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Zero placement for discrete-time systems using nonstationary extrapolators

by

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A new method for zero placement in single-input single-output discrete-time systems is given. It is done by means of a nonstationary-gain zero-order extrapolator. An algorithm for computing the gain is proposed.

1. Introduction

In discrete-time control systems with continuous plants the output $v(t)$ of the controller is extrapolated to change the sequence of impulses into a staircase function $u^*(t)$ (Fig. 1). T is a sampling period and α is a gain of the zero-order extrapolator. A designer can influence parameters of the discrete time model of the

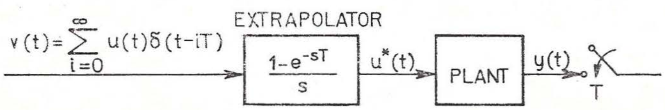

Fig. 1. Discrete-time system

system only by changing the sampling period T. Keviczky and Kumar [2] have proved for a wide class of linear plants and for practically acceptable small T that at least one zero of the transfer function of the discrete-time model of the system is outside the unit circle. This nonminimum phase effect is more often met in systems with delay in control. On the other hand there exist some control strategies (for example: exact model matching [3] and minimum variance control [1]) which for nonminimum phase systems lead to unstable modes in closed loop.

The aim of this paper is to show that if we replace the constant gain α of the zero-order extrapolator by an appropriate periodic staircase function $\alpha(t)$, the zeros of the model transfer function of the system can be placed into any prescribed position.

The problem is precisely formulated in the next section. In section 3 relations between continuous-time model of the plant and discrete-time model of the system are considered. A solution of the problem and an appropriate algorithm are given in section 2. A simple example to illustrate the method is presented in the last section.

2. Problem formulation

We will assume that $\alpha(t)$ changes its value in

$$t=\ldots -T_a, 0, T_a, 2T_a, \ldots .$$

T_a is a real defined by:

$$T_a=T/r \tag{2.1}$$

where r is an integer specified later.

Let Ω_r be a set of reals called modifying factors which are the values of $\alpha(t)$. So:

$$\Omega_r=\{\alpha_1, \ldots, \alpha_r\} \tag{2.2}$$

and:

$$\alpha(t) \stackrel{\text{def}}{=} \alpha_j \text{ if and only if } \exists\, i \in N\; t \in [iT+(j-1)\,T_a,\; iT+jT_a) \tag{2.3}$$

where:

$$N \text{ is a set of integers and } 0<j\leqslant r.$$

Note that:

$$u^*(iT+jT_a)=\alpha_{j+1}\, u(iT) \tag{2.4}$$

It is assumed throughout the paper that the plant is: single-input, single-output, stationary, linear, with delay in input. Three types of models of the plant and the system are considered:

a) continuous-time model of the plant denoted by S_c with transfer function:

$$G_c(s)=\frac{L_c(s)}{M_c(s)}\, v^{-sT_0} \tag{2.5}$$

b) discrete-time model of the system for sampling period T_a denoted by S_a with transfer function:

$$G_a(z)=\frac{L_a(z)}{z^{k+1}\, M_a(z)} \tag{2.6}$$

c) discrete-time model of the system for sampling period T denoted by S_T with transfer function:

$$G_T(z)=\frac{L_T(z)}{z^l\, M_T(z)}\,. \tag{2.7}$$

It is assumed for the last model that the gain of the zero-order holding member is defined by (2.3).

The above transfer-functions are assumed to satisfy following conditions:

i) $L_c(s)$, $M_c(s)$, $L_a(z)$, $M_a(z)$, $L_T(z)$, $M_T(z)$ are polynomials in s and z respectively

ii) $\deg M_c(s) = \deg M_a(z) = \deg M_T(z) = n$

iii) $\deg L_c(s) \leqslant n-1$, $\deg L_a(z) \leqslant n$, $\deg L_T(z) \leqslant n$.

The problem to be solved can be formulated as follows:

Given the S_c model of the plant, sampling period T, and a polynomial $B(z) = b_0 z^n + b_1 z^{n-1} + \ldots + b_n$. Choose such r and find such modifying factors $\alpha_1, \ldots, \alpha_r$ that the nominator of the S_T model transfer function of the system satisfies the following equation:

$$L_T(z) = B(z). \tag{2.8}$$

3. Relations between S_c, S_a and S_T models

3.1. Derivation of the S_a model from S_c.

We consider here the following problem: given S_c, T, r, find S_a.

The state space representation of the S_c transfer function (2.5) is given by:

$$\dot{x}(t) = Ax(t) + Bu^*(t - T_0) \tag{3.1}$$

$$y(t) = Cx(t) \tag{3.2}$$

where: $x(t)$ is n-dimensional state vector; A, B, C matrices of appropriate dimensions such that:

$$C^T (s1_n - A)^{-1} B = \frac{L_c(s)}{M_c(s)} \tag{3.3}$$

$u^*(\tau)$ is a staircase function changing its value in $\tau = iT_a$

Let:

$$T_0 = kT_a + T_s \tag{3.4}$$

where:

$$0 \leqslant T_s < T_a$$

then:

$$x((i+1)T_a) = e^{AT_a} x(iT_a) + \int_0^{T_a} e^{A(T_a - \tau)} Bu^*(\tau + (i-k)T_a - T_s)\, d\tau \tag{3.5}$$

$$y(iT_a) = C^T x(iT_a) \tag{3.6}$$

Taking into account that $u^*(\tau)$ is a piecewise constant function we have:

$$x((i+1)T_a) = e^{AT_a} x(iT_a) + e^{AT_a} \int_0^{T_s} e^{-A\tau}\, d\tau\, Bu^*((i-k-1)T_a) + $$
$$+ e^{AT_a} \int_{T_s}^{T_a} e^{-A\tau}\, d\tau\, Bu^*((i-k)T_a). \tag{3.7}$$

Let:

$$A_a = e^{AT_a} \tag{3.8a}$$

$$B_S = e^{AT_a} \int_0^{T_s} e^{-A\tau}\, d\tau\, B \tag{3.8b}$$

$$B_N = e^{AT_a} \int_{T_s}^{T_a} e^{-A\tau}\, d\tau\, B \tag{3.8c}$$

$$B_a = B_S + A_a B_N \tag{3.8d}$$

$$D_a = C^T B_N \tag{3.8e}$$

$$x_1(jT_a) = x(jT_a) - B_N u^*((j-k-1)T_a) \tag{3.8f}$$

Substituting (3.8a)—(3.8f) in (3.7) we get a state space representation of the S_a model:

$$x_1((j+1)T_a) = A_a x_1(jT_a) + B_a u^*((j-k-1)T_a) \tag{3.9}$$

$$y(jT_a) = C^T x_1(jT_a) + D_a u^*((j-k-1)T_a) \tag{3.10}$$

Its transfer function is given by:

$$G_a(z) = \frac{1}{z^{k+1}} \frac{C^T (z1_n - A_a)_{ad} B_a + D_a \det(z1_n - A_a)}{\det(z1_n - A_a)} = \frac{L_a(z)}{z^{k+1} M_a(z)}. \tag{3.11}$$

3.2. Derivation of the S_T model from S_a.

Let us assume that the S_a model and Ω_r are given (eq. (3.9), (3.10)) and S_T model is to be found.

Without loss of generality we can write equations (3.9) and (3.10) in the following form:

$$x_1(iT + jT_a) = A_a x_1(iT + (j-1)T_a) + B_a u^*((i-k_1)T + (j-1-k_2)T_a) \tag{3.12}$$

$$y(iT + (j-1)T_a) = C^T x_1(iT + (j-1)T_a) + D_a u^*((i-k_1)T + (j-1-k_2)T_a) \tag{3.13}$$

where:

k_1 and k_2 are integers satisfying the equation:

$$k+1 = k_1 r + k_2 \tag{3.14}$$

and $0 \leqslant k_2 < r, \quad 0 < j \leqslant r$

hence:

$$x_1((i+1)T) = A_a^r x_1(iT) + \sum_{j=1}^{r} A_a^{r-j} B_a u^*((i-k_1)T + (j-1-k_2)T_a) \tag{3.15}$$

$$y(iT) = C^T x_1(iT) + D_a u^*((i-k_1)T - k_2 T_a) \tag{3.16}$$

Substituting (2.4) in (3.15) and (3.16) we get:

$$x_1((i+1)T)=A_a^r x_1(iT)+\sum_{j=1}^{k_2} A_a^{r-j} B_a \alpha_{r-k_2+j} u((i-k_1-1)T)+ \\ +\sum_{j=k_2+1}^{r} A_a^{r-j} B_a \alpha_{j-k_2} u((i-k_1)T) \quad (3.17)$$

$$y(iT)=C^T x_1(iT)+D_a \alpha_{r-k_2+1} u((i-k_1-1)T) \quad (3.18)$$

if $k_2>0$, and:

$$x_1((i+1)T)=A_a^r x_1(iT)+\sum_{j=0}^{r-1} A_a^{r-1-j} B_a \alpha_{j+1} u((i-k_1)T) \quad (3.19)$$

$$y(iT)=C^T x_1(iT)+D_a \alpha_1 u((i-k_1)T) \quad (3.20)$$

if $k_2=0$

Let us substitute in (3.17) and (3.18) equations (3.21a)—(3.21g) and to (3.19) and (3.20) equations (3.22a)—(3.22e):

$$A_T=A_a^r \quad (3.21a)$$

$$B_r=\sum_{j=0}^{k_2-1} A_a^{r-1-j} B_a \alpha_{r-k_2+j+1} \quad (3.21b)$$

$$B_m=\sum_{j=k_2}^{r-1} A_a^{r-1-j} B_a \alpha_{j-k_2+1} \quad (3.21c)$$

$$B_T=B_r+A_T B_m \quad (3.21d)$$

$$D_T=D_a \alpha_{r-k_2+1}+C^T B_m \quad (3.21e)$$

$$l=k_1+1 \quad (3.21f)$$

$$x_2(iT)=x_1(iT)-B_M u((i-l)T) \quad (3.21g)$$

$$A_T=A_a^r \quad (3.22a)$$

$$B_T=\sum_{j=0}^{r-1} A_a^{r-1-j} B_a \alpha_{j+1} \quad (3.22b)$$

$$D_T=D_a \alpha_1 \quad (3.22c)$$

$$l=k_1 \quad (3.22d)$$

$$x_2(iT)=x_1(iT) \quad (3.22e)$$

In both cases we get:

$$x_2((i+1)T)=A_T x_2(iT)+B_T u((i-l)T) \quad (3.23)$$

$$y(iT)=C^T x_2(iT)+D_T u((i-l)T) \quad (3.24)$$

The above equations are a state space representation of the S_T model. Its transfer function is given by:

$$G_T(z)=\frac{C^T(z1_n-A_T)_{ad}B_T+\det(z1_n-A_T)D_T}{z^l\det(z1_n-A_T)}=\frac{L_T(z)}{z^l M_T(z)} \tag{3.25}$$

4. An algorithm for finding modifying factors

In this section we consider parallely two cases. The first one refers to $k_2>0$, and the other one to $k_2=0$. In order to simplify the notation equations are numbered with letter a (b) if they are related to the first case (second case) or without any letter if they concern both cases.

Note that:

$$B_T=[A_a^{r-k_2}B_a,\ A_a^{r-k_2+1}B_a,\ \dots,\ A_a^{2r-k_2-1}]\cdot\begin{bmatrix}\alpha_r\\ \alpha_{r-1}\\ \vdots\\ \alpha_1\end{bmatrix} \tag{4.1a}$$

$$B_T=[B_a,\ A_aB_a,\ \dots,\ A_a^{r-1}B_a]\cdot\begin{bmatrix}\alpha_r\\ \alpha_{r-1}\\ \vdots\\ \alpha_1\end{bmatrix} \tag{4.1b}$$

$$D_T=[D_a,\ C^TB_a,\ C^TA_aB_a,\ \dots,\ C^TA_a^{r-k_2-1}B_a]\cdot\begin{bmatrix}\alpha_{r-k_2+1}\\ \alpha_{r-k_2}\\ \alpha_{r-k_2-1}\\ \vdots\\ \alpha_1\end{bmatrix} \tag{4.2a}$$

$$D_T=D_a\cdot\alpha_1\,. \tag{4.2b}$$

Let:

$$M_1=\begin{bmatrix}A_a^{r-k_2} & 0\\ 0 & 1\end{bmatrix} \tag{4.3a}$$

$$M_1=1_{n+1} \tag{4.3b}$$

$$M_2=\begin{bmatrix}B_a, & A_aB_a, \dots, & A_a^{k_2-1}B_a, & A_a^{k_2}B_a, \dots, & A_a^{r-1}B_a\\ 0, & 0, \dots, & D_a, & C^TB_a, \dots, & C^TA_a^{r-k_2-1}B_a\end{bmatrix} \tag{4.4a}$$

$$M_2=\begin{bmatrix}B_a, & A_aB_a, \dots, & A_a^{r-2}B_a, & A_a^{r-1}B_a\\ 0, & 0, \dots, & 0, & D_a\end{bmatrix} \tag{4.4b}$$

$$M=M_1\cdot M_2 \tag{4.5}$$

Then:

$$\begin{bmatrix} B_T \\ D_T \end{bmatrix} = M \cdot \begin{bmatrix} \alpha_r \\ \alpha_{r-1} \\ \vdots \\ \alpha_1 \end{bmatrix} \tag{4.6}$$

Now we can formulate an algorithm for solving the problem stated in chapter 2.

Algorithm 1.

Step 1. Find any minimal realization (3.1) and (3.2) of (2.5).
Step 2. Choose $r \geqslant n+1$.
Step 3. Find T_a (2.1) and k, T_s (3.4).
Step 4. Find A_a, B_a, D_a (3.8a)—(3.8e).
Step 5. Find $A_T = e^{AT}$.
Step 6. Find k_1, k_2 (3.14).
Step 7. Find l (3.19f), ((3.20*d*)).
Step 8. Choose any B_T and any D_T such that:

$$B(z) = C^T (z1_n - A_T)_{ad} B_T + \det (z1_n - A_T) D_T$$

Step 9. Find M_1, M_2 (4.3a), (4.4a), ((4.3b), (4.4b)) and M (4.5).
Step 10. Solve the set of equations (4.6).
Step 11. Use the solution obtained in step 10 for modifying extrapolator.

Note that only step 8 and step 10 cannot always be executed. Propositions 1 and 5 provide sufficient conditions.

PROPOSITION 1. If (C^T, A_T) is an observable pair and rank M satisfies:

$$\operatorname{rank} M = n+1 \tag{4.7}$$

then the problem formulated in chapter 2 has a solution and moreover it can be found using the algorithm 1.

Proof. We should only show that under assumptions of this theorem steps 8 and 10 can be executed.

It is well known that if (C^T, A_T) is an observable pair then there exists such nonsingular matrix T that:

$$C^T T^{-1} (z1_n - TA_T T^{-1})_{ad} = [1, z, ..., z^{n-1}] \tag{4.8}$$

denoting:

$$\det (z1_n - A_T) = a_n + a_{n-1} z + ... + a_1 z^{n-1} + z^n \tag{4.9}$$

and:

$$\bar{B}_T = TB_T \tag{4.10}$$

we have:

$$B(z)=C^T(z1_n-A_T)_{ad}B_T+\det(z1_n-A_T)D_T=$$
$$=C^TT^{-1}(z1_n-TA_TT^{-1})_{ad}TB+\det(z1_n-A_T)D_T \quad (4.11)$$

hence:

$$B(z)=[1, z, ..., z^{n-1}]\cdot\bar{B}_T+a_nD_T+...+a_1D_Tz^{n-1}+D_Tz^n \quad (4.12)$$

so taking:

$$D_T=b_0 \quad \text{and} \quad \bar{B}_T=\begin{bmatrix} b_n-a_nb_0 \\ \vdots \\ b_1-a_1b_0 \end{bmatrix} \quad (4.13)$$

where $b_0, b_1, ..., b_n$ are coefficients of $B(z)$, and then solving (4.10) we obtain B_T and D_T needed in step 8 of the algorithm.

To complete this proof we note that if (4.7) is satisfied then there exists a solution of (4.6). Hence the step 10 can be executed, too. ■

Note that (4.7) is satisfied if and only if:

$$\text{rank } M_2=n+1 \quad (4.14)$$

The next propositions show that (4.14) is not a very restrictive condition.

M_2 is treated now as a matrix function in V where:

$$V=(A_a, B_a, C, D_a) \quad (4.15)$$

If M_2 is a square matrix then $f(V)=\det M_2(V)$ is a scalar function in V. $S(V, \varepsilon)$ denotes a sphere with center V and radius ε.

PROPOSITION 2. If $k_2>0$, $r=n+1$ and $f(V_1)=0$ then $\forall\varepsilon>0\ \exists\bar{V}\in S(V_1, \varepsilon)\ f(\bar{V})\neq 0$.

Proof. Note that $f(V)$ is a polynomial in elements of A_a, B_a, C and D_a. Hence assuming the thesis to be false i.e.:

$$\exists\varepsilon>0\ \forall\bar{V}\in S(V_1, \varepsilon) \quad f(\bar{V})=0$$

We get:

$$\forall V f(V)=0 \quad (4.16)$$

But taking $V=(\bar{A}_a, \bar{B}_a, \bar{C}, \bar{D}_a)$ where:

$$\bar{A}_a=\begin{bmatrix} 0, & 1, & 0, & ..., & 0 \\ 0, & 0, & 1, & ..., & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & ..., & 1 \\ 0, & 0, & 0, & ..., & 0 \end{bmatrix} \quad \bar{B}_a=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \bar{C}=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} k_2-1 \\ \\ \\ \\ \end{matrix} \quad \bar{D}_a=0 \quad (4.17)$$

we get:

$$f(\bar{V})=\left|\begin{array}{ccccc:c} 0, & 0, & \ldots, & 0, & 1 & 0 \\ 0, & 0, & \ldots, & 1, & 0 & 0 \\ & & \ddots & & & \\ 0, & 1, & \ldots, & 0, & 0 & 0 \\ 1, & 0, & \ldots, & 0, & 0 & 0 \\ \hdashline 0, & 0, & \ldots, & 0, & 0 & 1 \end{array}\right| \neq 0 \qquad (4.18)$$

(4.18) contradicts (4.16). Hence tne thesis must be true. ■

PROPOSITION 3. If $r=n+1$ and $f(V_1)=0$ then $\exists \varepsilon>0\ \forall V \in S(V_1, \varepsilon)\ f(V) \neq 0$. The above proposition results from the fact that $f(V)$ is a polynomial.

Note that if we replace in propositions 2 and 3 expressions like $f(V)=0$ and $f(V)\neq 0$ by rank $M_2(V)\leqslant n$ and rank $M_2(V)=n+1$ respectively they hold for $r \geqslant n+1$, too. Hence we see that (4.14) is satisfied for almost every V.

In the next proposition a necessary condition for (4.14) is given:

PROPOSITION 4. If rank $M_2(V)=n+1$ then (A_a, B_a) is a controllable pair.

The case $k_2=0$ is treated in one proposition:

PROPOSITION 5. If $k_2=0$ then rank $M_2=n+1$ if and only if $r\geqslant n+1$, (A_a, B_a) is a controllable pair and $D_a \neq 0$.

5. Example

Let a model transfer function of a continuously working plant be given by:

$$G_c(s)=\frac{3s+1}{s(s+0.5)}\, v^{-2.4s} \qquad (5.1)$$

and the sampling time $T=3$. Find such modifying factors that the nominator of the S_T model transfer function is of the form:

$$B(z)=z^2+0.1z \qquad (5.2)$$

Using the algorithm given in previous chapter we have:

Step 1.

$$G_c(s)=\left(\frac{3}{s}+\frac{1}{s+0.5}\right) e^{-2.4s} \qquad (5.3)$$

so:

$$x(t)=\begin{bmatrix} 0, & 0 \\ 0, & -0.5 \end{bmatrix} \cdot x(t)+\begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t-2.4) \qquad (5.4)$$

$$y(t)=[1,\ 1] \cdot x(t) \qquad (5.5)$$

Step 2.

$$r=3 \tag{5.6}$$

Step 3.

$$T_a=1, \quad k=2, \quad T_s=0.4 \tag{5.7}$$

Step 4.

$$A_a=\begin{bmatrix}1, & 0\\ 0, & 0.607\end{bmatrix}, \quad B_a=\begin{bmatrix}2\\ -0.584\end{bmatrix}, \quad D_a=[0.682] \tag{5.8}$$

Step 5.

$$A_T=\begin{bmatrix}1, & 0\\ 0, & 0.223\end{bmatrix} \tag{5.9}$$

Step 6.

$$k_1=1, \quad k_2=0 \tag{5.10}$$

Step 7.

$$l=k_1=1 \tag{5.11}$$

Step 8.

$$C^T\,(z1_2-A_T)_{ad}=[z-0.223,\ z-1] \tag{5.12}$$

$$\det\,(z1_2-A_T)=z^2-1.223z+0.223 \tag{5.13}$$

solving:

$$[z-0.223,\ z-1,\ z^2-1.223z+0.223]\cdot\begin{bmatrix}B_T\\ D_T\end{bmatrix}=z^2+0.1z \tag{5.14}$$

we get:

$$B_T=\begin{bmatrix}1.416\\ -0.093\end{bmatrix} \quad D_T=[1] \tag{5.15}$$

Step 9.

$$M=M_2=\begin{bmatrix}2, & 2, & 2\\ -0.584, & -0.354, & -0.215\\ 0, & 0, & 0.682\end{bmatrix} \tag{5.16}$$

Step 10.

$$\begin{bmatrix}2, & 2, & 2\\ -0.584, & -0.354, & -0.215\\ 0, & 0, & 0.682\end{bmatrix}\cdot\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix}=\begin{bmatrix}1.416\\ -0.093\\ 1\end{bmatrix} \tag{5.17}$$

hence:

$$\alpha_1=2.58 \quad \alpha_2=-3.34 \quad \alpha_3=1.47 \tag{5.18}$$

Note that if $\alpha_1=\alpha_2=\alpha_3=1$ (classical extrapolator) then:

$$G_T\,(z)=\frac{0.682z^2+4.013z-0.033}{z^2-1.223z+0.223} \tag{5.19}$$

zeros of this transfer function are: $z_1=5.89$ and $z_2=0.00821$.

6. Conclusions

The method presented in the paper can be applied for a wide class of linear systems. Unlike other known methods for zero placement it does not apply zero cancellation. From a practical point of view this fact is very important.

It can be proved that the algorithm presented in the paper always fails when $T_0=kT$. In this case we should modify the method assuming deg $B(z)\leqslant n-1$ and taking $r\geqslant n$. The appropriate algorithm and calculations become much simpler then.

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Umiejscawianie zer układów dyskretnych za pomocą ekstrapolatora z niestacjonarnym wzmocnieniem

W pracy przedstawiono nową metodę umiejscawiania zer obiektów dyskretnych posiadających jedno wejście i jedno wyjście. Wykazano, że cel ten można osiągnąć wprowadzając do ekstrapolatora zerowego rzędu niestacjonarne wzmocnienie. Podano również algorytm do wyznaczenia tego wzmocnienia.

Размещение нулей дискретных систем с помощью экстраполятора с нестационарным усилением

В работе представлен новый метод размещения нулей дискретных объектов, имеющих один вход и один выход. Показано, что эту цель можно достичь вводя в экcраполятор нулевого порядка нестационарное усиление. Дается также алгоритм для определения этого усиления.