Control
and Cybernetics
VOL. 13 (1984) No. 1-2

Eigenstructure assignment of multivariable systems with fast, medium and slow modes

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The arbitrary assignment of eigenvalues and eigenvectors via state variable feedback in the case of large multivariable systems is considered. The system analysed is assumed to be a structured one with state variables directly identifiable as belonging to fast, medium and slow modes of oscillation. The assignment of eigenvalues and eigenvectors are done individually for the smaller subsystems. Such an assignment is shown to approximately carry over to the composite system under certain mild restrictions. The computational and conceptual simplicity of this design procedure in modifying system dynamics is emphasized.

1. Introduction

One of the problems encountered in computing the control laws of a large scale system is the numerical instability of the computational processes. This instability may be directly traced to the ill conditioned nature of the system matrices involved. A deeper look, at this problem brings to light the important fact that computations are meaningful for dynamic systems only if the system to be controlled has time scales of comparable order of magnitude. To cite an example, if a system representation combines within itself nanosecond responses with phenomena whose occurrences are measured in hours, then the numerical computation on the system for determining optimal control laws invariably runs into difficulties. If it is possible to segregate this system into sets of differential equations with each set associated with time scales of comparable magnitude, then such a decomposition will considerably ease the computational problems mentioned above.

Such a system is considered here, with state variables $x(t)$, $y(t)$ and $z(t)$ associated with medium (i.e. intermediate between fast and low) fast and slow modes of oscillation respectively. The system is described by the equations

$$S_{\varepsilon\mu}: \begin{bmatrix} \dot{x}(t) \\ \varepsilon\dot{y}(t) \\ \mu\dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} u(t) \tag{1}$$

The sizes of the various vectors are as follows:

State variables x, y and z have dimensions n, p and q respectively. The input vector u is an m — vector. All matrices are real and of compatible order. ε and μ are tags attaches to denote that y and z are state vectors associated with fast and slow changing modes with ε very small (tending towards zero) and μ very large (tending towards infinity). For an unforced system with a structure similar to (1) Desoer and Shensa [1] have characterized the system dynamics into three subsystems wiz. S_F, S_M and S_S with reduced order governing equations as given in the sequel. Each of these subsystems dominates during particular interval of time with S_F, S_M and S_S holding sway respectively in the early, intermediate and final stage of the evolution of state in time. The equations are:

$$S_F: \varepsilon\dot{y}(t) = A_{22}\, y(t) \tag{2}$$

$$S_M: \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{3}$$

$$S_S: \begin{bmatrix} 0 \\ 0 \\ \mu\dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}. \tag{4}$$

2. Properties of reduced order subsystems

The matrices $[A_{22}]$ and $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ corresponding to reduced order subsystems S_F and S_M are assumed to be nonsigular. For such systems, we recall here some of the properties as derived in [1] and relevant in the present context. These are:

i) S_F, S_M and S_S are associated with p, n and q natural frequencies respecitvely.

ii) Let $\Lambda_F = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ denote the set of p natural frequencies of S_F obtained with $\varepsilon = 1$ substituted in equation (2) and let λ_i be a member of this set. λ_i is an eigenvalue of A_{22} and let y_i be the corresponding eigenvector. Then for the composite system $S_{\varepsilon\mu}$ p-number of eigenvalues, $\lambda_1', \lambda_2', \ldots, \lambda_p'$, are given by

$$\lambda_i' = \frac{\lambda_i}{\varepsilon} + r\left(\varepsilon, \frac{1}{\mu}\right), \qquad i = 1, 2, \ldots, p \tag{5}$$

where $r(\varepsilon, 1/\mu)$ is holomorphic, but $r(0, 0) = 0$ (1), uniformly for $\mu \geqslant 1$. Further the leading term of expansion of the mode of $S_{\varepsilon\mu}$ corresponding to λ_i' is given by

$$(x_i', y_i', z_i')^T \exp(\lambda_i' t) \text{ where}$$

$$x_i \triangleq \frac{1}{\lambda_i'} A_{12}\, y_i$$

$$z_i \triangleq \frac{1}{\mu\lambda_i'} A_{32}\, y_i\,, \qquad i=1, 2, \ldots, p\,. \tag{6}$$

It is clear from (5) and (6) that as $\varepsilon \to 0$ and $\mu \to \alpha$, the eigenvalues and eigenvectors of the composite system $S_{\varepsilon\mu}$ tend to values given below

$$\lambda_i' \to \lambda_i/\varepsilon$$

$$(x_i', y_i', z_i')^T \to (0, y_i', 0)^T.$$

In otherwords, the eigenvalues and eigenvectors of S_F (obtained after setting $\varepsilon=1$) carry over to the composite system $S_{\varepsilon\mu}$ with some modifications.

iii) Let $\Lambda_S=\{\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{p+q}\}$ denote the set of q natural frequencies of S_S obtained with $\mu=1$ substituted in equation (4). Let λ_i be a member of this set and let $(x_i', y_i', z_i')^T$ be the corresponding natural mode; thus

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33}-\lambda_i I \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{7}$$

Then for the composite system $S_{\varepsilon\mu}$ q-number of eigenvalues $\lambda'_{p+1}, \lambda'_{p+2}, \ldots, \lambda'_{p+q}$ are given by

$$\lambda_i' = \frac{\lambda_i}{\mu} + s\left(\frac{1}{\mu}, \frac{\varepsilon}{\mu}\right), \qquad i=p+1, \ldots, p+q \tag{8}$$

where s is holomorphic. However $s\,(1/\mu, \varepsilon/\mu)=0\,(1/\mu^2)$ uniformly for $\varepsilon \leqslant 1$. Further the leading term of the expansion of the mode of $S_{\varepsilon\mu}$ corresponding to λ_i' is given by $(x_i', y_i', z_i')^T \exp\,(\lambda_i'\, t)$. It is hence clear that as $\varepsilon \to 0$ and $\mu \to \infty$ the q-number of eigenvalues of the composite system $S_{\varepsilon\mu}$ become

$$\lambda_i' = \frac{\lambda_i}{\mu} \qquad i=p+1, p+2, \ldots, p+q \tag{9}$$

and the eigenvectors of S_S (corresponding to eigenvalues (Λ_S)) carry over to the composite system without any modifications.

iv) Let $\Lambda_M=\{\lambda_{p+q+1}, \ldots, \lambda_{p+q+n}\}$ denote the set of n-natural frequencies of S_M. Let λ_i be a member of this set and let $(x_i', y_i')^T$ be the corresponding natural mode, thus

$$\begin{bmatrix} A_{11}-\lambda_i I & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{10}$$

Let λ_i' be a natural frequency of the composite system $S_{\varepsilon\mu}$ close to λ_i, then

$$\lambda_i' = \lambda_i + v\left(\varepsilon, \frac{1}{\mu}\right), \qquad i=p+q+n, \ldots, p+q+n \tag{11}$$

where v is holomorphic in a neighbourhood of (0, 0), $v(0, 0)=0$, and $v(\varepsilon, 1/\mu)= =0(\varepsilon, 1/\mu)$. Further the leading term of expansion of the natural mode of $S_{\varepsilon\mu}$ corresponding to λ'_i is given by $(x'_i, y'_i, z'_i)^T \exp(\lambda'_i t)$ where

$$z_i = \frac{1}{\mu\lambda'_i}(A_{31} x_i + A_{32} y_i); \quad i=p+q+1, \ldots, p+q+n \tag{12}$$

It is clear from (11) and (12) that as $\varepsilon\to 0$ and $\mu\to\infty$ the eigenvalues and eigenvectors of the composite system $S_{\varepsilon\mu}$ tend to the following values

$$\lambda'_i \to \lambda_i \quad (x'_i, y'_i, z'_i)^T \to (x'_i, y'_i, 0)^T \quad i=p+q+1, \ldots, p+q+n \tag{13}$$

In otherwords, the eigenvalues and eigenvectors of S_M carry over to the composite system $S_{\varepsilon\mu}$, former without any change and latter with some modifications.

In case where submatrices $[A_{22}]$ and $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ are singular, the degrees of the characteristic polynomials of the reduced order systems S_M and S_S (with μ set equal to l in the latter) respectively become less than n and q, indicating thereby the presence of some eigenvalues which become infinity or multi-valued.

3. Problem statement and outline of procedure

We are concerned herewith simultaneous eigenvalue and eigenvector assignment of a large scale system $S_{\varepsilon\mu}$ described by equation (1), using state variable feedback of the form $u(t)=K_1 x(t)+K_2 y(t)+K_3 z(t)$. This feedback results in a new system $\bar{S}_{\varepsilon\mu}$ whose equations are as follows:

$$\bar{S}_{\varepsilon\mu}: \begin{bmatrix} \dot{x}(t) \\ \varepsilon\dot{y}(t) \\ \mu\dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_{11}+B_1 K_1 & A_{12}+B_1 K_2 & A_{13}+B_1 K_3 \\ A_{21}+B_2 K_1 & A_{22}+B_2 K_2 & A_{23}+B_2 K_3 \\ A_{31}+B_3 K_1 & A_{32}+B_3 K_2 & A_{33}+B_3 K_3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}. \tag{14}$$

The problem is to assign eigenvalues and eigenvectors for the above closed loop system when ε and $1/\mu$ are very small, which in turn involves the determination of the feedback matrix $K=[K_1\ K_2\ K_3]$. A straightforward determination of K for a given eigenstructure using Moore's [2] technique, involves a system of dimension $(p+n+q)$. We demonstrate below that the same problem may be tackled in a much simpler manner by assigning eigenvalues and eigenvectors separately to individual subsystems $\bar{S}_F$, $\bar{S}_M$ and $\bar{S}_S$ which have smaller dimensions. Hence there is considerable saving in computational efforts by adopting such a procedure. Further, the computational instability introduced because of widely varying time scales is circumvented in such an approach. From the physical view point also, the steps proposed for computation seem to be logical and appropriate. For example one may visualise how, at the start of a transient phenomenon, the fast modes completely dominate to the total exclusion of the medium and slow modes. Hence only the submatrix K_2 in the feedback matrix K is operative during this period. This means that K_2 may be directly calculated by considering $\bar{S}_F$ which is a dynamic system of order p. As time progresses, the medium modes take over and at this stage the feedback matrix involves both K_1 and K_2. The system model appropriate at this stage is $\bar{S}_M$ and we can use this model to evaluate both K_1 and K_2. Since K_2 has already been determined, the calculation of K_1 is straightforward. Coming to the third and final phase, system $\bar{S}_S$ represents the most appropriate model during this period. The feedback matrix now involves all the three submatrices K_1, K_2 and K_3 out of which K_1 and K_2 have already been determined, leaving the computation of K_3 alone at this stage. Thus as the system evolves in time, we have progressively represented the system dynamics by three different models $\bar{S}_F$, $\bar{S}_M$ and $\bar{S}_S$ each operative during a certain period. The reduced sizes of these models are thus fully exploited in calculating separately K_1, K_2 and K_3. Regarding the eigenvalues and eigenvectors associated with these subsystems, their properties as recalled in section 2, provide full justification for our procedure. Thus, the assigned eigenvectors of $\bar{S}_F$, $\bar{S}_M$ and $\bar{S}_S$ with some modifications (to the first order of small quantities ε and $1/\mu$) become identical to corresponding eigenvectors of the composite system $\bar{S}_{\varepsilon\mu}$. As far as eigenvalues of $\bar{S}_{\varepsilon\mu}$ are concerned the set associated with $\bar{S}_F$ tends towards negative infinity, the set associated with $\bar{S}_M$ remains virtually static and the set associated with $\bar{S}_S$ moves asymptotically towards the origin. For systems having only fast and slow modes, Porter [3] has assigned the eigenstructure using a slightly different approach and this paper is an extension of Porter's work to more general systems possessing fast, medium and slow modes.

4. Application of method

As already stated in section 3, we assign independently the eigenstructure (eigenvalues and eigenvectors) of each of the subsystems $\bar{S}_F$, $\bar{S}_M$ and $\bar{S}_S$. To begin with we have the subsystem $\bar{S}_F$ defined by,

$$\bar{S}_F\colon \varepsilon \dot{y}(t)=[A_{22}+B_2 K_2]\, y(t) \tag{15}$$

Let $y\, e^{\lambda t}$ be a solution of (15) with $\varepsilon=1$.

Hence

$$\lambda y e^{\lambda t}=[A_{22}+B_2 K_2]\, y e^{\lambda t} \tag{16}$$

$$(A_{22}-\lambda I_p)\, y+B_2 K_2 y=0$$

i.e.

$$(A_{22}-\lambda I_p\, B_2)\begin{bmatrix} y \\ K_2 y \end{bmatrix}=0. \tag{17}$$

Set

$$K_2 y=w \tag{18}$$

and define

$$S^F_{\lambda_i}=[A_{22}-\lambda_i I_p\, B_2]. \tag{19}$$

Obviously $[y'\, w']'$ belongs to the null space of $S^F_{\lambda_i}$. We assume that (A_{22}, B_2) is controllable. If it is not so, the uncontrollable eigenvalues may be included in the set $\{\lambda_1 \ldots \lambda_p\}$ chosen for the system. The eigenvectors $\{y_i\}$, $i=1, \ldots, p$ are arbitrary, subject only to the condition that they are linearly independent and belong to their respective null spaces of $S^F_{\lambda_i}$. It is easily seen that

$$K_2 [y_1 \, y_2 \ldots y_p] = [w_1 \, w_2 \ldots w_p] \tag{20}$$

or

$$K_2 \, Y = W.$$

Hence

$$K_2 = WY^{-1} \tag{21}$$

The feedback submatrix K_2 is thus evaluated. Next consider the subsystem $\bar{S}_M$. We have

$$\bar{S}_M: \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11}+B_1 K_1 & A_{12}+B_1 K_2 \\ A_{21}+B_2 K_1 & A_{22}+B_2 K_2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \tag{22}$$

Let $\begin{bmatrix} x \\ y \end{bmatrix} e^{\lambda t}$ be a solution of (22).

Hence

$$\lambda \begin{bmatrix} x \\ 0 \end{bmatrix} e^{\lambda t} = \begin{bmatrix} A_{11}+B_1 K_1 & A_{12}+B_1 K_2 \\ A_{21}+B_2 K_1 & A_{22}+B_2 K_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} e^{\lambda t}$$

$$\begin{bmatrix} (A_{11}-\lambda I_n) & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ K_1 x + K_2 y \end{bmatrix} = 0. \tag{23}$$

Set

$$K_1 x + K_2 y = v \tag{24}$$

and define

$$S^M_{\lambda_i} = \begin{bmatrix} (A_{11}-\lambda_i I_n) & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{bmatrix}. \tag{25}$$

Hence $[x' \, y' \, v']'$ belongs to the null space of $S^M_{\lambda_i}$.

We assume an arbitrary set of eigenvalues $\{\lambda_1 \ldots \lambda_n\}$ subject to the condition that unassignable eigenvalues are included in this set. The eigenvectors assumed as before belong to Ker $[S^M_{\lambda_i}]$ $i=1, 2, \ldots, n$ and they are further chosen to be linearly independent.

We have

$$K_1 [x_1 \, x_2 \ldots x_n] + K_2 [y_1 \, y_2 \ldots y_n] = [v_1 \, v_2 \ldots v_n] \tag{26}$$

or

$$K_1 X + K_2 Y = V.$$

Since K_2 is already determined from (21),

$$K_1 X = V - K_2 Y. \tag{27}$$

Hence

$$K_1=[V-K_2\,Y]\,X^{-1} \tag{28}$$

where the inverse of X is assured by choosing K_2 such that $(A_{22}+B_2\,K_2)$ is Hurwitz.

During the final stages of the response, the model appropriate for the system is $\bar{S}_S$. We have

$$\bar{S}_S:\begin{bmatrix}0\\0\\\mu\dot{z}(t)\end{bmatrix}=\begin{bmatrix}A_{11}+B_1\,K_1 & A_{12}+B_1\,K_2 & A_{13}+B_1\,K_3\\A_{21}+B_2\,K_1 & A_{22}+B_2\,K_2 & A_{23}+B_2\,K_3\\A_{31}+B_3\,K_1 & A_{32}+B_3\,K_2 & A_{33}+B_3\,K_3\end{bmatrix}\begin{bmatrix}x(t)\\y(t)\\z(t)\end{bmatrix}. \tag{29}$$

Let $\begin{bmatrix}x\\y\\z\end{bmatrix}e^{\lambda t}$ be a solution of (29) with $\mu=1$.

Hence

$$\begin{bmatrix}0\\0\\z\end{bmatrix}\lambda e^{\lambda t}=\begin{bmatrix}A_{11}+B_1\,K_1 & A_{12}+B_1\,K_2 & A_{13}+B_1\,K_3\\A_{21}+B_2\,K_1 & A_{22}+B_2\,K_2 & A_{23}+B_2\,K_3\\A_{31}+B_3\,K_1 & A_{32}+B_3\,K_2 & A_{33}+B_3\,K_3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}e^{\lambda t}$$

i.e.

$$\begin{bmatrix}A_{11} & A_{12} & A_{13} & B_1\\A_{21} & A_{22} & A_{23} & B_2\\A_{31} & A_{32} & A_{33}-\lambda I_q & B_3\end{bmatrix}\begin{bmatrix}x\\y\\z\\K_1\,x+K_2\,y+K_3\,z\end{bmatrix}=0. \tag{30}$$

As before set

$$K_1\,x+K_2\,y+K_3\,z=r$$

and define

$$S^S_{\lambda_i}=\begin{bmatrix}A_{11} & A_{12} & A_{13} & B_1\\A_{21} & A_{22} & A_{23} & B_2\\A_{31} & A_{32} & A_{33}-\lambda_i\,I_q & B_3\end{bmatrix} \tag{31}$$

Clearly $[x'\,y'\,z'\,r']'$ belongs to the null space of $S^S_{\lambda_i}$. There are q eigenvalues associated with (29). The choice of these eigenvalues are not critical (except that they should be stable) as in any case they migrate towards the origin for large μ in the composite system. We assume an arbitrary set of eigenvalues $\{\lambda_1, \ldots, \lambda_q\}$ subject to the condition that unassignable eigenvalues are included in this set. The corresponding eigenvectors may be arbitrarily chosen from Ker $[S^S_{\lambda_i}]$ $i=1, \ldots, q$ with the proviso that they are linearly independent. We have,

$$K_1\,[x_1\,x_2\,\ldots\,x_q]+K_2\,[y_1\,y_2\,\ldots\,y_q]+K_3\,[z_1\,z_2\,\ldots\,z_q]=[r_1\,r_2\,\ldots\,r_q]$$

$$\text{i.e. } K_1\,X+K_2\,Y+K_3\,Z=R. \tag{32}$$

Since K_1 and K_2 are already known

$$K_3=[R-K_1\,X-K_2\,Y]\,Z^{-1} \tag{33}$$

where inverse of Z is assured if the matrix

$$\begin{bmatrix} A_{11}+B_1 K_1 & A_{12}+B_1 K_2 \\ A_{21}+B_2 K_1 & A_{22}+B_2 K_2 \end{bmatrix} \quad \text{is nonsingular}.$$

Thus the three components of the feedback matrix K_1, K_2 and K_3 are determined step by step with minimum computational effort.

5. Illustrative example

The above results can be illustrated by considering the state feedback control of the multivariable linear system governed by respective state, output and control lew equations

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1/\varepsilon & 0 & 1/\varepsilon \\ 1/\mu & -1/\mu & 1/\mu \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1/\varepsilon \\ 1/\mu & 0 \end{bmatrix} u(t) \tag{34}$$

$$h(t) = \begin{bmatrix} 3 & -1 & -5 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \tag{35}$$

and

$$u(t) = [K_1 \; K_2 \; K_3] \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}. \tag{36}$$

It is required to determine feedback matrices K_1, K_2 and K_3 to simultaneously assign the eigenvalues of the medium, fast and slow subsystems as

$$\begin{aligned} \lambda_M &= -1 \\ \lambda_F &= -1 \\ \lambda_S &= -3 \end{aligned} \tag{37}$$

and their corresponding eigenvectors. Further it is required that the slow mode associated with slow system be made unobservable in the output.

We assign $\lambda_F = -1$ and choose the corresponding eigenvector from Ker $[S_\lambda^F]$ as follows:

For the problem, Ker $[S_\lambda^F] = \text{sp}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$. Since $\begin{bmatrix} y \\ K_2 y \end{bmatrix} \in$ Ker $[S_\lambda^F]$ according to equation (17) we choose

$$\left[\frac{y}{K_2 y}\right] = \left[\frac{1}{\begin{matrix} 1 \\ -1 \end{matrix}}\right]. \tag{38}$$

Therefore

$$K_2=\begin{bmatrix}1\\-1\end{bmatrix}. \tag{39}$$

We assign $\lambda_M=-1$ and choose the corresponding eigenvector from Ker $[S_\lambda^M]$ as follows:

For the problem, Ker $[S_\lambda^M]=\mathrm{sp}\left\{\begin{bmatrix}1\\1\\-4\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\-2\\-1\end{bmatrix}\right\}$.

Since $\begin{bmatrix}x\\y\\K_1\,x+K_2\,y\end{bmatrix}\in$ Ker $[S_\lambda^M]$ according to equation (23), we choose

$$\begin{bmatrix}x\\y\\\hline K_1\,x+K_2\,y\end{bmatrix}=\begin{bmatrix}1\\1\\\hline -4\\-1\end{bmatrix} \tag{40}$$

(i.e.)

$$K_1\,x+K_2\,y=\begin{bmatrix}-4\\-1\end{bmatrix}.$$

Therefore

$$K_1=\begin{bmatrix}-5\\0\end{bmatrix}. \tag{41}$$

We assign $\lambda_S=-3$ and choose the corresponding eigenvector from Ker $[S_\lambda^S]$ as follows:

For the problem

$$\text{Ker }[S_\lambda^S]=\mathrm{sp}\left\{\begin{bmatrix}1\\0\\-1/5\\-6/5\\-4/5\end{bmatrix}, \begin{bmatrix}1\\1\\2/5\\-13/5\\-7/5\end{bmatrix}\right\}.$$

Since $\begin{bmatrix}x\\y\\z\\K_1\,x+K_2\,y+K_3\,z\end{bmatrix}\in$ Ker $[S_\lambda^S]$ according to equation (30), we choose

$$\begin{bmatrix}x\\y\\z\\\hline K_1\,x+K_2\,y+K_3\,z\end{bmatrix}=\begin{bmatrix}1\\1\\2/5\\\hline -13/5\\-7/5\end{bmatrix} \tag{42}$$

(i.e.)

$$K_1\, x + K_2\, y + K_3\, z = \begin{bmatrix} -13/5 \\ -7/5 \end{bmatrix}.$$

Using (39) and (41) we get

$$K_3 = \begin{bmatrix} 7/2 \\ -1 \end{bmatrix}. \tag{43}$$

Further the slow mode

$$[x'\, y'\, z']' \in \operatorname{Ker} \begin{bmatrix} 3 & -1 & -5 \\ 1 & -1 & 0 \end{bmatrix} \tag{44}$$

and hence it becomes unobservable in the output.

The state feedback matrix becomes

$$K = [K_1\; K_2\; K_3] = \begin{bmatrix} -5 & 1 & 7/2 \\ 0 & -1 & -1 \end{bmatrix}. \tag{45}$$

The state equations of the closed loop system governed by equations (34), (36) and (45) thus become

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} -4 & 3 & 5/2 \\ 1/\varepsilon & -1/\varepsilon & 0 \\ -3/\mu & 0 & (9/2)/\mu \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \tag{46}$$

with

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 2/5 \end{bmatrix}$$

as the medium, fast and slow eigenvector respectively. The corresponding eigenvalues are $\lambda_M = -1$, $\lambda_F = -1/\varepsilon$ and $\lambda_S = -3/\mu$ when $\varepsilon \to 0$ and $\mu \to \infty$.

The above assignment holds for small values of ε and $1/\mu$. This is demonstrated by taking $\varepsilon = 10^{-3}$ and $\mu = 10^3$. Direct calculation for eigenvalues and eigenvectors of the closed loop system (46) gives the following results:

The fast eigenvalue of the composite system $\lambda_F' = -1003.0027$, is noted to be nearly equal to $\lambda_F/\varepsilon = -1/10^{-3} = -1000$. Similarly fast eigenvector of the composite system, given by

$$\begin{bmatrix} -0.0030 \\ 1.0000 \\ -0.0001 \end{bmatrix}$$

is almost identical in direction to the assigned eigenvector of $[0.0\ 1.0\ 0.0]'$. The medium eigenvalue of the composite system, $\lambda_M = -0.989483$, is noted to be nearly equal to the assigned value of -1.0000. The corresponding eigenvector of the composite system, given by

$$\begin{bmatrix} 0.999 \\ 1.000 \\ 0.003 \end{bmatrix}$$

is almost identical in direction to the assigned eigenvector of [1.0 1.0 0.0]'. The slow eigenvalue of the composite system, $\lambda_S' = -0.00335$, is noted to be nearly equal $\lambda_S/\mu = -3/10^3 = -0.003$. Similarly slow eigenvector, given by

$$\begin{bmatrix} 1.0000 \\ 1.0000 \\ 0.3986 \end{bmatrix}$$

is almost identical in direction to the assigned eigenvector of [1.0 1.0 0.4]'.

6. Conclusion

There are many engineering situations where a physical grasp of the problem enables one to group together system variables according to their temporal behaviour as fast, medium and slow modes. For such structured problems, we suggest a methodology for individually assigning eigenvalues and eigenvectors corresponding to the three different modes alluded to earlier. For small values of ε and large values of μ it turns out that the composite system $S_{\varepsilon\mu}$ under control laws derived for smaller subsystems possesses eigenvalues and eigenvectors which are almost identical to those for the smaller subsystems. Such a fortuitous property leads to considerable computational economy. There is also a conceptual simplicity in visualising the system transition from fast through medium to slow modes, with appropriate reduced order models representing the process at each stage. Further implications of this approach in the context of the control of large scale system are now being studied.

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Received, September 1983.

Rozmieszczenie wartości własnych systemów wielowymiarowych ze składowymi szybko-, średnio- i wolnozmiennymi

W pracy rozważane jest rozmieszczenie wartości własnych i wektorów własnych poprzez sprzężenie zwrotne zmiennych stanu dla przypadku wielkich systemów wielowymiarowych. Zakłada się, że analizowane systemy posiadają wyraźną strukturę wynikającą z bezpośredniego podziału zmiennych stanu na szybko-, średnio- i wolnozmienne. Rozmieszczenie wartości i wektorów własnych jest dokonywane osobno dla poszczególnycb mniejszych podsystemów. Pokazano, że rozmieszczenie takie w przybliżeniu przenosi się na całość złożonego systemu przy dość słabych warunkach. Podkreślić należy obliczeniową i metodyczną prostotę proponowanej procedury modyfikacji dynamiki systemu.

Распределение собственных значений многомерных систем с быстро, средне и медленно изменяющимися составляющими

В работе рассматривается распределенние собственных значений и векторов посредством обратной связи переменных состояния для случая больших многомерных систем. Предполагается, что анализируемые системы обладают явной структурой, вытекающей из непосредственного разделения переменных состояния на быстро, средне и медленно изменяющиеся. Распределение собственных значений и векторов производится раздельно для отдельных меньших подсистем. Показано, что такое распределение можно приблизительно перенести на всю сложную систему при довольно слабых условиях. Следует подчеркнуть вычислительную, методическую простоту предлагаемой процедуры модификации динамики системы.