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Existence of the solution for the oxygen diffusion consumption problem in a cylindrical domain *)

by

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The paper is concerned with a free boundary problem arising from the diffusion of oxygen in absorbing tissues. Assuming cylindrical symmetry of the problem, the local in time existence of a solution is proved by using a fixed point argument. The global existence is shown as well by continuation of the local solution. The paper extends known results concerning the case of plane symmetry.

1. Introduction

A kind of Cauchy type parabolic free boundary problems arising from the diffusion of oxygen in insulated living tissues, which simultaneously consume the oxygen has been considered in several papers (see e.g. [1], [2], [3]) mostly referring to the case of plane symmetry. A more realistic model of the diffusion consumption of oxygen in living tissues surrounding a blood vessel deals with cylindrical domains. Extension the results existing in the literature to the case of cylindrical symmetry is not trivial. This is the principal aim of the present paper.

We state our problem as [1] does: first, suppose some fixed concentration of oxygen u_0 is allowed to diffuse through a blood vessel into living tissues. A diffusion--consumption process will continue until a steady state is reached in which the oxygen does not penetrate further into the tissues. The supply of oxygen is then cut off and the surface is sealed. The living tissues go on consuming oxygen and as a consequence the surface bounding the oxygen containing region recedes towards the sealed surface. This process can be given the following mathematical scheme, using nondimensional variables

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$$L(u)=u_{rr}+\frac{u_r}{r}-u_t=1, \quad r_0<r<s(t), \quad 0<t<T; \tag{1.1}$$

$$u(r,0)=f(r), \quad r_0<r<r_1; \tag{1.2}$$

$$u_r(r_0,t)=0, \quad 0<t<T; \tag{1.3}$$

$$u\big(s(t),t\big)=0, \quad 0<t<T; \tag{1.4}$$

$$u_r\big(s(t),t\big)=0, \quad 0<t<T; \tag{1.5}$$

$$s(0)=r_1. \tag{1.6}$$

Here $f(r)$ represents the steady state concentration distribution of oxygen which solves:

$$f''(r)+\frac{f'(r)}{r}=1, \quad r_0<r<r_1; \tag{1.7}$$

$$f(r_0)=u_0, \tag{1.8}$$

$$f(r_1)=0, \tag{1.9}$$

$$f'(r_1)=0. \tag{1.10}$$

It is easy to find that

$$f(r)=\frac{r^2-r_1^2}{4}-\frac{r_1^2}{2}\ln\frac{r}{r_1} \tag{1.11}$$

where r_1 solves

$\frac{r_0^2-r_1^2}{4}-\frac{r_1^2}{2}\ln\frac{r_0}{r_1}=u_0$ (u_0, r_0 are given constants whose physical meaning is clear). (1.12)

By simple calculation, we can find a unique root $r_1>r_0$ for (1.12) and thus $f(r)$ is well defined.

In section 3, using the same approach as [3], based on a fixed point argument, we prove a local existence theorem for (1.1)–(1.6), which remains valid even for more general initial data. Before applying this approach, we establish in section 2 the existence of the solution to a Stefan type free boundary problem in a cylindrical domain. We give also some continuous dependence results to be used in section 3.

In section 4, using the results of [8], we investigate the continuation of the local solution to (1.1)—(1.6), and prove that the solution can be extended until all oxygen is absorbed.

2. Some results for a Stefan type free boundary problem in a cylindrical domain

We consider the existence property for a Stefan type problem in preparation for next section. But the results obtained here have their own interest.

LEMMA 2.1. *Suppose* $g(t)$ *is continuous in* $[0, T]$, $u_0(r)$ *is continuous in* $[r_0, r]$, $|u_0(r)| \leqslant M(r_1 - r)$, *where* M *is a constant. Then there exists a triple* $(u(r, t), s(t), T)$ *such that*

a) $T>0$,

b) $u(r, t)$ *is continuous in* $\bar{D}_T \setminus \{r_0, 0\}$ *and bounded in* $\bar{D}_T$;
$u_r(r, t)$ *is continuous for* $r_0 < r \leqslant s(t)$, $0<t<T$;
$u_{rr}(r, t)$, $u_t(r, t)$ *are continuous in* $D_T = \{r_0 < r < s(t),\ 0<t<T\}$.

c) $s(t)$ *is continuously differentiable in* $(0, T)$ *and continuous in* $[0, T]$, $s(0)=r_1$, $s(t)>r_0$.

d)

$$u_{rr} + \frac{1}{r} u_r - u_t = 0, \quad \text{in } D_T; \tag{2.1}$$

$$u(r, 0) = u_0(r), \quad r_0 \leqslant r \leqslant r_1; \tag{2.2}$$

$$u(r_0, t) = g(t), \quad 0<t<T; \tag{2.3}$$

$$u(s(t), t) = 0, \quad 0<t<T; \tag{2.4}$$

$$u_r(s(t), t) = -s'(t), \quad 0<t<T. \tag{2.5}$$

Proof. Take the Banach space

$$B(A, T_0) = \{\sigma(t) \in \text{Lip}\,[0, T_0];\ |\sigma(t_1) - \sigma(t_2)| \leqslant A\,|t_1 - t_2|,\ \sigma(0) = r_1\}$$

where $T_0>0$ and $A>0$ are to be determined.

Suppose T_0 is small enough, such that

$$T_0 \leqslant \frac{r_1 - r_0}{2A}, \tag{2.6}$$

For any $\sigma(t) \in B(A, T_0)$, we can solve an initial-boundary value problem

$$L(u) \equiv u_{rr} + \frac{1}{r} u_r - u_t = 0 \quad \text{in } D^{\sigma}_{T_0} = \{(r, t);\ r_0 < r < \sigma(t),\ 0<t<T_0\} \tag{2.7}$$

$$u(r_0, t) = g(t), \quad 0 \leqslant t \leqslant T_0; \tag{2.8}$$

$$u(r, 0) = u_0(r), \quad r_0 \leqslant r \leqslant r_1; \tag{2.9}$$

$$u(\sigma(t), t) = 0, \quad 0 \leqslant t \leqslant T_0. \tag{2.10}$$

In order to estimate $u_r(\sigma(t), t)$, we introduce the function $v(r, t)$ as the solution of

$$v_{rr} - v_t = 0, \quad \text{in } D^{\sigma}_{T_0}; \tag{2.11}$$

$$v(r_0, t) = H(r_1 - r_0), \quad 0<t<T_0; \tag{2.12}$$

$$v(r, 0) = H(r_1 - r), \quad r_0 < r < r_1; \tag{2.13}$$

$$v(\sigma(t), t) = 0, \quad 0<t<T_0 \tag{2.14}$$

where $H = \max\left\{\frac{\|g\|_{C[0, T_0]}}{f_1 - f_0}, M\right\}$.

It is easy to obtain, by the maximum principle applied in $D^{\sigma}_{t_0}$ that $v_r(r,t)\leqslant 0$. Thus, $L(u-v)=-\frac{1}{r}v_r(r,t)\geqslant 0$ and $u\leqslant v$ on the parabolic boundary of $D^{\sigma}_{T_0}$. Using the maximum principle once more, we deduce

$$u(r,t)\leqslant v(r,t), \quad (r,t)\in D^{\sigma}_{T_0}.$$

Similar arguments used for $-u(r,t)$ yield

$$-u(r,t)\leqslant v(r,t), \quad (r,t)\in D^{\sigma}_{T_0}.$$

These results imply that

$$u_r(\sigma(t),t)| \leqslant |v_r(\sigma(t),t)|, \tag{2.15}$$

because of (2.10) and (2.14).

From Lemma 1 in [4], we obtain

$$|v_r(\sigma(t),t)|\leqslant 2\left\{1+\left[A+\frac{2}{e}((r_1-r_0)-AT_0)^{-1}\right]\right\}\times$$
$$\times\left(\frac{T_0}{\pi}\right)^{1/2} H\exp\left\{\left[\frac{A}{2}+\frac{1}{e}((r_1-r_0)-AT_0)^{-1}\right]^2 T_0\right\}.$$

We see when T_0 tends to zero, the right-hand side of this inequality tends to $2H$. Thus, there exists a positive $\delta<1$ which depends only on A, r_0, r_1 such that:

$$|v_r(\sigma(t),t)|\leqslant 3H, \quad \text{if } T_0<\delta<1.$$

But $H=\max\left\{\frac{\|g\|_{c[0,T_0]}}{r_1-r_0}\right\}\leqslant\max\left\{\frac{\|g\|_{c[0,1]}}{r_1-r_0}, M\right\}$ so if we choose A large enough so that

$$A\geqslant\max\left\{\frac{9\|g\|_{c\,0,T]}}{r_1-r_0}, 9M\right\}$$

we finally have

$$|u_r(\sigma(t),t)|\leqslant\frac{A}{3}, \quad \text{provided } T_0<\delta. \tag{2.16}$$

Now we define an operator $\mathscr{J}$ on $B(A,T_0)$ such that:

$$[\mathscr{J}(\sigma)](t)=r_0+\left[(r_1-r_0)^2-2\int_0^t(\sigma(\tau)-r_0)\,u_r(\sigma(\tau),\tau)\,d\tau\right]^{1/2} \tag{2.17}$$

where $u(r,t)$ is the solution of (2.7)—(2.10). Because of (2.16), when T_0 is sufficiently small, (2.17) is well defined.

Setting $\mathscr{U}=u$, $\mathscr{V}=r-r_0$ in the Green's formula

$$\iint_{D^{\sigma}_{t}}(\mathscr{V}L\mathscr{U}-\mathscr{U}L^*\mathscr{V})\,dr d\tau=\int_{\partial D^{\sigma}_{t}}\mathscr{U}\mathscr{V}dr+\left(\mathscr{V}\mathscr{U}_r-\mathscr{U}\mathscr{V}_r+\frac{\mathscr{U}\mathscr{V}}{r}\right)d\tau, \tag{2.18}$$

we can obtain

$$-\int_0^t (\sigma(\tau)-r_0)\, u_r(\sigma(\tau),\tau)\, d\tau = -\int_{r_0}^{\sigma(t)} (r-r_0)\, u(r,t)\, dr + \int_0^t g(\tau)\, d\tau +$$

$$+\int_{r_0}^{r_1} (r-r_0)\, u_0(r)\, dr - r_0 \iint_{D_t^\sigma} \frac{u(r,\tau)}{r^2}\, dr\, d\tau .$$

Thus:

$$\mathscr{J}(\sigma)(t) = r_0 + \Big[(r_1-r_0)^2 - 2\int_{r_0}^{\sigma(t)} (r-r_0)\, u(r,t)\, dr + 2\int_0^t g(\tau)\, d\tau +$$

$$+2\int_{r_0}^{r_1} (r-r_0)\, u_0(r)\, dr - 2r_0 \iint_{D_t^\sigma} \frac{u(r,\tau)}{r^2}\, dr\, d\tau\Big]^{1/2}. \qquad (2.19)$$

We derive a few properties of $\mathscr{J}$:

(1) *$\mathscr{J}$ maps $B(A, T_0)$ into itself.*

Denote $h(t)=\mathscr{J}(\sigma(t))$, we have from (2.17)

$$h(0)=r_1 .$$

When T_0 is small enough, from (2.19) and (2.16) we also have

$$h(t) \geqslant \frac{r_1+r_0}{2}. \qquad (2.20)$$

We get from (2.19), after differentiating both sides,

$$h'(t) = \frac{-1}{h(t)-r_0} \cdot (\sigma(t)-r_0) \cdot u_r(\sigma(t), t),$$

then from (2.6), (2.16) and (2.20)

$$|h'(t)| \leqslant \frac{2}{r_1-r_0} \cdot \frac{3(r_1-r_0)}{2} \cdot \frac{A}{3} = A. \qquad (2.21)$$

Thus $h(t) \in B(A, T_0)$.

(2) *$\mathscr{J}$ is contractive.*

Suppose $\sigma_1(t), \sigma_2(t) \in B(A, T_0)$, the corresponding solutions of (2.7)—(2.10) are $u_1(r,t)$, $u_2(r,t)$. From (2.17), (2.20) and (2.21), it follows that

$$(r_1-r_0)\, |\mathscr{J}_{\sigma(t)}(\sigma_1) - \mathscr{J}(\sigma_2)| \leqslant |\mathscr{J}(\sigma_1)+\mathscr{J}(\sigma_2)-2r_0| \cdot |\mathscr{J}(\sigma_1)-\mathscr{J}(\sigma_2)| \leqslant$$

$$\leqslant 2\left|\int_{r_0}^{\alpha(t)} (r-r_0)\,[u_1(r,t)-u_2(r,t)]\, dr\right| + 2\left|\int_{\alpha(t)}^{\beta(t)} (r-r_0)\, u_j(r,t)\, dr\right| +$$

$$+2r_0\left|\iint_{A_t} \frac{u_1(r,\tau)-u_2(r,\tau)}{r^2}\, dr\, d\tau\right| + 2r_0\left|\iint_{B_t} \frac{u_j(r,\tau)}{r^2}\, dr\, d\tau\right|. \qquad (2.22)$$

Here we use the following notations

$$\alpha(t)=\min\big(\sigma_1(t), \sigma_2(t)\big),$$

$$\beta(t)=\max\big(\sigma_1(t), \sigma_2(t)\big),$$

$$j=k, \quad \text{if } \beta(t)=\sigma_k(t), \quad k=1,2,$$

$$A_t=\{(r,\tau);\ r_0<r<\alpha(t),\ 0<\tau<t\},$$

$$B_t=\{(r,\tau);\ \alpha(\tau)<r<\beta(\tau),\ 0<\tau<t\}.$$

We solve the problem:

$$\tilde{L}(w)=0, \quad -\infty<r<+\infty,\ 0<t<T_0;$$

$$w(r,0)=0, \quad -\infty<r<+\infty;$$

$$w\big(\alpha(t), t\big)=\frac{A}{3}\|\sigma_1-\sigma_2\|_{C[0,t]}, \quad 0<t<T_0.$$

where $\tilde{L}$ denotes a parabolic operator with smooth coefficients which is defined in the strip $[0, T_0]\times(-\infty, +\infty)$ and

$$\tilde{L}\equiv L, \quad \text{for } r=r_0.$$

The solution of this problem can be represented in the form

$$w(r,t)=\int_0^t \lambda_r(\tau)\,\Gamma\big(r, t, \alpha(\tau), \tau\big)\,d\tau \tag{2.23}$$

where Γ is the fundamental solution for the operator $\tilde{L}$ and

$$\lambda_r(t)=\begin{cases}\lambda^+(t), & \text{if } r<\alpha(t);\\ \lambda^-(t), & \text{if } r>\alpha(t).\end{cases}$$

while $\lambda^\pm(t)$ satisfies

$$w\big(\alpha(t)_{\pm 0}, t\big)=\mp\frac{1}{2}\lambda^\mp(t)+\int_0^t \lambda^\mp(\tau)\,\Gamma_\rho\big(\alpha(t), t, \alpha(\tau), \tau\big)\,d\tau.$$

An estimate for Γ_ρ (see [5], p. 406) and a useful lemma (see [6], Lemma 7) allow us to get

$$\lambda_1(t)\leqslant C\|\sigma_1-\sigma_2\|_{C[0,t]}, \tag{2.24}$$

where C depends on only A, T_0, r_0.

From (2.23), (2.24), it can be found:

$$\int_{r_0}^{\alpha(t)}(r-r_0)\,w(r,t)\,dr\leqslant C_1\int_0^t\frac{\|\sigma_1-\sigma_2\|_{C[0,\tau]}}{\sqrt{t-\tau}}\,d\tau=2C_1\sqrt{t}\,\|\sigma_1-\sigma_2\|_{C[0,t]},$$

$$\iint_{A_t}\frac{w(r,\tau)}{r}\,dr\,d\tau\leqslant C_2\int_0^t\frac{\|\sigma_1-\sigma_2\|_{C[0,\tau]}}{\sqrt{t-\tau}}\,d\tau=2C_2\sqrt{t}\,\|\sigma_1-\sigma_2\|_{C[0,t]},$$

$$\int_{\alpha(t)}^{\beta(t)} (r-r_0)\, w(r,t)\, dr = C_3 \int_0^t \frac{\|\sigma_1-\sigma_2\|_{C[0,\tau]}}{\sqrt{t-\tau}}\, d\tau = 2C_3 \sqrt{t}\, \|\sigma_1-\sigma_2\|_{C[0,t]},$$

$$\iint_{B_t} \frac{w(r,\tau)}{r}\, dr\, d\tau = C_4 \int_0^t \frac{\|\sigma_1-\sigma_2\|_{C[0,t]}}{\sqrt{t-\tau}}\, d\tau = 2C_4 \sqrt{t}\, \|\sigma_1-\sigma_2\|_{C[0,t]},$$

where C_i $(i=1, 2, 3, 4)$ depend only on r_0, r_1, A. But from the maximum principle, we have

$$|u_1(r,t)-u_2(r,t)| \leqslant w(r,t), \qquad (r,t) \in A_t$$

and

$$|u_j(r,t)| \leqslant w(r,t), \qquad (r,t) < B_t.$$

So, we assure that the right-hand side of (2.22) can be estimated by $\|\sigma_1-\sigma_2\|_{C[0,T_0]}$ and finally obtain:

$$\|\mathscr{J}(\sigma_1(t)) - \mathscr{J}(\sigma_2(t))\|_{C[0,T_0]} \leqslant K \|\sigma_1(t)-\sigma_2(t)\|_{C[0,T_0]}, \tag{2.25}$$

where $K<1$, whenever T_0 is small enough, say, $T_0 \leqslant \delta_0$, with δ_0 depending on r_0, r_1, A only. (2.25) means that $\mathscr{J}$ is contractive.

(3) *$\mathscr{J}$ has a fixed point $s(t)$ in $B(A, T_0)$.*

Owing to the properties (1), (2) of $\mathscr{J}$, applying the contractive mapping theorem, we conclude that there exists a unique $\sigma(t) \in B(A, T_0)$, such that

$$\sigma(t) = r_0 + \left[(r_1-r_0)^2 - 2\int_0^t (\sigma(\tau)-r_0)\, u_r(\sigma(\tau),\tau)\, d\tau\right]^{1/2},$$

or

$$\sigma'(t) = -u_r(\sigma(t), t), \qquad t \in (0, T_0),$$

where $u(r,t)$ solves (2.7)—(2.10) and the proof is complete. ■

LEMMA 2.2. *Suppose*

1) $g_1(t), g_2(t) \in C[0, T_0]$
2) $(u_1(r,t), s_1(t), T_0)$ *and* $(u_2(r,t), s_2(t), T_0)$ *solve* (2.1)—(2.5) *respectitæe to the boundary data* $g_1(t)$ *and* $g_2(t)$ *and the same initial data.*
3. $|s_i'(t)| \leqslant A$, $i=1, 2$; T_0 *is defined as in lemma* 2.1; $s_i(0)=r_1$, $i=1, 2$.

Then

$\|s_1(t)-s_2(t)\|_{C[0,T_0]} \leqslant M \|g_1(t)-g_2(t)\|_{C[0,T_0]}$, *where M is a constant which satisfies*

$$M \to 0 \ \text{as}\ T_0 \to 0.$$

Proof. As in (2.22), we now have

$$|s_1(t)-s_2(t)| \leqslant \frac{2}{(r_1-r_0)} \left\{ \int_{r_0}^{\alpha(t)} (r-r_0)\, |u_1(r,t)-u_2(r,t)|\, dr + \right.$$

$$+\int_{\alpha(t)}^{\beta(t)} (r-r_0)\,|u_j(r,t)|\,dr+\int_0^t |g_1(\tau)-g_2(\tau)|\,d\tau+$$

$$+r_0\left|\iint_{A_t} \frac{(u_1(r,\tau)-u_2(r,\tau))}{r^2}\,dr\,d\tau\right|+r_0\left|\iint_{B_t} \frac{u_j(r,\tau)}{r^2}\,dr\,d\tau\right|\Bigg\}.$$

The similar discussion as in the proof of (2.25) in Lemma 2.1 gives

$$\|s_1(t)-s_2(t)\|_{C[0,T_0]} \leqslant K\,\|s_1(t)-s_2(t)\|_{C[0,T_0]}+\|g_1(t)-g_2(t)\|_{C[0,T_0]}\,T_0,$$

where $K<1$, thus

$$\|s_1(t)-s_2(t)\|_{C[0,T_0]} \leqslant M\,\|g_1(t)-g_2(t)\|_{C[0,T_0]},$$

where $M=T_0/(1-K)$ depends on A and T_0. ∎

LEMMA 2.3. *Suppose $u(r,t)$ and $v(r,t)$ satisfy*

$$L(u)=0, \quad (r,t)\in\{r_0<r<s(t),\ 0<t<T\};$$
$$u(r,0)=0, \quad r_0<r<r_1;$$
$$u_r(r_0,t)=0, \quad 0<t<T;$$
$$u(s(t),t)=\varphi(t), \quad 0<t<T,$$

and

$$L(v)=0, \quad (r,t)\in\{r_0<r<\sigma(t),\ 0<t<T\};$$
$$v(r,0)=0, \quad r_0<r<r_1;$$
$$v_r(r,t)=0, \quad 0<t<T;$$
$$v(\sigma(t),t)=\psi(t), \quad 0<t<T,$$

where $s(t)$, $\sigma(t)$, $\varphi(t)$, $\psi(t)$ satisfy the conditions:

$$s(t),\ \sigma(t),\ \varphi(t),\ \psi(t)\in C^1(0,T)\cap C[0,T],$$
$$\varphi(0)=\psi(0)=0, \quad s(0)=\sigma(0)=r_1,$$
$$|s'(t)|\leqslant A,\ |\sigma'(t)|\leqslant A, \quad \text{for} \quad t\in(0,T),$$
$$|\varphi'(t)|\leqslant F,\ |\psi'(t)|\leqslant F, \quad \text{for} \quad t\in(0,T),$$
$$s_0=\inf_{[0,T]} s(t)>r_0, \quad \sigma_0=\inf_{[0,T]} \sigma(t)>r_0.$$

Then

$$|u(r,t)-v(r,t)|\leqslant K_1\,|s(t)-\sigma(t)|+|\varphi(t)-\psi(t)|,$$

where K_1 depends on A, F, s_0, σ_0 and T, only.

Proof. Let us compare $u(r,t)$ with the functions

$$w^{\pm}(r,t)=\varphi(t)\pm C\,(s(t)-r)\,(2b+r-s(t)) \tag{2.26}$$

in the domain $E_T=\{(r,t);\ s(t)-b<r<s(t),\ 0<t<T\}$

where $b=\min\{s_0,\sigma_0,1/2A\}$, $C=\max\{F/2\,(1-Ab),\ 2FT/b^2\}$.

It can be verified that

$$w^+\big(s(t),t\big)=\varphi(t)=u\big(s(t),t\big),$$

$$w^+\big(s(t)-b,t\big)=\varphi(t)+Cb^2\geqslant -FT+2FT=FT\geqslant\max_{D_T^s}|u|,$$

$$w^+(r,0)=C(r_1-r)(2b+r-r_1)\geqslant 0=u(r,0),\quad -b+r_1\leqslant r\leqslant r_1;$$

$$L(w^+)=2C\left[-1-\frac{1}{r}(b+r-s(t))-s'(t)(b+r-s(t))\right]+$$
$$-\varphi'(t)\leqslant -2C(1-Ab)+F\leqslant 0,\quad \text{in } E_T.$$

From the maximum principle it follows that

$$u(r,t)\leqslant w^+(r,t),\quad \text{in } E_T. \tag{2.27}$$

Similarly, we have

$$u(r,t)\geqslant w^-(r,t),\quad \text{in } E_T. \tag{2.28}$$

Thus

$$|u(r,t)-\varphi(t)|\leqslant 2Cb(s(t)-r),\quad \text{for } (r,t)\in E_T.$$

moreover, for $r_0\leqslant r\leqslant s(t)-b$, $0\leqslant t\leqslant T$, we have

$$2Cb\big(s(t)-r\big)\geqslant Cb^2\geqslant 2FT\geqslant\max_{D_T^s}|u|+\max_{[0,T]}|\varphi|\geqslant|u(r,t)-\varphi(t)|.$$

It follows then that

$$|u(r,t)-\varphi(t)|\leqslant 2Cb\big(s(t)-r\big),\quad (r,t)\in D_T^s. \tag{2.29}$$

The same results can be obtained for $v(r,t)$ and $\psi(t)$:

$$|v(r,t)-\psi(t)|\leqslant 2Cb\big(\sigma(t)-r\big). \tag{2.30}$$

Now, let us apply the maximum principle to the functions

$$\pm(u(r,t)-v(r,t))\quad \text{in } A_T=\{r_0<r<\alpha(t),\ 0<t<T\},$$

where

$$\alpha(t)=\min\{s(t),\sigma(t)\}.$$

Using (2.29) and (2.30), we can conclude

$$|u(r,t)-v(r,t)|\leqslant K_1\,|s(t)-\sigma(t)|+|\varphi(t)-\psi(t)|,$$

where K_1 depends on A, F, s_0, σ_0 and F, only. ∎

3. Local existence theorem for (1.1)—(1.6)

Before discussing the existence, let us mention that uniqueness for (1.1)—(1.6) can be shown in the same way as in [3], Thm. 3.1.

In this section we establish

THEOREM 3.1. *There exists a triple* $[u(r,t), s(t), T\}$ *such that*

(a) $T>0$;

(b) $s(t)$ *is continuously differentiable in* $(0, T)$ *and continuous in* $[0, T]$;

(c) $u(r, t)$ *is continuous in* $\bar{D}_T=\{(r,t);\ r_0\leqslant r\leqslant s(t),\ 0\leqslant t\leqslant T\}$,
$u_r(r,t)$ *is continuous in* $r_0\leqslant r\leqslant s(t),\ 0<t<T$,
$u_{rr}(r,t),\ u_t(r,t)$ *are continuous in* $D_T=\{r_0<r<s(t),\ 0<t<T\}$;

(d) $\{u(r,t), s(t), T\}$ *solves* (1.1)—(1.6).

Proof. Consider a Stefan-type free boundary problem

$$L(z)=z_{rr}+\frac{1}{r}z_r-z_t=0, \quad \text{in } D_T^b=\{r_0<b<r<s(t),\ 0<t<T\}; \tag{3.1}$$

$$z(r,0)=-1, \quad 0<t<T; \tag{3.2}$$

$$z(b,t)=V(t), \quad 0<t<T; \tag{3.3}$$

$$z\big(s(t),t\big)=-1, \quad 0<t<T; \tag{3.4}$$

$$z_r\big(s(t),t\big)=-s'(t), \quad 0<t<T; \tag{3.5}$$

$$s(0`=r_1 \tag{3.6}$$

where

$$b=\frac{r_0+r_1}{2},$$

$$V(t)\in\mathscr{B}(T,X)=\{V\in C[0,T],\ \|V\|_{C[0,T]}\leqslant X\},$$

X and T are constants to be determined later.

Existence of the solution to (3.1)—(3.6) can be got for some small T, from Lemma 2.1 when we substitute $z=z_1-1$. Put $\mathscr{U}=z$, $\mathscr{V}=b-r$ in (2.18). By a similar calculation we can deduce that when

$$T\leqslant\min\left\{\frac{(r_1-b)^2-k^2b^2}{2X}, T_1\right\}, \quad \left(\text{where } k=\frac{r_1-r_0}{2r_1}\right), \tag{3.7}$$

we have

$$\min_{0\leqslant t\leqslant T}\{s(t)-b\}\geqslant\frac{(r_1-r_0)}{2r_1}b. \tag{3.8}$$

Then let us solve the following initial-boundary value problem with $r=s(t)$ given by (3.1)—(3.6):

$$L_1(v)=v_{rr}+\frac{1}{r}v_r-\frac{v}{r^2}-v_t=0, \quad \text{in } D_T=\{0<t<T,\ r_0<r<s(t)\}; \tag{3.9}$$

$$v(r,0)=-\frac{r}{2}-\frac{r_1^2}{2r}, \quad r_0<r<r_1; \tag{3.10}$$

$$v(r_0,t)=-r_0, \quad 0<t<T; \tag{3.11}$$

$$v_r(s(t),t)+\frac{v(s(t),t)}{s(t)}=-1, \quad 0<t<T. \tag{3.12}$$

Classical results ensure the existence of the function $v(r,t)$. Set

$$V_1(t)=v_r(b,t)+\frac{v(b,t)}{b}.$$

It is easy to see that $V_1(t)\in C[0,T]$. Furthermore, using the Schauder's estimate of the solution of (3.9)—(3.12) up to the boundary portion (see [5] chapt. 5 p. 437, Thm. 3.1) we can prove that there exists a constant $X>1$, depending on r_0, r_1 and b only, such that

$$\|V_1(t)\|_{C^\alpha[0,T]}\leqslant X; \quad \text{for some } 0<\alpha<1. \tag{3.13}$$

irrespective of the choice of boundary $r=s(t)$, or more precisely, the choice of $V(t)$. Then, we obtain

$$V_1(t)\in\mathscr{B}_1(T,X)=\{V\in\mathscr{B}(T,X)\cap C^\alpha[0,T], \|V\|_{C^\alpha[0,T]}\leqslant X\}\subset\mathscr{B}(T,X).$$

Now we define an operator $\mathfrak{C}$ on $\mathscr{B}(T,X)$ by

$$V_1(t)=\mathfrak{C}(V(t))\in\mathscr{B}_1(T,X)\subset\mathscr{B}(T,X).$$

According to the illustration above, we see that $\mathfrak{C}$ maps a closed, convex and compact subset of $\mathscr{B}(T,X)$ into itself. In particular,

$$\mathfrak{C}:\mathscr{B}_1(T,X)\to\mathscr{B}_1(T,X).$$

The operator $\mathfrak{C}$ is also contractive on $C[0,T]$, because of the following estimates resulting from Lemma 2.2:

$$\|s^*(t)-s^{**}(t)\|_{C[0,T]}\leqslant K_1\|V^*(t)-V^{**}(t)\|_{C[0,T]}, \ \lim_{t\to 0}K_1=0. \tag{3.14}$$

where we denote by $s^*(t)$, $s^{**}(t)$ (and later $V_1^*(t)$, $V_1^{**}(t)$) the respective functions obtained after replacing V by V^* and V^{**}.

Denote by $E(r,t)$ the solution of the problem

$$L_1(E)=E_{rr}+\frac{1}{r}E_r-\frac{1}{r}E-E_t=0, \quad -\infty<r<+\infty, 0<t<T;$$

$$E(r,0)=-\frac{r}{2}-\frac{r_1^2}{2r}, \quad -\infty<r<+\infty;$$

$$E(r_0,t)=-r_0, \quad 0<t<T,$$

and $v_0^*(r,t)=(v-E)_r(r,t)+\dfrac{v^*-E}{r}$ which solves:

$$L(v_0^*)=(v_0^*)_{rr}+\frac{1}{r}(v_0^*)_r-(v_0^*)_t=0, \quad r_0<r<s^*(t), \ 0<t<T;$$

$$v_0^*(r, 0)=0, \quad r_0<r<r_1;$$

$$(v_0^*)_r(r_0, t)=0, \quad 0<t<T;$$

$$v_0^*(s^*(t), t)=-1-\left(E_r+\frac{1}{r}E\right)(s^*(t), t), \quad 0<t<T.$$

$v_0^{**}(r, t)=(v^{**}-E)_r+\dfrac{v^{**}-E}{r}$ which solves:

$$L(v_0^{**})=0, \quad r_0<r<s^{**}(t),\ 0<t<T;$$

$$v_0^{**}(r, 0)=0, \quad r_0<r<r_1;$$

$$v_0^{**}(r_0, t)=0, \quad 0<t<T;$$

$$v_0^{**}(s^{**}(t), t)=-1-\left(E_r+\frac{E}{r}\right)(s^{**}(t), t), \quad 0<t<T.$$

Using Lemma 2.3, we have

$$\|V_1^*(t)-V_1^{**}(t)\|_{C[0,T]}\leqslant \bar{K}_1\,\|s^*(t)-s^{**}(t)\|_{C[0,T]}+$$
$$+\left\|\left(E_r+\frac{E}{r}\right)(s^*(t), t)-\left(E_r+\frac{E}{r}\right)(s^{**}(t), t)\right\|_{C[0,T]}\leqslant$$
$$\leqslant K_2\,\|s^*(t)-s^{**}(t)\|_{C[0,T]}, \tag{3.15}$$

where K_2 depends only on r_0, r_1, A, X.

Combining (3.14) and (3.15), we obtain

$$\|V_1^*(t)-V_1^{**}(t)\|_{C[0,T]}\leqslant K_1 K_2\,\|V^*(t)-V^{**}(t)\|_{C[0,T]}=$$
$$=K\,\|V^*(t)-V^{**}(t)\|_{C[0,T]},$$

where $0<K<1$, provided T is small enough. Thus $\mathfrak{C}$ is contractive. By the contractive mapping theorem, we obtain a function $V(t)\in\mathscr{B}(T, X)$ such that $\mathfrak{C}(V(t))\equiv$ $\equiv V_1(t)=V(t)$ or

$$z(b, t)=v_r(b, t)+\frac{v(b, t)}{b}, \quad 0<t<T. \tag{3.16}$$

Let us compare $z(r, t)$ with $v_r(r, t)+\dfrac{v(r, t)}{r}$ in the region D_T^b. From (3.4) and (3.12) we have

$$z(s(t), t)=\left(v_r+\frac{v}{r}\right)(s(t), t), \quad 0<t<T.$$

Moreover, it is easy to check that

$$z(r, 0)=\left(v_r+\frac{v}{r}\right)(r, 0), \quad r_0<r<r_1,$$

$$L(z)=L\left(v_r+\frac{v}{r}\right) \quad \text{in } D_T^b.$$

Thus, from the uniqueness for problem (3.1)—(3.4), we get

$$z(r,t)=v_r(r,t)+\frac{v(r,t)}{r}, \qquad (r,t)\in D_T^b.$$

Differentiating both sides w.r.t. r, we have

$$z_r(r,t)=\left(v_r+\frac{v}{r}\right)_r=v_t(r,t), \qquad (r,t)\in D_T^b.$$

By the continuity of the derivatives of the solution $z(r,t)$ and $v(r,t)$ up to $r=s(t)$, we have

$$v_t(s(t),t)=z_r(s(t),t)=-s'(t), \qquad 0<t<T. \tag{3.17}$$

Denote $w(t)=v(s(t),t)$ and differentiate both sides,

$$w'(t)=v_r(s(t),t)\,s'(t)+v_t(s(t),t);$$

from (3.12) and (3.17), we have

$$w'(t)=\left(-1-\frac{w(t)}{s(t)}\right)s'(t)-s'(t)$$

or

$$w(t)\,s(t)=-s^2(t)+C.$$

By taking $t=0$, we can determine $C=0$. Thus, we have

$$v(s(t),t)=w(t)=-s(t), \qquad 0<t<T. \tag{3.18}$$

Now, take

$$u(r,t)=-\int_r^{s(t)}[\xi+v(\xi,t)]\,d\xi,$$

From (3.9)—(3.12) and (3.18), we can easily find that $u(r,t)$ satisfies the problem (1.1)—(1.6). The local existence is proved completely. ∎

Remark. Theorem 3.1 is still valid when the initial function $u(r,0)=f(r)$ is not the so-called equilibrium distribution

$$f(r)=\frac{r_1^2-r_1^2}{2}-\frac{r_1^2}{2}\ln\frac{r}{r_1},$$

if we only suppose that $f(r)$ satisfies:

$$f(r)\in C^{3+\alpha}[r_0,r_1], \tag{3.19}$$

$$\left|f''(r)+\frac{f'(r)}{r}-1\right|\leqslant M\,|r_1-r|. \tag{3.20}$$

The main reason for this is inherent in Lemma 2.1. where we have not supposed

$$u_0(r_0)=g(0).$$

Of course, the conditions (3.2) and (3.10) must be changed accordingly.

4. Global existence theorem for (1.1)—(1.6)

In Theorem 3.1 above, we proved the existence of a local solution for (1.1)—(1.6). We want to show how global existence follows from the analysis performed in [8] for problem (2.1)—(2.5).

We need the following lemmas.

LEMMA 4.1. *Suppose* $\{u(r,t), s(t), T\}$ *solves the problem* (1.1)—(1.6), *then*

$$s'(t)<0, \text{ for } t\in(0,\varepsilon), \text{ where } \varepsilon \text{ is small enough.}$$

P r o o f. From the proof of Theorem 3.1, it is easy to see that we need only to prove $z(r,t)\leqslant -1$ where $z(r,t)$ solves (3.1)—(3.6), or need only to prove

$$v_r+\frac{v}{r}\leqslant -1, \tag{4.1}$$

where $v(r,t)$ solves the problem (3.9)—(3.12).

Consider the approximating problems:

$$(v_n)_{rr}+\frac{(v_n)_r}{r}-\frac{v_n}{r^2}-(v_n)_t=0, \quad (r,t)\in\{r_0<r<s(t),\ 0<t<\varepsilon\}; \tag{4.2}$$

$$v_n(r,0)=-\frac{r}{2}-\frac{r_1^2}{2r}, \quad r_0<r<r_1; \tag{4.3}$$

$$v_n(r_0,t)=g_n(t), \quad 0<t<\varepsilon; \tag{4.4}$$

$$\left((v_n)_r+\frac{v_n}{r}\right)(s(t),t)=-1, \quad 0<t<\varepsilon, \tag{4.5}$$

where

$$g_n(t)=\begin{cases} -r_0, & t>1/n; \\ \text{smooth function with:} & \\ g_n(0)=-\dfrac{r_0}{2}-\dfrac{r_1^2}{2r_0}, \quad g_n(1/n)=-r_0, & 0\leqslant t\leqslant 1/n, \\ g_n'(t)>0. & \end{cases}$$

Set $(v_n)_r+\dfrac{v_n}{r}=w_n$, $w_n(r,t)$ satisfies:

$$(w_n)_{rr}+\frac{(w_n)_r}{r}-(w_n)_t=0, \quad (r,t)\in\{r_0<r<s(t),\ 0<t<\varepsilon\};$$

$$w_n(r,0)=-1, \quad r_0<r<r_1;$$

$$w_n(s(t),t)=-1, \quad 0<t<\varepsilon;$$

$$w_n(r_0,t)=\left((v_n)_r+\frac{v_n}{r}\right)(r_0,t), \quad 0<t<\varepsilon.$$

v_n attains its maximum on the boundary $r=r_0$. Otherwise, $v_n(r,t)$ should attain its maximum at some point $(s(t),t)$ on the boundary $r=s(t)$, then from the maximum principle $(v_n)_r(s(t),t)\geqslant 0$. By (4.5), we have

$$v_n(s(t),t)\leqslant -s(t)<-r_0=v_n(r_0,t),\ (t>1/n),$$

which is a contradiction to our assumption.

Then, we can easily obtain

$$(v_n)_r(r_0,t)<0, \quad t>1/n. \tag{4.6}$$

$w_n(r,t)$ cannot attain its maximum on the boundary $r=r_0$, because when $t>1/n$, $w_n(r_0,t)=(v_n)_r(r_0,t)-1<-1$, and when $0\leqslant t\leqslant 1/n$, we have $(w_n)_r(r_0,t)=$ $=\left((v_n)_r+\frac{v_n}{r}\right)_r(r_0,t)=(v_n)_t(r_0,t)=g_n'(t)>0$. Thus we obtain:

$$w_n(r,t)=(v_n)_r+\frac{v_n}{r}\leqslant -1. \quad (r,t)\in\{r_0<r<s(t),\ 0<t<\varepsilon\}.$$

But it can be easily seen that $v_n(r,t)$ tends to $v(r,t)$ together with its first and second derivatives when n tends to $+\infty$, so we finally have (4.1). ■

Lemma 4.2. *Suppose $u(r,t)$ satisfies (1.1)—(1.6), then we have*

$$u_{rt}(r,\varepsilon)>0, \quad \textit{for all } \varepsilon\in(0,T) \tag{4.7}$$

P r o o f. Denote $v(r,t)=u_r(r,t)$, $v(r,t)$ should satisfy:

$$L_1(v)=v_{rr}+\frac{v_r}{r}-\frac{v}{r^2}-v_t=0, \quad r_0<r<s(t),\ 0<t<T; \tag{4.8}$$

$$v(r,0)=\frac{r}{2}-\frac{r_1^2}{2r}, \quad r_0<r<r_1; \tag{4.9}$$

$$v(r_0,t)=0, \quad 0<t<T; \tag{4.10}$$

$$v(s(t),t)=0, \quad 0<t<T. \tag{4.11}$$

Suppose $e(r,t)$ solves the following problem:

$$L_1(e)=0, \quad r>r_0,\ t>0; \tag{4.12}$$

$$e(r,0)=0, \quad r>r_0; \tag{4.13}$$

$$e(r_0,t)=\frac{r_1^2}{2r_0}-\frac{r_0}{2}, \quad t>0. \tag{4.14}$$

and $v_0(r,t)$ solves:

$$L_1(v_0)=0, \quad r_0<r<s(t),\ 0<t<T; \tag{4.15}$$

$$v_0(r,0)=\frac{r}{2}-\frac{r_1^2}{2r}, \quad r_0<r<r_1; \tag{4.16}$$

$$v_0(r_0, t) = \frac{r_0}{2} - \frac{r_1^2}{2r_0}, \quad 0 < t < T; \tag{4.17}$$

$$v_0(s(t), t) = -e(s(t), t), \quad 0 < t < T. \tag{4.18}$$

Then we have

$$v(r, t) = e(r, t) + v_0(r, t).$$

In order to deduce (4.7), we first show

$$\liminf_{(r,t)\to(r_0,0)} v_t(r, t) \geq 0. \tag{4.19}$$

To do this, we consider the approximating problems for (4.12)—(4.14):

$$L_1(e_n) = 0, \quad r > r_0, \; t > 0; \tag{4.20}$$

$$e_n(r, 0) = 0, \quad r > r_0; \tag{4.21}$$

$$e_n(r_0, t) = g_n(t), \quad t > 0, \tag{4.22}$$

where $g_n(t) = \frac{r_1^2}{2r_c} - \frac{r_2}{2}$ for $t \geq \frac{1}{n}$, $g_n(t)$ is continuous in $\left[0, \frac{1}{n}\right]$ and $g_n(t) \in C^\infty(0, +\infty)$, $g_n(0) = g_n'(0) = 0$, and $g_n'(t) \geq 0$ for $t \in (0, +\infty)$.

The function $e_n(r, t)$ can be expressed as:

$$e_n(r, t) = \int_0^t \varphi(\tau)\, \Gamma_\rho(r, t, r_0, \tau)\, d\tau,$$

where Γ is a fundamental solution of L_1 and $\varphi(t)$ is a suitable continuous function.

We have then

$$(e_n)_t(r, t) = \int_0^t \varphi(\tau)\, \Gamma_{\rho t}(r, t, r_0, \tau)\, d\tau + \varphi(t)\, \Gamma_\rho(r, t, r_0, t).$$

From [5], p. 406 (16.3), we have

$$|\Gamma_\rho(r, t, \rho, \tau)| \leq c(t-\tau)^{-1} \exp\left(-c_1 \frac{(r-r_0)^2}{t-\tau}\right) \to 0, \text{ when } r \to +\infty;$$

$$|\Gamma_{\rho t}(r, t, \rho, \tau)| \leq c(t-\tau)^{-2} \exp\left(-c_1 \frac{(r-r_0)^2}{t-\tau}\right) \to 0, \text{ when } r \to +\infty.$$

So, we obtain

$$(e_n)_t(r, t) \to 0, \text{ when } r \to +\infty. \tag{4.23}$$

Using the maximum principle in a bounded parabolic region and (4.23), we can deduce

$$(e_n)_t(r, t) \geq 0, \quad r > r_0, \; t > 0.$$

Moreover, Theorem 10.1 in [5] p. 204 gives a Hölder estimate for $e_n(r, t)$:

$$|e_n(r, t)|_{Q'}^{\alpha} \leq K, \tag{4.24}$$

where Q' is a closed bounded domain belonging to the quarter plane $\{r>r_0, t>0\}$ separated from the point $(r_0, 0)$ by a positive distance d. The constant K depends on r_0, r_1 and d only. Therefore we can choose a subsequence $\{e_{n_k}\}$ which converges uniformly in Q'. By taking into account an invading sequence of domains $\{Q'_n\}$, and using a diagonal process, we find a subsequence, which we denote by $\{e_n\}$ again, converging to $e(r, t)$ in the quarter plane $\{r>r_0, t>0\}$ and uniformly in every compact subdomain which does not include $(r_0, 0)$.

By a result concerning families of solutions to parabolic equations ([7] p. 90), we affirm that $\{(e_n)_t (r, t)\}$ converges to $e_t(r, t)$, which turns out to be nonnegative. Thus, we easily obtain (4.19). From the strong maximum principle, the result of Lemma 4.1 and (4.19), we finally deduce (4.7). ∎

THEOREM 4.3. *There exists* $T_0>0$, *such that the problem* (1.1)—(1.6) *has a solution* $\{u(r, t), s(t), T_0\}$ *referring to the region* $D_{T_0}=\{0<t<T_0,\ r_0<r<s(t)\}$ *where* $s(t)$ *satisfies furthermore*:

$$\lim_{t\to T_0} s(t)=r_0. \tag{4.25}$$

Proof. By Theorem 3.1, there exists a triple $\{u(r, t), s(t), T\}$ which solves (1.1)—(1.6). Consider the solution of the Stefan-type free boundary problem:

$$L(z)=0, \quad \text{in } D_{T*}=\{r_0<r<s^*(t),\ \varepsilon<t<T^*\}; \tag{4.26}$$

$$z(r, \varepsilon)=u_t(r, \varepsilon), \quad r_0\leqslant r\leqslant s^*(\varepsilon); \tag{4.27}$$

$$z_r(r_0, t)=0, \quad \varepsilon<t<T^*; \tag{4.28}$$

$$z(s^*(t), t)=0, \quad \varepsilon<t<T^*; \tag{4.29}$$

$$z_r(s^*(t), t)=-s^{*\prime}(t), \quad \varepsilon<t<T^*; \tag{4.30}$$

$$s^*(\varepsilon)=s(\varepsilon), \tag{4.31}$$

where $\varepsilon\in(0, T)$.

From (1.1)—(1.6), it follows that the function u_t solves (4.26)—(4.31) referring to the region $D_T=\{r_0<r<s(t),\ \varepsilon<t<T\}$. Because of the uniqueness for the problem (4.26)—(4.31), we have:

$$z(r, t)=u_t(r, t), \quad (r, t)\in\{r_0<r<s(t)=s^*(t),\ \varepsilon<t<\min(T, T^*)\}. \tag{4.32}$$

Furthermore, we know from the results of [8] (Theorem 1) that only three cases can occur for the solution of (4.26)—(4.31):

(A) The problem has a solution with arbitrarily large T^*;

(B) There exists a constant $T_0>0$ such that $\lim_{t\to T_0^-} s(t)=r_0$;

(C) There exists a constant $T_1>0$ such that:

$$\inf_{(\varepsilon, T_1)} s(t)>r_0, \qquad \liminf_{t\to T_1^-} s'(t)=-\infty.$$

In addition, case (B) is valid under the following assumptions ([8] Theorem 3)

$$\frac{s(\varepsilon)^2}{2}-\frac{r_0^2}{2}+\int_{r_0}^{s(o)} rz(r,\varepsilon)\,dr=0, \tag{4.33}$$

$$z(r,\varepsilon) \text{ is increasing.} \tag{4.34}$$

Now, we can easily verify (4.33) by using equation $L(u)=1$, and (4.34) by Lemma 4.1. Then case (B) occurs for the problem (4.26)—(4.31), more precisely, there exists $T_0>0$, such that $z(r,t)$ satisfies (4.26)—(4.31) with T^* substituted by T_0 and

$$\lim_{t\to T_0^-} s^*(t)=r_0. \tag{4.35}$$

Let us define the function:

$$u^*(r,t)=u(r,\varepsilon)+\int_{\varepsilon}^{t} z(r,\tau)\,d\tau, \quad (r,t)\in D_{T_0}=\{r_0<r<s^*(t),\ \varepsilon<t<T_0\}. \tag{4.36}$$

From (4.26) and (1.1) we obtain:

$$L(u^*)=1, \quad \text{in } D_{T_0}=\{(v,t),\ v_0<v<s^*(t),\ \varepsilon<t<T_0\}, \tag{4.37}$$

Moreover, we have:

$$u_r^*(r_0,t)=u_r(r_0,\varepsilon)+\int_{\varepsilon}^{t} z_r(r_0,\tau)\,d\tau=0, \quad \varepsilon<t<T_0, \tag{4.38}$$

and

$$u^*(r,t)=u(r,t), \quad (r,t)\in\{r_0<r<s(t),\ \varepsilon\leqslant t<T\}. \tag{4.39}$$

By taking into account Green's identity (2.18) for the domain $\{(\rho,\tau); s^*(t)<\rho<s^*(\tau), \varepsilon<\tau<t\}$ with $\mathscr{U}=z(r,t)$, $\mathscr{V}=r$, we obtain after simple calculation:

$$u_r^*(s^*(t),t)=u_r(s^*(t),\varepsilon)+\int_{\varepsilon}^{t} z_r(s^*(t),\tau)\,d\tau=0. \tag{4.40}$$

Because of (4.29)

$$u_t^*(s^*(t),t)=z(s^*(t),t)=0, \tag{4.41}$$

Since $u^*(s^*(\varepsilon),\varepsilon)=u(s(\varepsilon),\varepsilon)=0$, (4.40) and (4.41) yield

$$u^*(s^*(t),t)=0, \quad \varepsilon<t<T_0. \tag{4.42}$$

Collecting (4.37), (4.38), (4.39), (4.40), (4.42) and (4.25), we actually extend a local solution $\{u(r,t), s(t), T\}$ to a global solution $\{u^*(r,t), s^*(t), T_0\}$ for (1.1)—(1.6). The proof of Theorem 4.3 is complete. ■

Remark. Owing to Theorem 5 of [8], we can get also $T_0\leqslant f(r_0)$, noting that:

$$\lim_{t\to 0}\int_{r_0}^{s(t)} r\ln r\, u_t(r,t)\,dr=f(r_0)-\frac{1}{4}(r_1^2\ln r_1^2-r_1^2-r_0^2\ln r_0^2+r_0^2).$$

References

[1] CRANK J., GUPTA R. S. A moving boundary problem arising from the diffusion of oxygen in absorbing tissues. *J. Inst. Math. Appl.* **10** (1972) 19–33.
[2] ROGERS J. C. W. A free boundary problem as diffusion with nonlinear absorbtion. *J. Inst. Math. Appl.* **20** (1977), 261–268.
[3] FASANO A., PRIMICERIO M. New results on some classical parabolic free boundary problem. *Quart. Appl. Math.* **38** (1981), 439–460.
[4] FASANO A., PRIMICERIO M. General free boundary problems for the heat equation. *J. Math. Anal. Appl.* **57** (1977), 694–723; **58** (1977), 202–231; **59** (1977), 1–14.
[5] LADYZHENSKAJA O. A., SOLONNIKOV V. A., URAL'CEVA N. N. Linear and quasilinear equations of parabolic type. *Amer. Math. Soc. Transl.* **23**, Providence R. I. 1968.
[6] CANNON J. R., HILL C. D. Existence, uniqueness, stability and monotone dependence in a Stefan problem for the heat equation. *L. Math. Mech.* **17** (1967), 1–20.
[7] FRIEDMAN A. Partial differential equations of parabolic type. Prentice Hall, Englewood Cliffs, N. J. 1964.
[8] ANDREUCCI D. Continuation of the solution to a free boundary problem in cylindrical symmetry. [to appear]

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Rozwiązanie zagadnienia dyfuzji tlenu w obszarze cylindrycznym

Praca dotyczy zagadnienia ze swobodną granicą występującego w procesach dyfuzji tlenu w żywych tkankach. Zakłada się radialną symetrię problemu. Dowodzi się istnienia rozwiązania lokalnego w czasie a następnie bada się możliwość jego przedłużenia na dany przedział czasu.

Решение задачи диффузии кислорода в цилиндрической области

Работа касается задачи со свободной границей, имеющей место в процессах диффузии кислорода в живых тканях. Предполагается радиальная симметрия проблемы. Доказывается существование локального решения во времени, а затем исследуется возможность его переноса для заданного интервала времени.