator problem of optimal control theory with a constrained final state. The above problem with $\varepsilon>0$ may be also considered as a relaxed version of the quadratic optimization problem with affine constraint $Sx=y$ (see [10]) when this equality constraint is enlarged for approximation or computational reasons.

In the first section of this paper we describe the convergence of the optimal solutions, Lagrange multipliers and the optimal values of the perturbed problems toward the corresponding objects of the unperturbed one. We show that this (strong or weak) convergent behavior is related to that of the equality constrained problem obtained in [10]. The weak convergence criterion thereby obtained (theorem 1) generalizes the G-convergence of quadratic forms in Banach spaces (see [9]) to the constrained setting and extends some of the convergence results of [10].

In the second section we obtain a convergence criterion for approximate or quasi-solutions. Moreover we show by an example that a suitable convergent behavior for quasi-solutions requires stronger assumptions about the perturbations acting on the objective functional than for the optimal solutions.

In the third section we prove explicit estimates for the strongly convergent behavior of optimal solutions and values depending upon problem data, both for a fixed $\varepsilon \geqslant 0$ and for $\varepsilon \to 0$ (uniformly with respect to the perturbations). Such quantitative informations may be useful in the analysis of the relaxation of the equality constraint ($\varepsilon = 0$) and in the sensitivity analysis for constrained quadratic problems. Related results in the finite dimensional setting may be found in [3] [6] and [7].

In the fourth section we give an application to a perturbed linear regulator problem in the optimal control of ordinary differential equations with fixed plant, perturbed weighting functions in the cost functional and linearly constrained final state.

Further results about the dependence of the solutions of quadratic optimization problems upon the data may be found in the references of [10].

Notations, problem statement and basic assumptions

We are given a real reflexive Banach space X with dual space X^*, an integer $m > 0$, a real constant $\varepsilon \geqslant 0$, a bounded linear surjective map

$$S: X \to R^m$$

two positive numbers $\bar{\alpha}$, ω and a sequence of linear bounded symmetric operators

$$A_n: X \to X^* \qquad n = 0, 1, 2, \ldots$$

such that

$$\bar{\alpha} \|x\|^2 \leqslant \langle A_n x, x \rangle \leqslant \omega \|x\|^2 \tag{1}$$

for every $n = 0, 1, 2, \ldots$, and $x \in X$.

Throughout this paper we shall denote by $\langle \cdot, \cdot \rangle$ either the duality between X^* and X or the usual scalar product on R^m, by $\rightharpoonup$ the weak convergence an by $\to$ the strong convergence.

Given $y \in R^m$ the n-th problem, $n = 0, 1, 2, \ldots$, is defined by minimizing

$$f_n(x) = \langle A_n x, x \rangle$$

subject to the constraint

$$\|Sx - y\| \leqslant \varepsilon \tag{2}$$

Here $\| \cdot \|$ denotes the euclidean norm on R^m. The same notation will be used for the norm of X. The unperturbed problem corresponds to $n = 0$, while the problems defined by $n=1, 2, \ldots$, should be considered as perturbations, or approximations, of the unperturbed one.

If $\varepsilon=0$ we get the problem of minimizing $f_n(x)$ subject to the affine constraint $Sx=y$, see [10].

Standard results

Given $n\geq 0$, $y \in R^m$, $\varepsilon>0$ we shall use throughout this paper the following results. The constraint (2) defines a non empty closed convex subset of X (since S is onto), and f_n is a strictly convex function (by (1)) with an unique absolute minimum point x_n on such a set. Therefore x_n is the only optimal solution of the convex programming problem of minimizing f_n subject to the inequality constraint

$$\|Sx-y\|^2\leqslant\varepsilon^2$$

Slater's condition ([1] p. 157) holds since S is onto (take x^* witn $Sx^*=y$). Then by the multiplier rule (see [1], corollary 1.2, p. 159) x_n is characterized by the existence of λ_n such that the following optimality conditions hold

$$\left[\begin{array}{ll}\lambda_n\geqslant 0; & \\ \langle A_n x_n, u\rangle+\lambda_n\langle Sx_n-y, Su\rangle=0, & \text{for every } u\in X; \\ \lambda_n(\|Sx_n-y\|^2-\varepsilon^2)=0. & \end{array}\right. \tag{3}$$

If $\|y\|\leqslant\varepsilon$ then obviously $x_n=0$, the problem becomes trivial, with $\lambda_n=0$ by (3) and injectivity of the adjoint operator S^* of S.

If $\|y\|>\varepsilon$ and $\lambda_n=0$, then by (3) and (1) we get $x_n=0$ which is not feasible. Therefore $\lambda_n>0$ and

$$\|Sx_n-y\|=\varepsilon, \qquad x_n\neq 0$$

by the complementarity condition in (3).
Moreover, in this case, by (3)

$$\langle A_n x_n, x_n\rangle+\lambda_n\langle Sx_n-y, Sx_n\rangle=0 \tag{4}$$

with

$$\langle Sx_n-y, Sx_n\rangle\neq 0.$$

Hence the Lagrange multiplier λ_n is always uniquely determined.

1. Convergence of exact solutions

We show in this section weak and strong convergence theorems for optimal solutions, multipliers and values if $\varepsilon>0$. We need the following.

LEMMA 1. *For any $y\in R^m$ both sequences x_n, λ_n are bounded.*

P r o o f. Letting $u=x_n$ in (3) and remembering (1) we get

$$0=\langle A_n x_n, x_n\rangle+\lambda_n\langle Sx_n-y, Sx_n\rangle\geqslant\bar{\alpha}\|x_n\|^2-\lambda_n\langle y, Sx_n\rangle$$

hence

$$\bar{\alpha}\ \|x_n\| \leqslant \lambda_n\ \|y\|\ \|S\|. \tag{5}$$

By Kuhn-Tucker's theorem (see [1], theorem 1.2, p. 157) we see that for every n

$$\langle A_n\, x_n,\, x_n\rangle \leqslant \langle A_n\, x,\, x\rangle + \lambda_n\, (\|Sx-y\|^2 - \varepsilon^2),$$

for every $x \in X$. In particular, by taking x^* with $Sx^* = y$,

$$0 \leqslant \langle A_n\, x^*,\, x^*\rangle - \lambda_n\, \varepsilon^2$$

so by (1)

$$\varepsilon^2\, \lambda_n \leqslant \omega\ \|x^*\|^2$$

and the proof is finished by using (5), q.e.d. ∎

THEOREM 1. *Assume that*

$$A_n^{-1}\, u \rightharpoonup A_0^{-1}\, u, \quad \textit{for every } u \in S^*(R^m). \tag{6}$$

Then the following holds

$$\langle A_n\, x_n,\, x_n\rangle \to \langle A_0\, x_0,\, x_0\rangle \quad (\textit{convergence of values}); \tag{7}$$

$$\lambda_n \to \lambda_0 \quad (\textit{convergence of multipliers}); \tag{8}$$

$$x_n \rightharpoonup x_0 \quad (\textit{weak convergence of optimal solutions}). \tag{9}$$

Proof. Given $y \in R^m$ such that $\|y\| > \varepsilon$, by lemma 1 we get for some subsequence

$$\lambda_n \to \bar{\lambda},\quad x_n \rightharpoonup \bar{x} \text{ with } \bar{\lambda} \geqslant 0,\ \bar{x} \in X.$$

From (1) we see that A_n is an isomorphism, so condition (6) makes sense. As shown in [10] (proof of theorem 1, p. 253) we have equiboundedness of $\|A_n^{-1}\|$. By (3)

$$A_n\, x_n + \lambda_n\, S^*\, (Sx_n - y) = 0$$

Thus

$$x_n = \lambda_n\, A_n^{-1}\, S^*\, y - \lambda_n\, A_n^{-1}\, S^*\, Sx_n. \tag{10}$$

By writing

$$\lambda_n\, A_n^{-1}\, S^*\, y = (\lambda_n - \bar{\lambda})\, A_n^{-1}\, S^*\, y + \bar{\lambda}\, (A_n^{-1}\, S^*\, y - A_0^{-1}\, S^*\, y) + \bar{\lambda} A_0^{-1}\, S^*\, y$$

we see from the convergence of λ_n, equiboundedness of A_n^{-1} and (6) that for a subsequence

$$\lambda_n\, A_n^{-1}\, S^*\, y \rightharpoonup \bar{\lambda} A_0^{-1}\, S^*\, y.$$

Moreover

$$\lambda_n\, A_n^{-1}\, S^*\, Sx_n = (\lambda_n - \bar{\lambda})\, A_n^{-1}\, S^*\, Sx_n + \bar{\lambda}\, A_n^{-1}\, S^*\, S\, (x_n - \bar{x}) +$$
$$+ \bar{\lambda}\, (A_n^{-1} - A_0^{-1})\, S^*\, S\bar{x} + \bar{\lambda} A_0^{-1}\, S^*\, S\bar{x}$$

and remembering $Sx_n \to S\bar{x}$, for a subsequence we get

$$\lambda_n\, A_n^{-1}\, S^*\, Sx_n \rightharpoonup \bar{\lambda} A_0^{-1}\, S^*\, S\bar{x}.$$

By compactness of S as well as by (3) and (10) we obtain

$$\bar\lambda \geqslant 0;$$
$$\bar\lambda\,(\|S\bar x - y\|^2 - \varepsilon^2) = 0$$
$$A_0\,\bar x + \bar\lambda S^*\,(S\bar x - y) = 0\,.$$

This means by the optimality conditions (3) that $\bar\lambda = \lambda_0$, $\bar x = x_0$. By uniqueness of λ_0, x_0, we get (8) and (9) for the original sequence. Finally (7) follows from (4) by compactness of S, q.e.d.. ■

A persual of the above proof gives the following

THEOREM 2. *If*

$$A_n^{-1}\,u \to A_0^{-1}\,u, \qquad \textit{for every } u \in S^*\,(R^m) \tag{11}$$

then for every $y \in R^m$ *we have*

$$x_n \to x_0 \qquad (\textit{strong convergence of optimal solutions}).$$

Conditions (7), (8), (9) are not independent, for we can show

PROPOSITION 1. *If* (7) *and* (9) *hold for every* y, *then we have* (8). *If* (8) *and* (9) *hold then we get* (7), *for every* y.

Proof. Assume (7) and (9), for every y.
By lemma 1 we have $\lambda_n \to \bar\lambda$ for some subsequence. Letting $n \to +\infty$ in (3) with $u = x_n$ we get

$$\langle A_0\,x_0, x_0\rangle + \bar\lambda\,\langle Sx_0 - y, Sx_0\rangle = 0\,.$$

Then $\bar\lambda = \lambda_0$ if $\|y\| > \varepsilon$, while $\lambda_n = \bar\lambda = 0 = \lambda_0$ if $\|y\| \leqslant \varepsilon$. This gives (8). If we assume (8) and (9), for every y, then $Sx_n \to Sx_0$, and by (4), if $\|y\| > \varepsilon$

$$\lambda_n = \langle A_n\,x_n, x_n\rangle\,\langle y - Sx_n, Sx_n\rangle^{-1} \to \lambda_0 = \langle A_0\,x_0, x_0\rangle\,\langle y - Sx_0, Sx_0\rangle^{-1}$$

thus giving (7), q.e.d. ■

REMARK. By theorem 1 of this paper and theorem 1 of [10] we see that condition (6) is sufficient to ensure the weak convergence of optimal solutions, multipliers and values both if $\varepsilon = 0$ (affinely constrained problems) and if $\varepsilon > 0$. As remarked in [10], condition (6) is an extension to this constrained setting of the convergence

$$A_n^{-1}\,u \rightharpoonup A_0^{-1}\,u, \qquad \text{for every } u \in X^*, \tag{12}$$

which is a necessary and sufficient condition for the weak convergence of (unconstrained) minimum points over X for the functionals

$$\langle A_n\,x, x\rangle - \langle u, x\rangle, \qquad u \in X^*.$$

Condition (12) is in turn equivalent to the G-convergence of A_n towards A_0 (see [2], [9], [11]). Analogously, condition (11) is sufficient for strong convergence of exact solutions both for $\varepsilon > 0$ (theorem 2) and for $\varepsilon = 0$ (theorem 2 in [10]).

For a comparison with the perturbation theory of unconstrained quadratic optimization problems see [10], section 3, p. 256—257.

2. Convergence of quasi-solutions

Motivated by practical and theoretical reasons, given $\alpha>0$ we consider for any $y \in R^m$ the set of quasi-solutions (of order α) for the n-th problem, defined by

$$Q_n(\varepsilon, \alpha)=\{u \in X: \|Su \times y\| \leqslant \varepsilon,$$

$$\langle A_n u, u\rangle \leqslant \langle A_n x_n, x_n\rangle+\alpha\}.$$

By (1), the set $Q_n(\varepsilon, \alpha)$ is non empty and bounded.

The convergence of quasi-solutions is described by the following

THEOREM 3. *Assume condition* (12). *Then*

(i) $u_n \in Q_n(\varepsilon, \alpha)$, $u_n \rightharpoonup u_0$ *for some subsequence imply that* $u_0 \in Q_0(\varepsilon, \alpha)$;

(ii) *for every* $v_0 \in Q_0(\varepsilon, \alpha)$, $\varepsilon^*>\varepsilon$, $\alpha^*>\alpha$ *there exist* $v_n \in Q_n(\varepsilon^*, \alpha^*)$ *such that* $v_n \rightharpoonup v_0$.

Proof. By (12) the quadratic forms $\langle A_n x, x\rangle$ G-converge towards $\langle A_0 x, x\rangle$ on X equipped with its weak convergence (see [2], teorema 4.1., p. 150, which can be generalized to the abstract case in a straightforward way). This mode of convergence amounts to the following (see [2], teorema 3.1., p. 145):

(a) $z_n \rightharpoonup z$ in X implies $\liminf \langle A_n z_n, z_n\rangle \geqslant \langle A_0 z, z\rangle$;

(b) for every $\omega \in X$ there exists a sequence $\omega_n \rightharpoonup \omega$ in X such that

$$\langle A_n \omega_n, \omega_n\rangle \rightarrow \langle A_0 \omega, \omega\rangle.$$

Hence if u_n is as in (i) then by (a)

$$\liminf \langle A_n u_n, u_n\rangle \geqslant \langle A_0 u_0, u_0\rangle$$

thus by (7)

$$\alpha+\langle A_0 x_0, x_0\rangle=\alpha+\lim \langle A_n x_n, x_n\rangle \geqslant \liminf \langle A_n u_n, u_n\rangle \geqslant \langle A_0 u_0, u_0\rangle$$

moreover $Su_n \rightarrow Su_0$. This proves conclusion (i).

Let v_0 be as in (ii). By G-convergence of the quadratic forms defined by A_n we can find points $v_n \rightharpoonup v_0$ such that (b) holds and $Sv_n \rightarrow Sv_0$. Hence for large n

$$\|Sv_n-y\| \leqslant \varepsilon^*, \quad \langle A_n v_n, v_n\rangle \leqslant \langle A_0 x_0, x_0\rangle+\alpha^*$$

thus proving (ii), q.e.d. ■

The following example shows that conclusion (ii) in theorem 3 does not follow by replacing (12) with (6).

EXAMPLE. Let $m=1$, $X=R^2$, $Su=(1, 0)\, u$, $A_0=\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$,

$$A_n=\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{if } n \geqslant 1.$$

Then $S^* y=\begin{pmatrix} y \\ 0 \end{pmatrix}$, and $A_n^{-1} S^* v=A_0^{-1} S^* v=\begin{pmatrix} 2v \\ 0 \end{pmatrix}$, for every v and n so condition (6) is satisfied.

If $y=0$, given $\alpha>0$ and $\varepsilon\geqslant 0$, we see that

$$Q_0(\varepsilon, \alpha)=\left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2 : |x_1|\leqslant\varepsilon,\ x_1^2+2x_2^2\leqslant 2\alpha\right\}$$

while for $n\geqslant 1$

$$Q_n(\varepsilon, \alpha)=\left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2 : |x_1|\leqslant\varepsilon,\ x_1^2+4x_2^2\leqslant 2\alpha\right\}$$

so conclusion (ii) of Theorem 3 does not hold.

3. Error estimates

Given $y\in R^m$ we denote by $x_n(\varepsilon)$ the unique optimal solution of the n-th problem, and by

$$v_n(\varepsilon)=\langle A_n x_n(\varepsilon),\ x_n(\varepsilon)\rangle$$

its value, $\varepsilon\geqslant 0$, $n=0, 1, 2, \ldots$.

We wish to estimate $\|x_n(\varepsilon)-x_0(\varepsilon)\|$ and $|v_n(\varepsilon)-v_0(\varepsilon)|$ under strong convergence of the optimal solutions (as in theorem 2). Moreover we want to obtain information about the quantitative behavior of $x_n(\varepsilon)$ as $\varepsilon\to 0$, estimating $\|x_n(\varepsilon)+ -x_n(0)\|$ uniformly with respect to n.

In the following lemma let us collect some standard results we shall need in the following estimates.

LEMMA 2.

(i) *There exists a constant $c>0$ such that*

$$\|S^* x\|^2\geqslant c\,\|x\|^2, \quad \text{for every } x\in R^m. \tag{13}$$

(ii) *Assume condition* (11). *Then*

$$q_n=\sup\left\{\|A_1^{-1} S^* v - A_0^{-1} S^* v\| : \|v\|\leqslant\frac{\omega\,\|y\|}{c}\right\}\to 0 \quad \text{as } n\to+\infty .$$

(iii) *For every n we have*

$$\|(SA_n^{-1} S^*)^{-1}\|\leqslant\frac{\omega}{c}. \tag{15}$$

Proof. Condition (13) is well known since S is onto (see [8], p. 206). From (11) we get strong pointwise convergence of $A_n^{-1} S^*$ toward $A_0^{-1} S^*$, which is uniform on compact sets by equiboundedness of $\|A_1^{-1}\|$. This proves (14). Given n, let $T= =SA_n^{-1} S^*$. Then S is a symmetric operator such that

$$\langle Tx, x\rangle=\langle A_n^{-1} S^* x,\ S^* x\rangle\geqslant\frac{c}{\omega}\,\|x\|^2$$

by (13) and (1).

Hence

$$\langle T^{-1} x, x\rangle \leqslant \frac{\omega}{c} \|x\|^2$$

moreover

$$\|T^{-1}\| = \sup \{\langle T^{-1} x, x\rangle : \|x\|=1\}$$

and this gives (15), q.e.d. ■

By using the constant c of (13) and q_n defined by (14) we get the following error estimates if $\varepsilon=0$.

THEOREM 4. *Assume condition* (11). *Then*

$$\|x_n(0)-x_0(0)\| \leqslant \left(1+\frac{\omega}{\bar{\alpha} c}\|S\|^2\right) q_n,$$

moreover

$$|v_n(0)-v_0(0)| \leqslant \frac{\omega}{c} \|S\|\, q_n.$$

Proof. For short notation put $x_n = x_n(0)$. By (3), p. 251, of [10], for any $y \in R^m$ we get

$$x_n - x_0 = A_n^{-1} S^* [(SA_n^{-1} S^*)_0^{-1} y - (SA_0^{-1} S^*)^{-1} y] +$$
$$+ (A_0^{-1} - A_0^{-1}) S^* (SA_0^{-1} S^*)^{-1} y$$

If U, V are isomorphisms then for all y

$$\|U^{-1} y - V^{-1} y\| = \|U^{-1}(U-V) V^{-1} y\|$$

and remembering (15) and (1)

$$\|A_n^{-1} S^* [(SA_n^{-1} S^*)^{-1} y - (SA_0^{-1} S^*)^{-1} y]\| \leqslant$$
$$\leqslant \frac{\omega \|S\|^2}{\bar{\alpha} c} \|(A_n^{-1} S^* - A_0^{-1} S^*)(SA_0^{-1} S^*)^{-1} y\| \leqslant \frac{\omega \|S\|^2}{\bar{\alpha} c} q_n.$$

Moreover from (15)

$$\|(A_n^{-1} - A_0^{-1}) S^* (SA_0^{-1} S^*)^{-1} y\| \leqslant q_n$$

hence the first estimate follows. By (12) p. 255 of [10] we obtain

$$|v_n - v_0| \leqslant |\langle [(SA_n^{-1} S^*)^{-1} - (SA_0^{-1} S^*)^{-1}] y, y\rangle| \leqslant \leqslant \|(SA_n^{-1} S^*)^{-1}\| \times$$
$$\times \|(SA_n^{-1} S^* - SA_0^{-1} S^*)(SA_0^{-1} S^*)^{-1} y\| \leqslant \frac{\omega}{c} \|S\|\, q_n,$$ q.e.d. ■

Given $\varepsilon > 0$ and $y \in R^m$ the following estimate holds for optimal solutions.

THEOREM 5. *Assume that*

$$A_n u \to A_0 u, \quad \textit{for every } u \in X.$$

Then for $\varepsilon>0$ and any n

$$\|x_n(\varepsilon)-x_0(\varepsilon)\|\leqslant\frac{1}{\bar{\alpha}}\|(A_n-A_0)\,x_0(\varepsilon)\|.$$

REMARK. For every g and $x\in X^*$

$$A_n^{-1}x-A_0^{-1}x=A_n^{-1}(A_0-A_n)\,A_0^{-1}x\,.$$

This shows that the assumption in theorem 5 is equivalent (by equiboundedness of A_n^{-1}) to

$$A_n^{-1}x\to A_0^{-1}x,\qquad\text{for every } x\in X^*$$

and this is equivalent in turn to Mosco's convergence of the quadratic forms defined by A_n (see [9]), of course implying (11).

P r o o f o f t h e o r e m 5. The functionals

$$f_n(x)=\langle A_n x, x\rangle,\qquad n=0, 1, 2, \ldots$$

are equi-strongly convex on X by (1) as seen by

$$f_n(tu+(1-t)\,v)\leqslant tf_n(u)+(1-t)\,f_n(v)-\bar{\alpha}t\,(1-t)\,\|u-v\|^2$$

for every u and v in X and $t\in[0, 1]$. Then the conclusion is immediate by corollary 2.1. p. 89 of [4], q.e.d. ■

We end this section by estimating $\|x_n(\varepsilon)-x_n(0)\|$ and $|v_n(\varepsilon)-v_n(0)|$ uniformly in n. This relates in a quantitative way the linearly constrained quadratic problem ($\varepsilon=0$) with the problems having $\varepsilon>0$. The following estimate may be combined with the conclusions of theorem 5 to relate the error estimates about optimal solutions with constraint relaxation.

The following lemma is well known ([5], VI.6.1, p. 487).

LEMMA 3. *There exists a constant $k>0$ such that for every $z\in R^m$ we can find $v\in X$ satisfying $Sv=z$ and $\|v\|\leqslant k\,\|z\|$.*

By using the constant k of lemma 3 we get

THFOREM 6. *For every $\varepsilon>0$, $y\in R^m$ such that $\|y\|>\varepsilon$ and every n*

$$\|x_n(\varepsilon)-x_n(0)\|^2\leqslant\frac{k\omega}{\bar{\alpha}c}\,\|S\|\cdot\|y\|\,\varepsilon\,.$$

P r o o f. By proposition 2.2 p. 88 of [4]

$$\bar{\alpha}\,\|x_n(\varepsilon)-x_n(0)\,\|^2\leqslant\|A_n\,x_n(0)\|\,\inf\,\{\|z-x_n(\varepsilon)\|:Sz=y\}\tag{16}$$

since by (1) the quadratic forms defined by A_n are equi-strongly convex on X with a common strong convexity constant $\bar{\alpha}$ (and the choice $u_n=x_n(0)$, notations of (7) p. 88 of [4], is admissible).

By the optimality conditions (3) we know that

$$\|Sx_n(\varepsilon)-y\|=\varepsilon.$$

Let $w \in X$ be such that $Sw=y$. By lemma 3 there exists $w_n \in X$ with

$$Sw_n=S\left(x_n(\varepsilon)-w\right), \qquad \|w_n\| \leqslant k\varepsilon,$$

therefore $S\left(x_n(\varepsilon)-w_n\right)=y$, hence

$$\inf\left\{\|z-x_n(\varepsilon)\| : Sz=y\right\} \leqslant \|w_n\| \leqslant k\varepsilon.$$

Thus by (16)

$$\bar{\alpha}\,\|x_n(\varepsilon)-x_n(0)\|^2 \leqslant k\varepsilon\,\|A_n\,x_n(0)\|.$$

Using (3) p. 251 of [10] and (15) we get

$$\|A_n\,x_n(0)\| \leqslant \|S\|\,\|(SA_n^{-1}\,S^*)^{-1}\|\,\|y\| \leqslant \frac{\omega}{c}\,\|S\|\,\|y\|, \tag{17}$$

q.e.d. ∎

COROLLARY 1. *For some constant L, any n, any ε with $0<\varepsilon\leqslant 1$ and a fixed $y \in R^m$ such that $\|y\|>\varepsilon$,*

$$0 \leqslant \tau_n(\varepsilon)-\tau_n(0) \leqslant L\sqrt{\varepsilon}\,.$$

P r o o f. We have

$$\langle A_n\left(x_n(\varepsilon)-x_n(0)\right),\; x_n(\varepsilon)-x_n(0)\rangle = v_n(\varepsilon)-v_n(0)+2\langle A_n\,x_n(0),\; x_n(0)-x_n(\varepsilon)\rangle$$

The conclusion follows combining (1), theorem 6 and (17), q.e.d. ∎

4. Application to the linear regulator problem

We apply the results of the first section to the following constrained regulator problem. We wish to minimize the perturbed cost

$$\int_0^T u(t)'\,Q_n(t)\,u(t)\,dt, \quad n=0, 1, 2, \ldots, \tag{18}$$

subject to the state equations

$$\begin{cases} \dot{x}(t)=G(t)\,x(t)+B(t)\,u(t) & \text{a.e. in } [0, 1], \\ x(0)=0 \end{cases} \tag{19}$$

and the final value constraint

$$\|Hx(T)-y\| \leqslant \varepsilon \tag{20}$$

Here a prime denotes transpose, $u(t) \in R^q$ is the control variable, $u \in L^2(0, T)$, $x(t) \in R^p$ is the state variable, and $Q_n(t)$, $G(t)$, $B(t)$ are given matrices of the appropriate dimensions, H is a given $m \times p$ constant matrix, $m \leqslant p$, and $y \in R^m$.

We assume that $G \in L^1(0, T)$, $B \in L^2(0, T)$; $Q_n \in L^\infty(0, T)$, $Q_n(t)$ is symmetric, and there exist constants $\bar{\alpha}>0$, ω such that for every $z \in R^q$, $n=0, 1, 2, \ldots$, and a.e. t

$$\bar{\alpha}\,|z|^2 \leqslant z'\,Q_n(t)\,z \leqslant \omega\,|z|^2$$

Let F be a fundamental matrix for the uncontrolled plant $\dot{x}=G(t)x$. Then given the control $u \in L^2(0, T)$ the corresponding state x is given by

$$x(t)=(Lu)(t)=F(t)\int_0^t F^{-1}(s)B(s)u(s)\,ds, \qquad 0 \leqslant t \leqslant T.$$

By setting

$$\begin{gathered} X=L^2(0,T) \quad (\text{of } R^q\text{— valued vectors}) \\ (A_n(u))(t)=Q_n(t)u(t) \\ Su=H(Lu)(T), \qquad u \in X \end{gathered} \tag{21}$$

we obtain a particular case of the problem studied in section 1.

We wish to obtain direct sufficient conditions about the data of problem (18), (19), (20) to get weak or strong $L^2(0, T)$ convergence of optimal controls and of optimal values under the perturbations acting on the costs (18) and described by the sequence Q_n.

Let us denote by C the linear subspace of R^q spanned by the rows of the matrix H.

Corollary 2. *Assume that the control system* (19) *is completely controllable at time T, and that* rank $H=m$. *Then a sufficient condition of weak convergence in $L^2(0, T)$ of the optimal controls and values for every y is given by*

$$c' F(T) F^{-1} B(Q_n^{-1}-Q_0^{-1}) \rightharpoonup 0 \tag{22}$$

in $L^2(0, T)$, for every $c \in C$.

Strong convergence of the optimal controls is obtained if strong convergence holds in (22).

Proof. By theorem 1 we must check that (22) implies (6), with S defined by (21). Of course S is onto. Given $u \in L^2(0, T)$ and $v \in R^p$ we compute

$$L(u)(T)'v=\langle u, L(\cdot)(T)^* v\rangle=\int_0^T u(t)' B(t)' F(t)'^{-1} F(T)' v\,dt.$$

Thus, for every $w \in R^m$, and a.e. $t \in [0, T]$

$$(S^* w)(t)=B(t)' F(t)'^{-1} F(T') H' w.$$

By (18) condition (6) becomes (22), q.e.d. ∎

Of course condition (22) may be far weaker than weak convergence of Q_n^{-1} towards Q_0^{-1} as in the following

Example. Let $p=q=2$, $m=T=1$, $G=B\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ $H=(1, 0)$, $Q_n(t)=\begin{pmatrix}w_n(t) & 0\\ 0 & z_n(t)\end{pmatrix}$, where w_n, z_n are measurable functions such that

$$0<\bar{\alpha} \leqslant w_n(t), z_n(t) \leqslant \omega$$

for every n, and a.e. t. Then it is easily seen that (22) amounts to $\dfrac{1}{w_n} \to \dfrac{1}{w_0}$ in $L^2(0, 1)$, while no convergence condition is required about z_n.

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O zaburzonych funkcjonałach kwadratowych z ograniczeniami

Minimalizowana jest zaburzona względem A forma kwadratowa $\langle Ax, x\rangle$ przy ograniczeniu $\|Sx-y\|\leqslant\varepsilon$. Dowodzi się warunków zbieżności rozwiązań optymalnych i przybliżonych, wartości formy i mnożników, rozszerzając wyniki z pracy [10], w której rozważa się przypadek $\varepsilon=0$. Stanowi to rozszerzenie warunków G-zbieżności dla form kwadratowych na przypadek z ograniczeniami. Otrzymano analityczne oceny błędu zarówno dla ustalonego $\varepsilon<0$, jak i dla $\varepsilon\to 0$ (jednostajnie względem zaburzeń oddziaływujących na A), w odniesieniu do dokładnych rozwiązań i wartości formy.

О нарушенных квадратичных функционалах с ограничениями

Миномизируется нарушенная по отношению к А квадратичная форма $\langle Ax, x\rangle$ при ограничении $\|Sx-y\|\leqslant\varepsilon$. Доказываются условия сходимости оптимальных и приближенных решений, значений формы и множителей, расширяя результаты работы [10], в которой рассматривается случай $\varepsilon=0$. Это является расширением условий G — сходимости для квадратических форм на случай с ограничениями. Получены аналитические оценки ошибки как для определенного $\varepsilon>0$, так и для $\varepsilon\to 0$ равномерно по отношению к нарушениям, воздействующим на A, в сравнении с точными решениями и значениями формы.