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Structure and logic in optimization

by

CONSTANTIN V. NEGOITA

City University of New York
695 Park Avenue
New York, NY 10021
USA

Optimization is viewed as a movement in an order structure. Described by a set and a function, there are two possibilities to speak about this movement. If the set is seen as an object and the function as a transformation, then exploring of a set means ranking its elements in time, and a two-valued logic is sufficient for observation. Multicriteria optimization needs another framework where the pair set-function is an object. Changing the world of sets for the world of their evaluations leads to partial membership and a continuous logic to perceive it. Turning from observation to representation means to recognize things by similarity instead of recognizing them by difference. A categorial language makes explicit this approach and the role of fuzzy sets.

1. Introduction

The interest in the foundation of fuzzy systems theory arose by difficulties encountered when faced with application of a conventional theory to multiciteria decision problems. Efforts to understand and resolve these problems led to the study of the foundations of fuzziness and its modeling.

2. Optimization means ordering

Since World War II there has been an explosive development in the quantitative methods of optimization, but not too many attempts to go to the fundamentals. The fuzzy set approach opened a door for such an enterprise.

In ordinary language, to optimize means to do the best one can under the circumstances. In order to refine this colloquial formulation into a mathematical statement, we must precisely define "best" and "under circumstances".

If we take the word "best" to mean either "maximal" or "minimal", then the problem is seriously simplified and the optimization process is represented as finding the maximal or the minimal value of some numerically valued function. In other words, an optimization problem consists of a set S and a function f such that S is in the domain of f and

$$f: S \rightarrow R$$

where R is the real line. The problem now is to find some or all the elements of S for which the function value is either maximum or minimum.

After making this very general definition we must state immediately that the solutions of very special instances of the optimization problem, in themselves, constitute entire branches of mathematics. A good example is linear programming, a subject on which many books have been written, and in which both the set S and the function f are severely restricted.

Approaching the optimization phenomena, the thought of the past years has systematically used the idea of natural order and by that means has obtained rules for dealing with matters of great practical utility.

Examining the meaning of the function f, one can see that the elements of the set S are mapped into numbers and in this way the order structure of the real line can be used to define the "best". The order structure of the real line is therefore an instrument of our perceiving apparatus. Through observation of its points only two directions can be perceived: ahead and behind, and these two directions are characterized by two relations: greater and lower. The points are perceived in time, moving along the real line, which is viewed as a space.

Underlying the optimization problem, there are two ideas: the idea of a space and the idea of a motion. This can be illustrated by the linear programming problem where the extreme points of a polyhedron representing some constraints are explored sequentially, and the exploration is guided by an objective function.

It is precisely on the perceiving apparatus that we must focus when speaking about the logic of optimization. Moving along the real line, we observe its elements. A two-valued logic to assess the existence of any element is quite sufficient. The path which opens immediately to us is that of applying the same philosophy when another structure is used to define "the best". In the following it will be shown that when time is considered with its historical integrity, a multivalued logic will be necessary to assess degrees of existence.

3. Any logic is internalized in a structure

Reformulating the standard optimization construction in the language of sets, one can say that the solution of any optimization problem is a subset of S as defined by the characteristic function

$$S \rightarrow \{0, 1\}$$

The internal membership structure is characterized externally by reference to connections with a special set $\{0, 1\}$ having only two elements. This correspondence between subset and characteristic function can be captured by the following diagram

$$\begin{array}{ccc} \text{solution} & \rightarrow & S \\ \downarrow & & \downarrow \\ \{0\} & \rightarrow & \{0, 1\} \end{array}$$

where the arrow $\{0\} \rightarrow \{0, 1\}$ gives the meaning of "true", and $\{0\}$ is a special set, having only one element, with the property that there is only one possible arrow to it.

Because the elements of the solution are uniquely linked with $\{0\}$, the diagram describes how elements in the solution are observed and selected from the elements of S according to a precise meaning of what is true. The set $\{0, 1\}$ has the role of a subset classifier, and its elements are the truth values of the logic of classification. If an element of S is in the solution, then the characteristic function takes the value 1; otherwise it takes the value 0.

There are only two elements in the subset classifier: zero (0)-"false" and one (1)-"true". "Truth" and "falsity" should not be taken in the literal or ordinary sense but in a more specific sense, generally, we dispense with the clumsy phraseology of "it is true" or "it is false" and say "equals 1" or "equals 0", respectively.

With these preambles, we can set down the problem of multicriteria optimization and see what happens if we face a problem defined by many classifications, i.e. by more than one ordering according to the real line.

Let us consider a family of such classifications (solutions) of which no two have any elements in common. That is, any two members of the family are sets that are disjoint. For each index $i \in I$, there is a set S_i that belongs to our collection. We can visualise these sets (solutions) as "sitting over" the index set I. A set over I is a fibre over i, and we have to consider a bundle of ordinary sets. The subbundle classifier is a bundle of set-classifiers $\{0, 1\} \times I$, and the classifier arrow can be thought as a bundle of copies of the set function "true".

An element of the bundle is a global section $I \rightarrow S$ picking one germ out of each fibre. Such things are called partial elements. We might regard a bundle as a "set--like" entity consisting of potentially existing (partially defined) elements.

Let e_1 and e_2 be two partial elements of a bundle and put

$$(e_1 = e_2) = \{i \in I,\ e_1(i) = e_2(i)\}$$

Then $(e_1 = e_2)$ being a subset of I is a truth value of the statement "$e_1 = e_2$" or, alternatively, a measure of the extent to which an element e_1 looks like an element e_2.

Emerging from this discussion is a generalized concept of a "set" as consisting of (partial) elements, with some Heyting algebra-valued measure of the degree of equality of these elements.

4. Some families of crisp sets can be modeled as fuzzy sets

Let us consider again the optimization problem

$$f: S \to R$$

and instead of observing and selecting the maximal (or minimal) values consider the representation of all the elements of S ordered with the reference to the extremal value of the function f. This can be achieved, for instance, by dividing all the values by the maximal (or minimal) value. This new ordering will be governed by the unit interval

$$S \to R \to [0, 1]$$

The new function fz:

$$S \to [0, 1]$$

is a fuzzy set, and according to the representation theorem it is equivalent to a family of level sets $(S_i)_{i \in [0,1]}$

$$S_i = \{s: fz(s) \geqslant i,\ i \in [0, 1],\ s \in S\}$$

$$S_i \supset S_j \Leftrightarrow j < i$$

For illustration, consider the fuzzy set given in the table

A	0.9	F	0.4
B	0.8	G	0.3
C	0.7	H	0.2
D	0.6	I	0.1
E	0.5	J	0.0

The level sets of this table are

A
$A\ B$
$A\ B\ C$
$A\ B\ C\ D$
$A\ B\ C\ D\ E$
$A\ B\ C\ D\ E\ F$
$A\ B\ C\ D\ E\ F\ G$
$A\ B\ C\ D\ E\ F\ G\ H$
$A\ B\ C\ D\ E\ F\ G\ H\ I$
$A\ B\ C\ D\ E\ F\ G\ H\ I\ J$

All the level sets are included in each other and form a „set-through-time"

$$\{A\} \subset \{A, B\} \supset \{A, B, C\} \subset \ldots$$

indexed by the membership degrees which we think of as the time of existence measured by the height of each column. The elements of the unit interval are, therefore, the truth values of an internalized logic in the category whose objects are sets-through-time.

Another possibility for looking at the level sets is to interpret the columns as sections of a bundle-like object indexed by the unit interval. Then, the heights of the columns can be measured by open sets in the unit interval, and the heights of the columns can represent the degree of equality between $A, B, \ldots, J$, or the distance between them.

No matter how we consider the family of level sets, the fact is that they are objects scattered in topoi, and in this way we can explain why we can use a non--two-valued logic to handle them according to what we call "fuzzy optimization".

5. Fuzzy optimization is knowledge engineering

The very core of the theory of topoi models a classification situation by allowing one to speak about partial elements. However, the theory can be difficult to use directly. Fortunately, the fuzzy sets modeling families of level sets offer a viable alternative. Handling degrees of membership instead of degrees of existence, they allow a precise model for subjective evaluations, and this is a turning point in the methodology of science.

Subjective means existing in the mind, belonging to the person thinking rather than to the object thought of. Any subjective evaluation is the result of a classification, conscious or not.

The notions of subjective evaluations and of fuzzy sets are not one and the same but rather have the relationship of goal and tool: having precisely manipulatable subjective evaluations is the goal, and fuzzy set theory is a tool to achieve the goal.

In any fuzzy programming model, knowledge is procedural, in the sense that it tells how the data for a problem can be manipulated in order to go about solving a problem. Fuzzy programming is logic programming. One can express knowledge in fuzzy programming in terms of either constraints or objective functions. The basic units of building constraints and objective functions are predications, i.e. expressions that say simple things about a universe of discourse. Predications are represented by fuzzy sets.

Once a set of constraints and objectives has been defined, one can use them to find information that can be deduced from those constraints and objectives. This is done by writing a query, an expression of the form

$$P_1 \text{ and } P_2 \text{ and } \ldots \text{ and } P_n\,?$$

where P_i are predicates represented as fuzzy sets. Inference take place automatically. The fact that answers are automatically extracted from data descriptions by a logic procedure results in a great degree of data independence. Articulation of a query to the knowledge base results in the generation of a knowledge tree. The answer procedure results in a reduction of the tree. The derivation of the knowledge tree about constraints and objectives is a forward process, while the evaluation process is a backward contraction process, a pullback in the structure of fuzzy sets.

In fuzzy optimization "the best" is viewed as a new evaluation in the structure of all evaluations, pulling back towards a synthesis. Any fuzzy optimization problem is an exploration of a knowledge tree. The constraints are partial sources of knowledge, or partial descriptions of a common universe of discourse — the universe of alternatives — and by conjunction a complete knowledge is derived.

This confirms the previously advanced supposition that our idea of movement is essentially composite, containing two notions: the space of the evaluations and the time to explore this space.

With the capacity of representing evaluations in addition to the ability of experiencing sets we get another dimension and land in knowledge engineering. In classical optimization logical operators are simply deduced from observation of facts and embrace nothing but the content of facts. They are no laws of thinking but merely laws of the external world as it is perceived by us, or laws of our relationship with the external world. As we have seen, logic is not something absolute, something existing outside and apart from us. Yet, in actual fact, it is merely the laws of the relations we have with the external world.

The first difference between the world of sets and the world of their evaluations is that the first is not general. It is a particular difference which counts. For mere observation, there exists no classification according to common properties. This element and that element are totally different things. Generally speaking, in the case of evaluations objects are recognized by their similarity while in the case of observation they are recognized by their difference.

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Struktura i logika w optymalizacji

Optymalizację rozpatruje się jako przemieszczenie w strukturze porządkowej. Przy pomocy zbiorów i funkcji można to przemieszczenie określić na dwa sposoby. Jeśli zbiór jest obiektem, zaś funkcja transformacją, to badanie zbioru oznacza porządkowanie jego elementów w czasie i logika dwuwartościowa jest do tego wystarczająca. Optymalizacja wielokryterialna wymaga innego podejścia, w którym para zbiór-funkcja staje się obiektem. Zmiana przestrzeni zbiorów na przestrzeń ich ocen prowadzi do częściowej przynależności i związanej z tym logiki ciągłej. Przejście od obserwacji przemieszczenia do reprezentacji pojmuje się jako oznaczające rozpoznawanie poprzez podobieństwo, a nie przez różnicę. Język kategorii wyjawia działanie takiego podejścia i rolę zbiorów rozmytych.

Структура и логика в оптимизации

Оптимизация рассматривается как перестановка в структуре упорядочения. С помощью множеств и функций можно эту перестановку определить двумя способами. Если множество является объектом, а функция — образованием, то исследование множества означает упорядочение его элементов во времени и двузначная логика является в этом случае достаточной. Многокритериальная оптимизация требует другого подхода, в котором пара множество-функция становится объектом. Преобразование пространства множеств в пространство их оценок ведет к частичной инцидентности и связанной с этим непрерывной логике. Переход от наблюдения перестановки к представлению понимается как распознавание посредством подобия, а не посредством различия. Язык категорий показывает суть такого подхода и роль нечетких множеств.