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Optimal finite parameter observer. An application to synthesis of stabilizing feedback for a linear system

by

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General problem of optimal observer (with minimal norm) in Hilbert spaces is considered. On the basis of the measurements of control and output of a linear system the observer estimates a certain unknown, finite parameter vector of this system. We formulate the problem of existence for the observer and the problem of its optimality with respect to the "worst" disturbances of measurements of output and control. We show an application of our results to state observation in dynamic systems and analyse properties of the closed loop system with the optimal observer in the feedback.

1. Introduction

In the theory of feedback control the reconstruction of inaccessible state vector is an important problem. This problem was formulated by R.E. Kalman and others [1], [3], [4], for linear dynamic systems.

D. G. Luenberger [2] considered a special class of observers given by differential equations. The asymptotic behavior of their solution gives an estimate of unknown present state of the observed system.

If disturbances with known probabilistic properties occur in the measured signals identification theory [5] enables us to obtain different types of estimators of state and parameters.

In this paper we will state and solve a general problem of optimal observer for system whose output depends linearly on the pair: finite parameter vector and control. The control and the output are from Hilbert spaces. On the basis of the disturbed measurements of control and output the optimal observer will estimate the unknown finite parameter vector. The optimality of the observer means that it has a minimal norm, hence it is least sensitive with respect to the "worst" disturbances of measurements belonging to the unit balls in these spaces. In Section 4 we also analyse the behaviour of the system with the observer in feedback loop in a stabilizing regime.

2. Statement of the problem

2.1. The description of the linear system and the observer problem

Consider a linear system given by two linear continuous maps: $\mathscr{H}_1: X \to Y$, $\mathscr{H}_2: U \to Y$, where U, Y are Hilbert spaces, and $X = R^n$.

We call X, Y, U the parameter, output, and control spaces, respectively.

An element $y \in Y$ such that

$$y = \mathscr{H}_1 x + \mathscr{H}_2 u \tag{1}$$

and

$$\mathscr{H}_1 x = h_1' x = \sum_{i=1}^{n} h_1^i x_i, \qquad h_1 \in Y^n, \qquad h_1^i \in Y,$$

$$x = [x_1, \ldots, x_n]' \in R^n$$

is called the output connected with the parameter x and control u, where $x \in X$ and $u \in U$.

Denote by $(\cdot \mid \cdot)$ the inner product in Hilbert space. By an observer for system (1) we mean two linear continuous maps:

$$\mathscr{G}_1: Y \to X, \qquad \mathscr{G}_2: U \to X \text{ such that}$$

$$\mathscr{G}_1 y = (g_1 \mid y) = [(g_1^1 \mid y), \ldots, (g_1^n \mid y)]', \qquad g_1 \in Y^n,\ g_1^i \in Y,\quad i = 1, \ldots, n$$

$$\mathscr{G}_2 u = (g_2 \mid u) = [(g_2^1 \mid u), \ldots, (g_2^n \mid u)]', \qquad g_2 \in U^n,\ g_2^i \in U,\quad i = 1, \ldots, n$$

and

$$x = \mathscr{G}_1 y + \mathscr{G}_2 u \tag{2}$$

holds for any triple (y, x, u) fulfilling (1). Combining (1) with (2) we have that a pair $(\mathscr{G}_1, \mathscr{G}_2)$ is an observer for system (1) iff $\ker \mathscr{H}_1 = 0$, and $y = \mathscr{H}_1 \mathscr{G}_1 y$, $\forall y \in Y_H$

$$\mathscr{H}_1 \mathscr{G}_2 = -\mathscr{G}_2 \tag{3}$$

or, alternatively, iff

$$\begin{aligned} \mathscr{G}_1 \mathscr{H}_1 &= I_n \\ \mathscr{G}_1 \mathscr{H}_2 &= -\mathscr{G}_2 \end{aligned} \tag{4}$$

where Y_H is the range of operator $\mathscr{H}_1$: $Y_H \subset Y$, and I_n is the $n \times n$ identity matrix.

2.2. The problem of existence and optimality of the observer.

Denote by S the set of all observers for system (1). From continuity and linearity of mappings in (1), (4) it follows that if S is non-empty then it is a closed, linear manifold in the space $Y^n \times U^n$. In this space we introduce a seminorm

$$\|(g_1, g_2)\|_{Y^n \times U^n} = \sqrt{\sum_{i=1}^{n} \alpha_i (g_1^i \mid g_1^i)_Y + \sum_{i=1}^{n} \beta_i (g_2^i \mid g_2^i)_U} \tag{5}$$

$$\alpha_i > 0, \quad \beta_i \geqslant 0, \quad i = 1, \ldots, n$$

By the optimal observer we mean a pair $(g_1^{\text{opt}}, g_2^{\text{opt}}) \in S$ such that

$$\min_{(g_1, g_2) \in S} \|(g_1, g_2)\|^2 = \|(g_1^{\text{opt}}, g_2^{\text{opt}})\|^2 . \tag{6}$$

Let z_1, z_2, $\|z_1\|_Y \leqslant 1$, $\|z_2\|_U \leqslant 1$, denote disturbances of output and control respectively. We have $x = \mathscr{G}_1 y + \mathscr{G}_2 u$, and $\hat{x} = \mathscr{G}_1 (y + z_1) + \mathscr{G}_2 (u + z_2)$ where $\hat{x}$ is the estimate of the parameter vector x. The observer optimal with respect to disturbances should be such that $\|\hat{x} - x\|^2$ is minimal for the "worst" z_1, z_2 from unit balls:

$$\min_{(g_1, g_2)} \max_{\|z_1\|, \|z_2\| \leqslant 1} \|\hat{x} - x\|^2 = \min_{(g_1, g_2)} \max_{\|z_1\|, \|z_2\| \leqslant 1} \|\mathscr{G}_1 z_1 + \mathscr{G}_2 z_2\|^2 .$$

The following inequality is fulfilled

$$\min_{(g_1, g_2)} \max_{\|z_1\|, \|z_2\| \leqslant 1} \|\hat{x} - x\|^2 \leqslant \min_{(g_1, g_2)} \|(g_1, g_2)\|^2 = \|(g_1^{\text{opt}}, g_2^{\text{opt}})\|^2 .$$

In the sequel we will deal with the minimization of the right-hand side of the above inequality although this is not strictly equivalent to finding the observer optimal with respect to disturbances.

It is easy to see that an observer (g_1, g_2) exists iff there exists $g_1 \in Y^n$ such that

$$(g_1 \mid h_1') = I_n . \tag{7}$$

LEMMA 1. *An observer for system* (1) *exists iff the elements $h_1^i \in Y$, $i = 1, \ldots, n$ are linearly independent.*

Proof. If h_1^i, $i = 1, \ldots, n$ are linearly independent we take $g_1 = (h_1 \mid h_1')^{-1} h_1$ and $g_2 = -\mathscr{H}_2^* g_1$ where $(h_1 \mid h_1')$ is the Gram matrix of elements h_1^i. If h_1^i, $i = 1, \ldots, n$ are linearly dependent, then evidently (7) is not fulfilled. ∎

THEOREM 1. *Let elements h_1^i, $i = 1, \ldots, n$ be linearly independent. Let α, β be diagonal matrices with reals $\alpha_i > 0$ and $\beta_i \geqslant 0$, $i = 1, \ldots, n$ on diagonals, respectively.*

Then there exists a unique optimal observer $(g_1^{\text{opt}}, g_2^{\text{opt}})$ for system (1).

Moreover $(g_1^{\text{opt}}, g_2^{\text{opt}})$ is a unique solution of equations

$$\alpha g_1 - \beta \mathscr{H}_2 g_2 = -\lambda h_1 \tag{8}$$

$$-g_2 = \mathscr{H}_2^* g_1 \tag{9}$$

$$(g_1 \mid h_1') = I_n \tag{10}$$

where λ is an $n \times n$ real matrix of Lagrange multipliers.

Let us define an operator P

$$P=\alpha+\beta\mathcal{H}_2\mathcal{H}_2^*.$$

Then

$$\begin{aligned} g_1^{\text{opt}}&=(P^{-1}h_1\,|\,h_1')^{-1}P^{-1}h_1 \\ g_2^{\text{opt}}&=-\mathcal{H}_2^*\,g_1^{\text{opt}}. \end{aligned} \tag{11}$$

P r o o f. Observe that functional $\|(g_1, \mathcal{H}_2^* g_1)\|^2$ is coercive and quadratic on the space Y^n, if $\alpha_i>0$, $\beta_i\geq 0$, $i=1, \ldots, n$. By Lax-Milgram theorem [6] there exists a unique minimum point of this functional with respect to variable g_1 described uniquely by Lagrange conditions (8). We can verify that (g_1, g_2) given by (11) fulfill the Lagrange conditions. From Lax-Milgram theorem we also have continuous dependence of the minimal point on operators defining the linear system. ■

3. Applications

3.1. An optimal observer for the linear time-independent system

Consider an observable system

$$\begin{aligned} \dot{x}(t)&=Ax(t)+Bu(t) \\ y(t)&=Cx(t) \\ x(o)&=x_0 \end{aligned} \tag{12}$$

$$x(t)\in R^n, \quad u(t)\in R^r, \quad y(t)\in R^m$$

A, B, C are constant matrices of compatible dimensions.

Assume that we measure the control u and output y on an interval $[0, T]$. Our purpose is to determine the best estimate of state $x(T)$ at time T.

We define $X=R^n$, $Y=(L_2(0, T))^m$, $U=(L_2(0, T))^r$. In our case equation (1) takes on the form

$$y(t)=Ce^{-A(T-t)}x(T)-C\int_t^T e^{A(t-\tau)}Bu(\tau)\,d\tau. \tag{13}$$

From observability of system (12) it follows that $\ker\mathcal{H}_1=0$

$$(\mathcal{H}_1 x(T))(t)=Ce^{-A(T-t)}x(T), \quad \forall x\in R^n, \quad \forall t\in[0, T].$$

The adjoint operator $\mathcal{H}_2^*$ has the form

$$(\mathcal{H}_2^* y)(t)=-\int_0^t B'e^{A'(\tau-t)}C'y(\tau)\,d\tau.$$

From the definition of spaces Y and U it follows that operators $\mathcal{G}_1$ and $\mathcal{G}_2$ are of the form.

$$\mathcal{G}_1 y=\int_0^T G_1(\tau)y(\tau)\,d\tau, \qquad \mathcal{G}_2 u=\int_0^T G_2(\tau)u(\tau)\,d\tau \tag{14}$$

where G_1 and G_2 are matrices of functions from $L_2(0, T)$ of dimensions $n\times n$ and $n\times r$, respectively. Rows of matrices G_1 and G_2 are transpositions of elements g_1^i and g_2^i from relation (2).

The norm of an observer in the space $Y^n\times U^n$ is defined as in formula (5)

$$\|(g_1, g_2)\|_{Y^n\times U^n}=\sqrt{\int_0^T\left[\sum_{i=1}^n \alpha_i\left(\sum_{j=1}^m G_{1,ij}^2(\tau)\right)+\sum_{i=1}^n \beta_i\left(\sum_{j=1}^r G_{2,ij}^2(\tau)\right)\right]d\tau} \tag{15}$$

where $G_{1,ij}$ (resp. $G_{2,ij}$) is the element of the i-th row and j-th column of matrix G_1 (resp. G_2) determined by formulas (14).

Introducing diagonal matrices α, β with reals α_i and β_i, $i=1, \ldots, n$ on diagonals, respectively according to formula (8) we have the following existence and optimality conditions for matrices G_1, G_2.

$$\int_0^T G_1(\tau)\, Ce^{-A(T-\tau)}\, d\tau=I_n \tag{16}$$

$$\alpha G_1(t)=-\beta\int_t^T G_2(\tau)\, B'\, e^{A'(t-\tau)}\, C'\, d\tau-\lambda e^{-A'(T-t)}\, C' \tag{17}$$

$$G_2(t)=\int_0^t G_1(\tau)\, Ce^{A(\tau-t)}\, B\, d\tau\,. \tag{18}$$

System of equations (17), (18) is of Volterra type, so it has a unique solution for an arbitrary λ, α, β. Moreover $G_1(T)=-\alpha^{-1}\lambda C'$, $G_2(o)=0$.

Now we can give another characterisation of the solution of equations (17), (18).

Assuming, that G_1, G_2 can be expressed by means of some matrices $K_1(t)$ and $K_2(t)$ of dimensions $n\times n$

$$\begin{aligned} G_1(t)&=K_1(t)\cdot C'\\ G_2(t)&=K_2(t)\cdot B \end{aligned} \tag{19}$$

and by substitution this into (17) we get

$$K_1(t)\, C'=-\alpha^{-1}\beta\int_t^T K_2(\tau)\, B\, B'\, e^{A'(t-\tau)}\, d\tau\, C'-\alpha^{-1}\lambda e^{-A(T-t)}\, C'\,. \tag{20}$$

Omitting C' we have the equation for $K_1(t)$

$$\begin{aligned} K_1(t)&=\left[-\alpha^{-1}\lambda e^{-A'T}-\alpha^{-1}\beta\int_0^T K_2(\tau)\, BB'\, e^{-A'\tau}\, d\tau\right]e^{A't}+\\ &\qquad+\alpha^{-1}\beta\int_0^t K_2(\tau)\, BB'\, e^{A'(t-\tau)}\, d\tau\\ K_1(T)&=-\alpha^{-1}\lambda\,. \end{aligned} \tag{21}$$

Similarly from (18) and (19)

$$K_2(t)=\int_0^t K_1(\tau)\, C'\, Ce^{A(\tau-t)}\, d\tau, \qquad K_2(o)=0\,. \tag{22}$$

The solution of system (21), (22) is also a solution of the system of differential equations

$$\dot{K}_1'(t)=AK_1'(t)+BB'K_2'(t)\,\alpha^{-1}\beta \tag{23}$$

$$\dot{K}_2'(t)=C'CK_1'(t)-A'K_2'(t) \tag{24}$$

with split boundary conditions $K_1'(T)=-\lambda'\alpha^{-1}$, $K_2'(o)=0$.

By $k_{1i}(t)$ and $k_{2i}(t)$ we mean the i-th columns of matrices $K_1'(t)$ and $K_2'(t)$, respectively. We define reals

$$\gamma_i=\frac{\beta_i}{\alpha_i} \qquad i=1, \dots, n.$$

Introducing the fundamental matrices

$$\Phi^i(t)=e^{W_i t} \quad \text{for } i=1, \dots, n$$

where

$$W_i=\begin{bmatrix} A, & \gamma_i BB' \\ C'C, & -A' \end{bmatrix},$$

we obtain for the column vectors $k_{1i}(t)$ and $k_{2i}(t)$

$$\begin{bmatrix} k_{1i}(t) \\ k_{2i}(t) \end{bmatrix}=\begin{bmatrix} \Phi_{11}^i(t), & \Phi_{12}^i(t) \\ \Phi_{21}^i(t), & \Phi_{22}^i(t) \end{bmatrix}\begin{bmatrix} k_{1i}(o) \\ k_{2i}(o) \end{bmatrix}=\begin{bmatrix} \Phi_{11}^i(t)\,k_{1i}(o) \\ \Phi_{21}^i(t)\,k_{1i}(o) \end{bmatrix}.$$

Vectors $k_{1i}(o)$ can be found from condition (16) as the i-th column of matrix D_i,

$$D_i=\left[\int_0^T e^{-A'(T-\tau)}\,C'C\Phi_{11}^i(\tau)\,d\tau\right]^{-1}.$$

Nonsingularity of D_i follows from the fact that the system (12) is observable.

Hence

$$\begin{aligned} k_{1i}(t)&=\Phi_{11}^i(t)\cdot D_i\cdot e_i \\ k_{2i}(t)&=\Phi_{21}^i(t)\cdot D_i\cdot e_i \end{aligned} \tag{25}$$

where e_i — basis vectors in R^n, $i=1, \dots, n$.

The rows of the optimal observer matrices $G_1(t)$ and $G_2(t)$ are transpositions of the vectors

$$\begin{aligned} g_1^i&=C\Phi_{11}^i(t)\cdot D_i\,e_i \\ g_2^i&=B'\,\Phi_{21}^i(t)\cdot D_i\,e_i \end{aligned} \tag{26}$$

$i=1, \dots, n$

If $\gamma_i=1$, $i=1, \dots, n$, then $\Phi^i(t)=\Phi(t)=e^{Wt}$ where

$$W=\begin{bmatrix} A, & BB' \\ C'C, & -A' \end{bmatrix} \tag{27}$$

andthe solution of equations (23), (24) can be written in a simpler form.

The formulas for the optimal observer matrices are

$$G_1(t)=e^{AT}\left[\int_0^T \Phi'_{11}(\tau)\, C'\, Ce^{A\tau}\, d\tau\right]^{-1}\cdot \Phi'_{11}(t)\cdot C',$$
$$G_2(t)=e^{AT}\left[\int_0^T \Phi'_{11}(\tau)\, C'\, Ce^{A\tau}\, d\tau\right]^{-1}\cdot \Phi'_{21}(t)\cdot B. \tag{28}$$

The observers (26) and (28) are optimal in the sense of minimal norm in $L_2(0, T)$.

3.2. A numerical example

$$\dot{x}(t)=\begin{bmatrix}0, & 1\\ 0, & 0\end{bmatrix} x(t)=\begin{bmatrix}0\\ 1\end{bmatrix} u(t), \quad t\in[0, T],$$
$$y(t)=[2, 0]\, x(t), \tag{29}$$
$$x(o)=x_0.$$

We will calculate the optimal observer which reconstructs the state $x(T)$. For simplicity, we assume that α and β in (15) are identity matrices. We obtain matrix W from (27), hence submatrices $\Phi_{11}(t)$, $\Phi_{21}(t)$ in (28) are of the form

$$\Phi_{11}(t)=\begin{bmatrix}\operatorname{ch} t\cos t, & 0.5(\operatorname{sh} t\cos t+\operatorname{ch} t\sin t)\\ \operatorname{sh} t\cos t-\operatorname{ch} t\sin t, & \operatorname{ch} t\cos t\end{bmatrix}$$

$$\Phi_{21}(t)=\begin{bmatrix}2(\operatorname{sh} t\cos t+\operatorname{ch} t\sin t), & 2\operatorname{sh} t\sin t\\ -2\operatorname{sh} t\sin t, & \operatorname{sh} t\cos t-\operatorname{ch} t\sin t\end{bmatrix}.$$

Matrix $R=e^{AT}\left[\int_0^T \Phi'_{11}(\tau)\, C'\, Ce^{AT}\, d\tau\right]^{-1}$ is given by

$$R=\frac{1}{2(\operatorname{sh}^2 T-\sin^2 T)}\begin{bmatrix}\operatorname{sh} T\cos T-\operatorname{ch} T\sin T, & 2\operatorname{sh} T\sin T\\ -2\operatorname{sh} T\sin T, & 2(\operatorname{sh} T\cos T+\operatorname{ch} T\sin T)\end{bmatrix},$$

The optimal observer (28) is determined by two vector functions

$$G_1(t)=R\cdot\begin{bmatrix}2\operatorname{ch} t\cos t\\ \operatorname{sh} t\cos t+\operatorname{ch} t\sin t\end{bmatrix},$$
$$G_2(t)=R\cdot\begin{bmatrix}-2\operatorname{sh} t\sin t\\ \operatorname{sh} t\cos t-\operatorname{ch} t\sin t\end{bmatrix}. \tag{30}$$

The norm of this observer $\|(G_1, G_2)\|(T)$ is the following function of time T

$$\|(G_1, G_2)\|(T)=\sqrt{\frac{3\operatorname{sh} 2T+\sin 2T}{4(\operatorname{sh}^2 T-\sin^2 T)}}. \tag{31}$$

This function is strictly decreasing and tend to $\sqrt{1.5}$ when $T\to\infty$.

4. The observer in the closed loop system

Consider the system:

$$\begin{aligned} \dot{x}(t)&=Ax(t)+Bu(t)\\ y(t)&=Cx(t). \end{aligned} \tag{32}$$

Let us assume that for the stabilization of the system a state feedback

$$u(t)=Px(t)$$

is needed where P is a constant matrix with suitable dimensions.

If the state is not directly accessible we can apply the observer to obtain the desired feedback. We have the following system of equations describing the behaviour of the closed loop system with an observer:

$$\begin{aligned} &\dot{x}(t)=Ax(t)+Bu(t)\\ &y(t)=Cx(t)\\ &\hat{x}(t)=\int_0^T G_1(s)\,y(t-T+s)\,ds+\int_0^T G_2(s)\,u(t-T+s)\,ds\\ &u(t)=P\hat{x}(t) \end{aligned} \tag{33}$$

where $\hat{x}(t)$ denotes the observed estimate of $x(t)$. This system can be reduced to the following form

$$\begin{aligned} &\dot{x}(t)=Ax(t)+Bu(t),\\ &u(t)=P\left[\int_0^T G_1(s)\,Cx(t-T+s)\,ds+\int_0^T G_2(s)\,u(t-T+s)\,ds\right] \quad \text{for } t\geqslant T. \end{aligned} \tag{34}$$

After change of the variable of integration we obtain

$$\begin{aligned} &\dot{x}(t)=Ax(t)+Bu(t),\\ &u(t)=P\cdot\left[\int_{-T}^0 G_1(s+T)\,Cx(t+s)\,ds+\int_{-T}^0 G_2(s+T)\,u(t+s)\,ds\right]. \end{aligned} \tag{35}$$

Without loss of generality we can assume that in equation (35) $t\geqslant 0$ and initial conditions

$$\begin{aligned} u(t)&=\varphi_1(t), \quad t\in[-T,0]\\ x(t)&=\varphi_2(t), \quad t\in[-T,0] \end{aligned} \tag{36}$$

are given. System (35) includes a distributed delay and its second component is of integral character.

LEMMA 2. *For any* $T_1\times 0$, $x\in L_2(-T,T_1;R^n)$, $\varphi_1\in L_2(-T,0;R^r)$, *there exists a unique absolutely continuous solution* u *of the equation*

$$u(t)=P\int_{-T}^0 G_1(T+s)\,Cx(t+s)\,ds+P\int_{-T}^0 G_2(T+s)\,u(s+t)\,ds \tag{37}$$

for $t\geqslant 0$

$$u(t)=\varphi_1(t) \quad \text{for} \quad t\in[-T,0)$$

where G_1, G_2 are square integrable matrices.

Proof. Let us denote the first component of the right-hand side of (37) by $w(t)$; w is absolutely continuous by definition of x, G_1.

Thus

$$u(t)=w(t)+P\int_{-T}^{-t} G_2(T+s)\,u(s+t)\,ds+P\int_{-t}^{0} G_2(T+s)\,u(s+t)\,ds=$$

$$=w(t)+P\int_{t-T}^{0} G_2(T-t+s)\,\varphi_1(s)\,ds+$$

$$+P\int_{0}^{t} G_2(T-t+s)\,u(s)\,ds \quad \text{for } 0\leqslant t\leqslant T. \tag{38}$$

This equation is of Volterra type, hence it has a unique solution on interval $[0,T]$. Arguing similarly for intervals $[(i-1)T, iT]$, $i\geqslant 2$ we have the thesis. ■

Let us assume that G_1, G_2 are of class $C^1(-T,0)$ and x is fixed.

LEMMA 3. *Every solution u of (37) fulfills the following FDE equation on the interval $[0,\infty)$.*

$$\dot u(t)=-P\int_{-T}^{0}\dot G_1(T+s)\,Cx(t+s)\,ds-P\int_{-T}^{0}\dot G_2(T+s)\,u(t+s)\,ds+$$
$$+PG_1(T)\,Cx(t)-PG_1(o)\,Cx(t-T)+PG_2(T)\,u(t)+$$
$$-PG_2(o)\,u(t-T), \tag{39}$$

for almost all t

$$u(t)=\varphi_1(t) \quad \text{for } t\in[-T,0]$$

The proof can be obtained by the differentiation of both sides of (37) and will be omitted. It can be noted that the solution of (39) is unique.

Equation (39) together with the state equation (32) gives a full description of the behaviour of the closed loop system. Finally, we have the equation

$$\begin{bmatrix}\dot x(t)\\ u(t)\end{bmatrix}=\begin{bmatrix}A, & B\\ PG_1(T)\,C, & PG_2(T)\end{bmatrix}\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}+\begin{bmatrix}0, & 0\\ -PG_1(o)\,C, & -PG_2(o)\end{bmatrix}\times$$
$$\times\begin{bmatrix}x(t-T)\\ u(t-T)\end{bmatrix}+\begin{bmatrix}0\\ -P\int_{-T}^{0}\dot G_1(T+s)\,Cx(t+s)\,ds-\\ 0\\ -P\int_{-T}^{0}\dot G_2(tT+s)\,u(t+s)\,ds\end{bmatrix} t\geqslant 0, \tag{40}$$

with the initial condition

$$\begin{matrix} x(t)=\varphi_2(t) \\ u(t)=\varphi_1(t) \end{matrix} \qquad t\in[-T,0].$$

For stability analysis of (40) we compute the eigenvalues of the FDE system (40). To this end we will seek all the exponential solutions of (40) of the form:

$$\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} e^{\lambda t}, \quad x_0\in R^n, \quad u_0\in R^r, \quad t\in[-T,0].$$

After some calculations we have the relation

$$\begin{aligned} &\lambda x_0=Ax_0+Bu_0, \\ &u_0=P\int_{-T}^{0} G_1(T+s)\,Cx_0\,e^{\lambda s}\,ds+P\int_{-T}^{0} G_2(T+s)\,u_0\,e^{\lambda s}\,ds. \end{aligned} \tag{41}$$

From the definition of the observer we know that

$$x(t)=\int_{-T}^{0} G_1(T+s)\,Cx(t+s)\,ds+\int_{-T}^{0} G_2(T+s)\,u(t+s)\,ds,$$

for every control u and the solution x of system (32) corresponding to this u. This is also true for $u(t)=u_0\,e^{\lambda t}$, $x(t)=x_0\,e^{\lambda t}$. Hence formula (41) reduces to:

$$\begin{aligned} &\lambda x_0=Ax_0+Bu_0, \\ &u_0=Px_0. \end{aligned}$$

And next

$$\lambda x_0=(A+BP)\,x_0.$$

Thus we have

LEMMA 4. *Equation* (40), *describing the system with the observer in a closed loop, has the same eigenvalues as the system*

$$\dot{x}(t)=Ax(t)+BPx(t) \tag{42}$$

and system (42) *is asymptotically stable iff system* (40) *is symptotically stable. On the other hand analysing the dynamics of the system and making use of the definition of observer one can see that every solution x of* (40) *fulfills* (42) *for $t\geqslant T$ and for arbitrary functional initial conditions in* (40).

This kind of stabilizing feedback can be easily realized using microprocessor techniques. Values of the matrices G_1, G_2 can be stored in the memory and the dalays (distributed or not) can be formed in shift registers.

5. Conclusions

Our approach makes it possible to formulate any finite parameter observation problem for linear system (1) in a uniform way. We have given an explicit formula for the observer, optimal with respect to disturbances in appropriate Hilbert spaces. We have applied our results to a simple example. The case where the observer is in the stabilizing feedback is also analysed.

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Optymalny obserwator skończonej ilości parametrów. Zastosowanie do syntezy stabilizującego sprzężenia w układach liniowych.

Rozważono ogólny problem optymalnego obserwatora (z minimalną normą) w przestrzeniach Hilberta. Na podstawie pomiarów sterowania i wyjścia systemu liniowego obserwator odtwarza nieznany, skończenie wymiarowy wektor parametrów tego systemu. Przedstawiono problem istnienia obserwatora i jego optymalności względem najbardziej niebezpiecznych zakłóceń w pomiarach. Teorię zastosowano do zagadnienia obserwacji stanu liniowych układów dynamicznych. Przeanalizowano własności zamkniętego układu sterowania z obserwatorem optymalnym w pętli sprzężenia zwrotnego.

Оотимальный наблюдатель конечного числа параметров. Применение к синтезу стабилизирующей связи в линейных системах

Рассматривается общая проблема оптимального наблюдателя (с минимальной нормой) в гильбертовом пространстве. На снове измерений управления и выхода линейной системы наблюдатель воспроизводит неизвестный, конечномерный вектор параметров этой системы. Представлена проблема существования наблюдателя и гео оптимальности по отношению к наиболее опасным помехам в измерениях. Теория применена к вопросу наблюдения состояния линейных динамических систем. Анализируются свойства замкнутой системы управления с оптимальным наблюдателем в цепи обратной связи.