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Fuzzy-set-theoretic differences and inclusions and their use in the analysis of fuzzy equations*)

by

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A survey of fuzzy-set-theoretic operations is proposed with emphasis on set-differences and inclusions. General procedures for a systematic generation of such connectives are described. Their importance for a proper statement and analysis of equations involving fuzzy numbers is stressed. The study of such equations leads to the definition of new types of fuzzy arithmetics where errors (or imprecision) compensate. Such "optimistic" fuzzy arithmetics are based on inclusion (i.e. implication) connectives while the usual "pessimistic" fuzzy arithmetics are based on intersection connectives.

Keywords: fuzzy set, logical connective, fuzzy number, fuzzy equation.

Introduction

The problem of finding suitable representations of fuzzy-set-theoretic operations has been often debated. Significant progress along this line has been recently observed. The first part of this paper provides a structured presentation of multiple-valued logical connectives which correspond to the intersection, union, complementation, difference and inclusion of fuzzy sets. Systematic generation procedures are developed.

Already acknowledged as basic in natural language and approximate reasonning modeling techniques, these connectives shed also some light on fuzzy arithmetics as demonstrated in the second part of this paper. Namely, a general formulation of a solution of equations involving fuzzy numbers is proposed, based on the use of the introduced implication connectives.

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While fuzzy arithmetics, as studied in [4], extend the interval analysis [13] to fuzzy intervals, and allow for no compensation of imprecision, the study of fuzzy equations leads to another concept of fuzzy arithmetics where a maximal compensation of imprecision is allowed.

Although mainly concerned with theory, this paper contains results which are first steps towards finding fuzzy solutions to imprecisely specified equations which may model some optimization problems.

1. Differences and inclusions of fuzzy sets: a generation procedure

In this section we consider a set X and fuzzy sets $A, B, \ldots$ on X defined by membership functions $\mu_A, \mu_B \ldots$ from X to the unit interval, $I=[0, 1]$.

1.1. A note on, intersection the union and complementation

Given two fuzzy sets A and B on X, their intersection, denoted $A \cap^* B$ is pointwisely defined on the basis of the membership functions μ_A and μ_B by:

$$\forall x \in X, \ \mu_{A \cap^* B}(x) = \mu_A(x) * \mu_B(x) \tag{1}$$

where $*$ is a triangular norm. A triangular norm $*$(t-norm for short) [20] is a mapping from I^2 to I such that $\forall (a, b, c, d) \in I^4$, i) $a*b=b*a$, ii) $a*(b*c)=(a*b)*c$, iii) $0*0=0$; iv) $1*b$, v) if $a \leqslant b$ and $c \leqslant d$, then $a*c \leqslant b*d$. These axioms are meant to preserve the basic properties of set-intersection, i.e. commutativity (i), associativity (ii), $\emptyset \cap \emptyset = \emptyset$ (iii), $X \cap B = B$ (iv), a lack of monotonicity (v) would also be very much counterintuitive. The greatest triangular norm is $a*b=\min(a, b)$ and the least one is defined by

$$a*b=T_W(a, b)=\begin{cases} a \text{ if } b=1 \\ b \text{ if } a=1 \\ 0 \text{ otherwise} \end{cases}$$

The basic triangular norms are, in an increasing order

$$T_W(a, b) \leqslant T_m(a, b) = \max(0, a+b-1) \leqslant a \cdot b \leqslant \min(a, b) \tag{2}$$

The suitability of triangular norms as models of fuzzy set intersections was shown in [15], [2], [11], etc. Of course, they are proper models of fuzzy Cartesian products too [5].

Similarly, the union $A \cup^{\perp} B$ of the two fuzzy sets A and B is defined by (see [15], [2], [11] for justifications)

$$\forall x \in X, \ \mu_{A \cup^{\perp} B}(x) = \mu_A(x) \perp \mu_B(x) \tag{3}$$

where $\perp$ is a triangular conorm [20], i.e. a mapping from I^2 to I which satisfies the same axioms as a triangular norm except that iii) and iv) are changed into $1 \perp 1 = 1$ and $0 \perp b = b$, respectively. Note that while t-norms are semigroups of [0, 1] with identity 1, and absorbing element 0, conorms are semi-groups of [0, 1] with identity 0 and absorbing element 1.

The basic conorms are, in an increasing order:

$$\max(a, b) \leqslant a+b-ab \leqslant \min(1, a+b) \leqslant T_w^*(a, b) = \begin{cases} a \text{ if } b=0 \\ b \text{ if } a=0 \\ 1 \text{ otherwise.} \end{cases} \tag{4}$$

A continuous triangular co-norm $\perp$, such that vi) $\forall a \in (0, 1)$, $a \perp a > a$, is said to be Archimedean. It can be expressed in terms of a generator φ which is a continuous strictly increasing function from I to $[0, +\infty)$ such that $\varphi(0)=0$, as, [12]:

$$a \perp b = \varphi*(\varphi(a)+\varphi(b)) \tag{5}$$

where φ^* is the pseudo-inverse of φ, defined by

$$\varphi*(a) = \begin{cases} \varphi^{-1}(a) \text{ if } a \in [0, \varphi(1)] \\ 1 \text{ if } a \in [\varphi(1), +\infty) \end{cases}$$

A similar result holds for continuous Archimedean t-norms $(a*a<a)$.

The complement A^c of a fuzzy set A is defined by

$$\forall x \in X,\ \mu_A\, c(x) = c(\mu_A(x)) \tag{6}$$

where c is a strong negation, see [21]. A strong negation is a continuous strictly decreasing function c from I to I, such that i) $c(0)=1$, ii) $c(c(a))=a$ (involution). For instance, $c(a)=1-a$ is a strong negation. Any strong negation c can be expressed by means of a continuous strictly increasing mapping t from I to $[0, +\infty]$, such that $t(0)=0$ and $t(1)$ is finite, as, [21]

$$c(a) = t^{-1}(t(1)-t(a)) \tag{7}$$

With a triangular norm $*$ and a strong negation c, we can associate a triangular co-norm defined by

$$a \perp b = c(c(a) * c(b)) \tag{8}$$

Then, we have $a*b = c(c(a)) \perp (c(b))$, and $*$ and $\perp$ are said to be c-dual [1]; and to be just for $c(a)=1-a$.

The main families of intersections and union of fuzzy sets are obtained from the following classes of t-norms and t-co-norms.

a) Strict operations

They are continuous, Archimedean, strictly increasing t-norms (resp. t-co-norms). They are isomorphic to the product $a \cdot b$ (resp. $a+b-ab$). Strict co-norms can be represented by (5) with $\varphi(1)=+\infty$; then it is clear that $\varphi^*=\varphi^{-1}$ holds. Any c-dual of a strict t-co-norm is a strict t-norm and conversely. Such a strict t-norm can be represented by (5), changing φ into $f=\varphi \circ c$. Intersections and unions of this family are neither idempotent nor mutually distributive, nor are the excluded middle and non-contradiction laws valid for any choice of a complementation.

b) Nilpotent operations

They are continuous, Archimedean t-norms (resp: t-co-norms) such that for any sequence $(a_n)_{n\in N}$ of numbers in (0, 1),

$$\exists n_0,\ a_1 * a_2 * \ldots * a_{n_0} = 0$$

$$(\text{resp.: } a_1 \perp a_2 \perp \ldots \perp a_{n_0} = 1)$$

Nilpotent t-co-norms (or : t-norms) are isomorphic to the bounded sum $\min(1, a+b)$ (or : I_m). They can be represented by (5) where φ is such that $\varphi(1) < +\infty$. Note that φ also generates a strong negation c_φ, by changing t into φ in (7). The c_φ-dual of $*$ is a nilpotent t-norm generated by (5), changing φ into $\varphi(1)-\varphi$. Intersection and union operations generated by c_φ-dual t-norms and t-co-norms satisfy the excluded middle and non-contradiction laws, if the complementation is based on c_φ. But they are neither idempotent nor mutually distributive. For $* = T_m$, $\perp =$ $=$bounded sum, $c_\varphi(a) = 1-a$.

c) Idempotent operations $(A \cup A) = A = (A \cap A)$

The only idempotent fuzzy set union and intersection are "max" and "min", respectively. Then all usual properties of union and intersection are recovered (for any choice of c) except for the excluded middle and non-contradiction laws. Note that max and min cannot be represented by (5).

1.2. Set differences and inclusions: a first construction

In the classical set theory the set difference is defined by

$$\forall A \in 2^X,\ \forall B \in 2^X,\ A - B = A \cap B^c \tag{9}$$

where B^c denotes the complement of B in X. The complementary operation $(A-B)^c =$ $= A^c \cup B$ is directly related to the set inclusion since

$$A \subseteq B \Leftrightarrow A^c \cup B = X. \tag{10}$$

Turning to fuzzy sets, given a strong negation c and a triangular norm $*$, we can define

$$\forall x \in X,\ \mu_{A-B}(x) = \mu_A(x) * c(\mu_B(x)) \tag{11}$$

Fuzzy set inclusions can be defined by generalizing (10), using a strong negation c and a triangular co-norm $\perp$, if we define:

$$A \subseteq B \Leftrightarrow \forall x \in X,\ c(\mu_A(x)) \perp \mu_B(x) = 1 \tag{12}$$

When $*$ and $\perp$ are c-dual, we have $(A-B)^c = A^c \cup^\perp B$.

Table 1 gives the operations which correspond to $A-B$ and to $A^c \cup^\perp B$ for basic triangular norms and co-norms, c being defined by $c(a) = 1-a$. The derivation of more general families based on strict or nilpotent operations, according to (11) or (12) is left to the reader (see [3, 7]).

Table 1. Differences and inclusions ($\mu_A(x)=a$, $\mu_B(x)=b$).

$A \cap^* B$	$A-B$	$A \cup^{\perp} B$	$A^c \cup^{\perp} B$
$\min(a, b)$	$\min(a, 1-b)$	$\max(a, b)$	$\max(1-a, b)$ (Dienes)
a, b	$a-a.b$	$a+b-ab$	$1-a+ab$ (Reichenbach)
$\max(0, a+b-1)$	$\max(0, a-b)$ (bounded difference, Zadeh)	$\min(1, a+b)$	$\min(1, 1-a+b)$ (Łukasiewicz)

On the right side of table 1 we recognize several implication functions used in multi-valued logic.

It can be checked that with definition (12) for the inclusion, we have:

- If the triangular co-norm $\perp$ is nilpotent and has the same generator as the strong negation c, then

$$A \subseteq B \Leftrightarrow \forall x \in X,\ \mu_A(x) \leqslant \mu_B(x) \tag{13}$$

which is the usual definition of inclusion in the fuzzy set theory, originally proposed by Zadeh.

- If $\perp=\max$ or if $\perp$ is a strict triangular co-norm, whatever c is, then (12) yields

$$\begin{aligned} A \subseteq B &\Leftrightarrow \forall x \in X,\ \mu_A(x)=0 \text{ or } \mu_B(x)=1 \\ &\Leftrightarrow \operatorname{supp} A \subseteq \operatorname{core} B \end{aligned} \tag{14}$$

where $\operatorname{supp} A=\{x \in X,\ \mu_A(x)>0\}$ and $\operatorname{core} B=\{x \in X,\ \mu_B(x)=1\}$. (14) still holds if we use in (12) a triangular co-norm $\perp$ such that $\forall a, \forall b$, $\max(a, b) \leqslant a \perp b \leqslant a \perp' b$ where $\perp'$ is a strict triangular co-norm. (14) corresponds to a very strong form of the inclusion.

REMARK 1. We may think of relaxing (14) by choosing a threshold a, such that

$$A \subseteq_\alpha B \Leftrightarrow \forall x \in X,\ \mu_A c \cup^{\perp} \mu_B(x) \geqslant \alpha \tag{15}$$

where $\perp$ is max or a strict triangular co-norm and c a strong negation.

However, $\subseteq_\alpha$ is generally either not transitive, or not consistent with the usual definition of inclusion (13). For a discussion on this point, see [4, 7]. A slight modification of (15), for $\alpha=\frac{1}{2}$, $\perp=\max$, $c(a)=1-a$, ensures transitivity and consistency (i.e. $A \subseteq B \Rightarrow A \subseteq_{\frac{1}{2}} B$), by stating [(4, p. 22]):

$$A \subseteq_{\frac{1}{2}} B \Leftrightarrow \forall x \in X,\ \mu_A(x) \leqslant \tfrac{1}{2} \text{ or } \mu_B(x) > \tfrac{1}{2} \tag{16}$$

N.B.2. Symmetrical differences can be defined as $(A-B) \cup^{\perp} (B-A)$, among other possibilities. See [6] for a preliminary discussion.

1.3. Set differences: a second construction

In this section we take the standard definition of inclusion, i.e. (13), for granted, and build set differences on this basis.

In the classical set theory the set difference $A-B$ can also be defined as the smallest S such that $B\cup S$ contains A, i.e.

$$A-B=\bigcap\{S, A\subseteq B\cup S\} \tag{17}$$

The complement operation $(A-B)^c$ can also be similarly defined by

$$(A-B)^c=\bigcup\{S, A\cap S\subseteq B\} \tag{18}$$

Both formulae can be readily extended to fuzzy sets taking (13) as a definition of inclusion and choosing a triangular co-norm $\perp$ and a triangular norm $*$ for extending $B\cup S$ in (17) and $A\cap S$ in (18), respectively, hence we get

$$\forall x\in X,\ \mu_{A-B}(x)=\inf\{s\in[0,1],\ \mu_B(x)\perp s\geqslant\mu_A(x)\} \tag{19}$$

and

$$\forall x\in X,\ \mu_{A\to B}(x)=\sup\{s\in[0,1],\ \mu_A(x)*s\leqslant\mu_B(x)\} \tag{20}$$

where s represents $\mu_S(x)$; inf and sup are used for extending $\bigcap S$ in (17) and $\bigcup S$ in (18) respectively, since min and max are the only triangular norm and co-norm which are idempotent and can easily be extended to a possibly non-finite number of arguments. (19) and (20) are usual in Brouwerian and dual Brouwerian lattices, and define relative pseudo-complements (e.g., see Sanchez [18]). (19) and (20) are symbolically written as

$$\forall x\in X,\ \mu_{A-B}(x)=\mu_A(x)\overset{\perp}{<}\mu_B(x) \tag{21}$$

and

$$\forall x\in X,\ \mu_{A\to B}(x)=\mu_A(x)*\!\!\to\mu_B(x) \tag{22}$$

Table 2 gives the analytic expressions of $\overset{\perp}{<}$ and $*\!\!\to$ (which are mappings from I^2 to I), for the basic triangular norms and co-norms.

In case of triangular norms or co-norms which are not continuous (e.g. T_w or T_w^*), it may occur that the greatest lower bound in (19) or the least upper bound in (20) does not belong to the set defined between brackets. See also Pedrycz [14] for the operation $\overset{\perp}{<}$ and Pedrycz [14], Prade [16], Dubois, Prade [4] for the operation $*\!\!\to$. The operation $*\!\!\to$ corresponds to implication functions which are encountered in multi-valued logics.

A noticeable result is that T_m and the bounded sum yield the same set difference and inclusion in tables 1 and 2 (i.e. Zadeh's bounded difference and Łukasiewicz's implication). This remark still applies to operations generated from c_φ-dual t-norms and t-co-norms, where the complementation is based on c in (11) and (12) (see [3, 7]).

Table 2a. Differences of the second kind

$a \perp b$	$a \overset{\perp}{-} b$
$\max(a, b)$	$\begin{cases} 0 \text{ if } a \leqslant b \text{ (Sanchez [18])} \\ a \text{ if } a > b \end{cases}$
$a+b-a \cdot b$	$\begin{cases} 0 \text{ if } b=1 \\ \max\left(0, \dfrac{a-b}{1-b}\right) \text{ if } b \neq 1 \end{cases}$
$\min(1, a+b)$	$\max(0, a-b)$ (Zadeh [22])
$T_w^*(a, b)$	$\begin{cases} 0 \text{ if } b > 0 \\ a \text{ if } b = 0 \end{cases}$

Table 2b. Inclusions of the second kind

$a * b$	$a \overset{*}{\rightarrow} b$
$\min(a, b)$	$\begin{cases} 1 \text{ if } a \leqslant b \text{ (Gödel, see} \\ b \text{ if } a > b \text{ Sanchez [18])} \end{cases}$
$a \cdot b$	$\begin{cases} 1 \text{ if } a = 0 \text{ (Goguen [10])} \\ \min(1, b/a) \text{ if } a \neq 0 \end{cases}$
$\max(0, a+b-1)$	$\min(1, 1-a+b)$ (Łukasiewicz)
$T_w(a, b)$	$\begin{cases} 1 \text{ if } a < 1 \\ b \text{ if } a = 1 \end{cases}$

REMARK 1. It can be checked that $(A-B) \cup^{\perp} B \subseteq A \cup^{\perp} B$ where $A-B$ is issued from $\perp$ in the sense of (19) and where $\subseteq$ is defined by (13); the equality holds for $\perp = \max$.

REMARK 2. Since (17) and (18), respectively, yield

$$A^c = \bigcap \{S, X = A \cup S\} \tag{23}$$

$$A^c = \bigcup \{S, A \cap S = \emptyset\} \tag{24}$$

as particular cases in the classical set theory, we can use (19) and (20) for defining fuzzy set complementation operations without the t-norms or *t*-co-norms. New operations are obtained, which are generally not involutive. However, nilpotent intersections and unions yield back strong negations introduced earlier. See [7].

1.4. Dual set-differences

Let c be a strong negation and let $(*, \perp)$ be a pair of a c-dual triangular norm and co-norm. Then it can be easily checked that

$$(A-B)^c = B^c \rightarrow A^c \tag{25}$$

$$(A \rightarrow B)^c = B^c - A^c \tag{26}$$

where $A-B$ is defined by (19) and $A \rightarrow B$ by (20).

However, the usual identities of set theory:

$$A-B=B^c-A^c \tag{27}$$

$$A\to B=B^c\to A^c \tag{28}$$

no longer hold generally for set-differences and inclusions introduced in 1.3. More specifically, (27) and (28) never hold for the following choices of generative connectives: $(*, \perp)=(\min, \max)$, (product, $a+b-ab$) and any pair of c-dual strict operations (see [7]). Hence a new family of set differences and inclusions can be obtained under the form $c(\mu_B) \not\leqslant c(\mu_A)$ and $c(\mu_B) \overset{*}{\to} c(\mu_A)$ instead of (21) and (22) respectively.

(27) and (28) are valid for set differences and inclusions built from c_φ-dual nilpotent operations, with $c=c_\varphi$ in (27) and (28). This is true especially for $c(a)=1-a$, $a*b=\max(0, a+b-1)$, $a\perp b=\min(1, a+b)$. Table 3 gives the expressions of set-differences an inclusions obtained from basic t-norms and t-co-norms other than the nilpotent ones, and for $c(a)=1-a$.

Table 3. Dual set-differences and inclusions

$a*b$	$a\perp b$	set difference operations $(1-b)\not\leqslant(1-a)$	implication operations $(1-b)\overset{*}{\to}(1-a)$
$\min(a, b)$	$\max(a, b)$	$\begin{cases} 0 \text{ if } a\leqslant b \\ 1-b \text{ if } a>b \end{cases}$	$\begin{cases} 1 \text{ if } a\leqslant b \\ 1-a \text{ if } a>b \end{cases}$
$a\cdot b$	$a+b-ab$	$\begin{cases} 0 \text{ if } a=0 \\ \max\left(0, \dfrac{a-b}{a}\right) \text{ if } a\neq 0 \end{cases}$	$\begin{cases} 1 \text{ if } b=1 \\ \min\left(1, \dfrac{1-a}{1-b}\right) \text{ if } b\neq 1 \end{cases}$
$T_w(a, b)$	$T_w^*(a, b)$	$\begin{cases} 0 \text{ if } a<1 \\ 1-b \text{ if } a=1 \end{cases}$	$\begin{cases} 1 \text{ if } b>0 \\ 1-a \text{ if } b=0 \end{cases}$

1.5. Pseudo-intersections and unions

Since we have in the classical set theory

$$A\cap B=A-B^c \tag{29}$$

$$A\cup B=A^c\to B \tag{30}$$

where $\to$ denotes a set operation corresponding to the implication connective, we may think of extending (29) and (30) to fuzzy sets in order to define the intersection and the union from a set difference or an implication operation and a strong negation. It is clear that, in the framework of the first construction (see 1.2), the application of (29) and (30) just yields the intersection and the union operations which generate the set difference and the operation $\to$. However, when $A-B$ and $A\to B$ are defined by (19) and (20) (or also by the dual approach, as in table 3), new intersection-like or union-like operations can be obtained by applying (29) and (30). The results of this procedure are given in table 4, for basic t-norms and

Table 4. Pseudo-intersections and unions

triangular co-norm from which $\mathrel{\underset{<}{\leftarrow}}$ is issued	Pseudo-intersections defined from table 2 as $a \mathrel{\underset{<}{\leftarrow}} (1-b)$	Pseudo-intersections defined from table 3 as $b \mathrel{\underset{<}{\leftarrow}} (1-a)$
$\max(a, b)$	$\begin{cases} 0 \text{ if } a+b \leqslant 1 \\ a \text{ if } a+b > 1 \end{cases}$	$\begin{cases} 0 \text{ if } a+b \leqslant 1 \\ b \text{ if } a+b > 1 \end{cases}$
$a+b-a \cdot b$	$\begin{cases} 0 \text{ if } b=0 \\ \max\left(0, \dfrac{a+b-1}{b}\right) \text{ if } b \neq 0 \end{cases}$	$\begin{cases} 0 \text{ if } a=0 \\ \max\left(0, \dfrac{a+b-1}{a}\right) \text{ if } a \neq 0 \end{cases}$
$T_w^*(a, b)$	$\begin{cases} 0 \text{ if } b<1 \\ a \text{ if } b=1 \end{cases}$	$\begin{cases} 0 \text{ if } a<1 \\ 0 \text{ if } a<1 \end{cases}$

triangular norm from which $*\!\!\rightarrow$ is issued	Pseudo-unions defined from table 2 as $(1-a) *\!\!\rightarrow b$	Pseudo-unions defined from table 3 as $(1-b) *\!\!\rightarrow a$
$\min(a, b)$	$\begin{cases} 1 \text{ if } a+b \geqslant 1 \\ b \text{ if } a+b < 1 \end{cases}$	$\begin{cases} 1 \text{ if } a+b \geqslant 1 \\ a \text{ if } a+b < 1 \end{cases}$
$a \cdot b$	$\begin{cases} 1 \text{ if } a=1 \\ \min\left(1, \dfrac{b}{1-a}\right) \text{ if } a \neq 1 \end{cases}$	$\begin{cases} 1 \text{ if } b=1 \\ \min\left(1, \dfrac{a}{1-b}\right) \text{ if } b \neq 1 \end{cases}$
$T_w(a, b)$	$\begin{cases} 1 \text{ if } a>0 \\ b \text{ if } a=0 \end{cases}$	$\begin{cases} 1 \text{ if } b>0 \\ a \text{ if } b=0 \end{cases}$

t-co-norms, except nilpotent operations which are self-generated through (29) and (30) (and $c=c_\varphi$ where φ is the common generator).

Note that the operations which appear in table 4 are not commutative, and that is why we call them pseudo-intersections and pseudo-unions. Nevertheless, starting from a "symmetrized" form of the set difference operation, i.e. $s\left(a \mathrel{\underset{<}{\leftarrow}} b, (1-b) \mathrel{\underset{<}{\leftarrow}} (1-a)\right)$ and from a "symmetrized" form of the implication operation, i.e. $s\left(a *\!\!\rightarrow b, (1-b) *\!\!\rightarrow (1-a)\right)$, where s is an operation such as min or max for instance, then we obtain the commutative operations $s\left(a \mathrel{\underset{<}{\leftarrow}} (1-b), b \mathrel{\underset{<}{\leftarrow}} (1-a)\right)$ and $s\left((1-a) *\!\!\rightarrow b, (1-b) *\!\!\rightarrow a\right)$, which respectively coincide with the binary conjunction and the binary disjunction when $(a, b) \in \{0, 1\}^2$. It is worth noticing that the pseudo-intersections and the pseudo-unions of table 4 satisfy De Morgan's law, since $1-\left(a \mathrel{\underset{<}{\leftarrow}} (1-b)\right) = b *\!\!\rightarrow (1-a)$ from (25) and $1-\left((1-a) *\!\!\rightarrow b\right) = (1-b) \mathrel{\underset{<}{\leftarrow}} a$ from (26). Besides, the operations of table 4 are not always associative:

e.g. $\begin{cases} 0 \text{ if } a+b \leqslant 1 \\ a \text{ if } a+b > 1 \end{cases}$ is not associative while $\begin{cases} 0 \text{ if } b<1 \\ a \text{ if } b=1 \end{cases}$ is associative. A pseudo-intersection such as $\begin{cases} 0 \text{ if } a+b \leqslant 1 \\ a \text{ if } a+b > 1 \end{cases}$ would define an "intersection" made of the elements of A which sufficiently belong to B with respect to their membership degree in A, where $a=\mu_A(x)$ and $b=\mu_B(x)$.

1.6. The generation process is closed

In section 1.5 we obtained some new intersections and unions. From them, using the first or the second construction, it is possible to define set difference and implication operations. Surprisingly, we do not get new operations, as is indicated now.

Let us define the following transformations, for any 2-place operation $\odot$ on I:
a) $\mathscr{J}_c(\odot)$ is a 2-place operation on I such that $a\mathscr{J}_c(\odot)\,b=c\,(a\odot c\,(b))$. For $c(a)=$ $=1-a$, $\mathscr{J}_c$ is denoted $\mathscr{J}$, for short. Note that $\mathscr{J}_c$ is involutive ($\mathscr{J}_c\circ\mathscr{J}_c$=identity).
b) $\mathscr{C}(\odot)$ is a 2-place operation on I such that

$$a\mathscr{C}(\odot)\,b=\sup\{s,\ s\in[0,1],\ a\odot s\leqslant b\}$$

Then the following diagram holds for the t-norm min, or any continuous Archimedean t-norm (cf (5)), and any strong negation c: Notice that the second construction of implications applied to pseudo-intersections yields implications of the

$$\begin{tikzcd}
\text{intersection} \quad a * b \arrow[rr, "C"'] \arrow[d, shift left, "\mathscr{J}_c"] & & a \overset{*}{\rightarrow} b \ \text{implication (second kind)} \arrow[d, shift left, "\mathscr{J}_c"] \\
\text{implication (first kind)} \quad a \rightarrow b \arrow[u, shift left, "\mathscr{J}_c"] & & a * b = b \leftarrow c\,(a) \ \text{pseudo-intersection} \arrow[ll, "C"] \arrow[u, shift left, "\mathscr{J}_c"]
\end{tikzcd}$$

Fig. 1.

first kind: $\mathscr{J}_c(*)=\mathscr{C}\circ\mathscr{J}_c\circ\mathscr{C}(*)$, and do not produce new operations. For a nilpotent intersection and a strong negation having the same generator, we have $\mathscr{C}\circ\mathscr{J}_c$=id., i.e. $\wedge^*=*$. Lastly, considering a dual implication $a \overset{*}{\Rightarrow} b \triangleq c\,(b) \overset{*}{\rightarrow} c\,(a)$, generated from t-norms. we have, under the same assumptions

$$\mathscr{J}\,(a \overset{*}{\leftarrow} c\,(b))=c\,(a \overset{*}{\leftarrow} b)=\mathscr{C}\,(a \overset{*}{\leftarrow} c\,(b))=a \overset{*}{\Rightarrow} b$$

where

$$a\odot b=a \overset{*}{\leftarrow} c\,(b)=b \wedge^* a \text{ (the other pseudo-intersection)}$$

From a nilpotent intersection and a strong negation based on the same generator we can derive: $a \overset{*}{\Rightarrow} b=a \overset{*}{\rightarrow} b=a\Rightarrow b$.

Proofs of these statements are easily obtained by a direct check for min and T_w, and using the functional representations (5) and (7) in other cases *).

2. Fuzzy equations and "optimistic" fuzzy arithmetics

Using Zadeh [22]'s extension principle, it is possible to introduce fuzzy quantities in arithmetic-algebraic expressions, and to perform calculations with them. A theory of fuzzy numbers, which parallels that of random variables, is now quite

*) Details of proofs can be found in Dubois, D., Prade, H. A theorem on implication functions defined from triangular norms. BUSEFAL n° 18, L.S.I., Univ. P. Sabatier, Toulouse, April 1984.

well-developed and efficient methods for practical calculation exist (see [4], and [3, chap. 4] for an up-to-date survey). Fuzzy arithmetics, in its standard form, involves error interval analysis, [13], i.e. generalizes best and worst case calculation. In that sense, usual fuzzy arithmetics is "pessimistic". This section presents the "optimistic" approach to dealing with fuzzy quantities and relates it to solving equations involving fuzzy parameters.

2.1. Minkowski operations on crisp sets

Here we first consider the case of crisp, albeit imprecise quantities. Let A and B be two subsets of X. X is closed under some operation denoted by '$\cdot$'. The quantity $A\,(\cdot)\,B$ is defined by

$$A\,(\cdot)\,B=\{z, \exists x\in A,\ \exists y\in B,\ z=x\cdot y\} \tag{31}$$

Denoting $A\cdot y\triangleq\{x\cdot y\,|\,x\in A\}$, (31) can be expressed as

$$A\,(\cdot)\,B=\bigcup\{A\cdot y\,|\,y\in B\} \tag{32}$$

(31) yields interval analysis when A and B are real intervals. $A\,(\cdot)\,B$ is the set of elements which may be attained by $A\cdot y$, when $y\in B$. It is then natural to consider the set of elements which are contained in $A\cdot y$, for any choice of y in B, i.e.

$$A)\cdot(B\triangleq\bigcap\{A\cdot y\,|\,y\in B\} \tag{33}$$

$$=\{z\,|\,\forall y\in B,\ \exists x\in A,\ x\cdot y=z\} \tag{34}$$

Operations $(\cdot)$ and $)\cdot($ have been first studied by Minkowski, when $\cdot$ is the usual addition. $)+($ is usually called "subtraction" although both operations extend the addition to set-valued arguments. The following properties hold

i) $A)\cdot(B\subseteq A\,(\cdot)\,B$
i.e. $)\cdot($ gives more precise results than $(\cdot)$

ii) $[A)\cdot(B]^c\supseteq A^c\,(\cdot)\,B$ when $\cdot$ is left-reducible $(x\cdot y=x'\cdot y\Rightarrow x=x')$
If, moreover, $\forall z$, $\forall y$, $\exists x$, $x\cdot y=z$, i.e. $\forall y$, $X\cdot y=X$, then $[A)\cdot(B]^c=A^c\,(\cdot)\,B$.

iii) if $(X,\cdot)$ has a group structure, denoting $B^{-1}\triangleq\{y^{-1}|y\in B\}$ where y^{-1} is the inverse of y, and $z\cdot B^{-1}=\{z\cdot y^{-1}, y\cdot B\}$, we have
$A\,(\cdot)\,B=\{z, (z\cdot B^{-1})\cap A\neq\emptyset\}$; $A)\cdot(B=\{z\,|\,(z\cdot B^{-1})\subseteq A\}$

iv) If $\cdot$ is commutative, then $(\cdot)$ is also commutative but $)\cdot($ is not.

Assume now that A and B are closed intervals $[a, a']$, and $[b, b']$ of the real line. The operation $\cdot$ is supposed to be continuous, isotonic $(x\leqslant x', y\leqslant y'\Rightarrow x\cdot y\leqslant x'\cdot y')$, and provides a subset X of the real line, containing A and B, with a group structure. Under such assumptions [3]:

$$A\,(\cdot)\,B=[a\cdot b,\ a'\cdot b']$$

$$A\,)\cdot(\,B=\begin{cases}[a\cdot b',\ a'\cdot b] & \text{if } a\cdot b'\leqslant a'\cdot b\\ \emptyset & \text{otherwise}\end{cases}$$

If $\cdot$ is the usual addition, note that length $(A\ (+)\ B=$length $(A)+$length (B), length $(A\ )+(\ B)=$length (A) — length (B). Hence $(+)$ provides pessimistic results (no compensation of imprecision) while $)+($ provides optimistic results (maximal compensation of imprecision). Note that $(+)$ is only a semi-group $(A\ (+)-A\neq 0)$ while $A\ )+(\ -A=0$, where $-A=[-a', -a]$.

Considering the equation with imprecise coefficients

$$S\,(\cdot)\,A=B \tag{35}$$

The greatest set $\hat{S}$ such that (35) holds for $S=\hat{S}$ is, when non-empty,

$$\begin{aligned}\hat{S}&=\{x\mid \forall z\in B,\ \exists y\in A,\ x\cdot y=z\}\\ &=\{x\mid x\cdot A\subseteq B\}\end{aligned} \tag{36}$$

When $\cdot$ is a group operation, then (36) can be written $S=B\,)\cdot(\,A^{-1}$ following (34). Hence "optimistic" operations are useful for solving imprecisely specified equations.

2.2. Equations involving fuzzy numbers [3, 7, 8]

Any operation $\cdot$ on the real line can be extended to fuzzy-valued operands, A and B, fuzzy set of numbers, in the spirit of (31) (see [4, 5]. $A\,(\cdot)\,B$ is a fuzzy quantity with membership function:

$$\mu_{A\,(\cdot)\,B}\,(z)=\begin{cases}\sup\,\{\mu_A\,(x)*\mu_B\,(y)\mid x\cdot y=z\}\\ 0 \text{ if not } \exists\,(x, y),\ x\cdot y=z\end{cases} \tag{37}$$

where the triangular norm $*$, supposedly continuous, models a fuzzy Cartesian product. Indeed, (37) yields (31) when A and B are crisp sets of numbers, and generalizes error interval analysis [13]. Let $x\cdot B$ be the fuzzy quantity defined by

$$\forall\, x\in \boldsymbol{R},\qquad \mu_{x\circ B}\,(z)=\begin{cases}\sup\,\{\mu_B\,(y)\mid x\cdot y=z\}\\ 0 \text{ if not } \exists\, y,\ x\cdot y=z\end{cases} \tag{38}$$

which is a particularization of (37), then (37) may read

$$\mu_{A\,(\cdot)\,B}\,(z)=\sup\,\{\mu_A\,(x)*\mu_{x\cdot B}\,(z)\mid x\in \boldsymbol{R}\} \tag{39}$$

What is obtained is an extension of (32).

If $\cdot$ is a group operation on a subset of R containing the supports of A and B, then for $*$, being continuous or not, (37) may also read

$$\mu_{A\,(\cdot)\,B}\,(z)=\sup\,\{\mu_A\,(z\cdot y^{-1})*\mu_B\,(y)\mid y\in \boldsymbol{R}\} \tag{40}$$

Consider now the equation in S:

$$S\,(\cdot)\,A=B \tag{41}$$

It is a natural generalization of imprecisely specified equations (35) to fuzzy coefficients. Defining a fuzzy relation RA by $\mu_{RA}\,(x, z)=\mu_{x\cdot A}\,(z)$, $\forall x, z$, this equation is actually equivalent to the fuzzy relational equation

$$S\circ RA=B \tag{42}$$

Where $\circ$ denotes *a* sup-$*$ composition as in (39). It is easy to figure out that if S_1 and S_2 are solutions of (41), then $S_1 \cup S_2$ (in the sense of maximum) is also a solution, i.e. if (41) has a solution, then it has a greatest solution in the sense of Zadeh's inclusion. Greatest solutions of fuzzy relational equations have been found by Sanchez [18] ($*=\min$) and Pedrycz [14]. Applying their results, the greatest solution $\hat{S}$ of (41), when it exists, is defined by the extension of (36):

$$\forall x, \mu_{\hat{S}}(x)=\inf\{\mu_{x\cdot A}(z) \ast\!\!\rightarrow \mu_B(z) \mid z \in \boldsymbol{R}\} \tag{43}$$

provided that $*$ is continuous. $\ast\!\!\rightarrow$ is the implication operation $\mathscr{C}(*)$ associated with $*$, introduced in 1.3. eqn. (20). If $\cdot$ is a group operation, then (43) can be expressed as

$$\hat{S}=B\,)\cdot(\,A^{-1} \tag{44}$$

provided $A\,)\cdot(\,B$ is defined by

$$\begin{aligned}\mu_{A\,).(\,B}(z)&=\inf\{\mu_{z\cdot B}-1\,(x) \ast\!\!\rightarrow \mu_A(x) \mid x \in \boldsymbol{R}\}\\&=\inf\{\mu_B(y) \ast\!\!\rightarrow \mu_A(x) \mid x\cdot y=z\}\end{aligned} \tag{45}$$

where $\mu_{B^{-1}}(x)=\mu_B(x^{-1})$. Note that (45) reduces to (34) when A and B are crisp. Thus a canonical definition of "optimistic" operation on fuzzy quantities requires the use of implications introduced earlier.

In the following, A and B are supposed to be fuzzy intervals, i.e. convex ($\forall\alpha$, $A_\alpha=\{x \mid \mu_A(x)\geqslant\alpha\}$ is convex), normalized ($\exists x$, $\mu_A(x)=1$) fuzzy quantities with upper-semi-continuous membership functions. Closed intervals belong to this class. Also, operation '$\cdot$' is supposed to be continuous and isotonic.

If $B\,)\cdot(\,(A^{-1})=\emptyset$, the equation $S\,(\cdot)\,A=B$ has no solution; however, it can be checked that we may have $B)\cdot(\,(A^{-1})\neq\emptyset$ while the equation $S\,(\cdot)\,A=B$ has no solution. Indeed, it is clear that $\hat{S}$ given by (44) is normalized if and only if

$$\exists x \in \boldsymbol{R},\ \ \forall y \in \boldsymbol{R},\ \ \mu_A(y)\leqslant\mu_B(x\cdot y). \tag{46}$$

since $a \ast\!\!\rightarrow b=1 \Leftrightarrow a\leqslant b$. Thus (46) is a necessary condition for the existence of a solution for $S\,(\cdot)\,A=B$, since S must be normalized, A and B being supposed to be normalized (as it can be easily proved from (37)). (46) is not a sufficient condition. Indeed if A and B are fuzzy intervals, then if $*=\min$,

$$S\,(\cdot)\,A=B \Leftrightarrow \forall\alpha \in (0, 1],\ S_\alpha\,(\cdot)\,A_\alpha=B_\alpha \tag{47}$$

i.e. a fuzzy equation is equivalent to an infinite set of crisp-set equations. (46) ensures that $\forall\alpha$, $\exists\hat{S}_\alpha$ solving $S_\alpha\,(\cdot)\,A_\alpha=B_\alpha$. However, the $\hat{S}_\alpha$, are not necessarily α-cuts of a fuzzy set, since the monotonicity condition $\alpha\leqslant\beta\Rightarrow\hat{S}_\alpha\supseteq\hat{S}_\beta$ may not hold. Hence the fuzzy set $\bar{S}$ defined by

$$\forall x, \mu_{\bar{S}}(x)=\sup\{\alpha \mid x \in \hat{S}_\alpha\} \tag{48}$$

may not be a solution of (41). If the equality in (41) is weakened into an inclusion (in Zadeh's sense),

$$S'\,(\cdot)\,A\subseteq B \tag{49}$$

then $\hat{S}$ solving (49) always exists (it is possibly the empty set) and it is $B)\cdot(A^{-1}$. N.B. Using $c(a)=1-a$ for the complementation, the equality $(A)\cdot(B)^c=A^c(\cdot)B$ holds, provided $(\cdot)$ is defined via (37), where the t-norm $*$ is changed into the pseudo-intersection $\wedge^*=\mathscr{I}(*\to)=\mathscr{I}\circ\mathscr{C}(*)$. This remark leads to the consideration of new pessimistic operations on fuzzy numbers. The corresponding "optimistic" operations are based on implications $\mathscr{I}(*)$, i.e. of the 1st kind. More details appear in [7].

2.3. Linear equations and optimistic sum and product of fuzzy numbers

This section reports results about practical computation of operation $)\cdot($ when $\cdot=$ sum (on $\boldsymbol{R}$) or product (on $\boldsymbol{R}^+$) and $*=\min$. A fuzzy interval A can be represented by two decreasing functions L and R from $\boldsymbol{R}^+$ to $[0, 1]$ such that $L(0)=$ $=R(0)=1$, and four parameters, a, a', α, β, where $[a, a']$ is the peak of A (i.e. $\{x \mid \mu_A(x)=1\}$), α, α' are the spreads and are positive numbers. Then μ_A is of the form (see [4]):

$$\begin{aligned}\mu_A(x)&=L\left(\frac{a-x}{\alpha}\right) \quad x\leqslant a\\ &=R\left(\frac{x-a'}{\alpha'}\right) \quad x\geqslant a'\end{aligned} \tag{50}$$

A is said to be of the $L-R$ type. For simplicity, $a=a'$ in the following, and A is then represented by $(a, \alpha, \alpha')_{LR}$ as in [4].

It is well known [4] that if A and B are fuzzy intervals of the $L-R$ type, then the sum $A(+)B$, in the sense of (37), is also of the $L-R$ type and:

$$(a, \alpha, \alpha')_{LR}(+)(b, \beta, \beta')_{LR}=(a+b, \alpha+\beta, \alpha'+\beta')_{LR} \tag{51}$$

As a consequence the greatest solution $\hat{S}$ of equation $S(+)A=B$ where A and B are $L-R$ fuzzy numbers is, when it exists, also of the $L-R$ type i.e. $\hat{S}=(\hat{s}, w, w')_{LR}$ and $s=b-a$, $w=\beta-\alpha$, $w'=\beta'-\alpha'$. The necessary condition of existence (46) is also sufficient, and is equivalent to both $\beta\geqslant\alpha$ and $\beta'\geqslant\alpha'$ (positivity of w and w'). Note that for $\cdot=$ sum, (46) expresses that it is possible to shift A and have the result included in B. The corresponding optimistic sum is $)+($ such that [8]:

$$A)+(B=(a, \alpha, \alpha')_{LR})+((b, \beta, \beta')_{LR}=(a+b, \alpha+\beta', \alpha'-\beta)_{LR} \tag{52}$$

it is clear that $\hat{S}=B)+(-A$, where $-A=(-a, \alpha', \alpha)_{RL}$.

Turning to the product of fuzzy numbers on $\boldsymbol{R}^+$, denoted $A\,()\,B$ (the product of two numbers is denoted ab), it is known [4] that even if A and B are of the $L-R$ type, then $A\,()\,B$ is not; however, the peak of $A\,()\,B$ is ab, and $\mu_{A\,()\,B}$ is obtained by solving the simple equations ([4]), whose unknown is $\lambda\in[0, 1]$

$$z=(a-\alpha L^{-1}(\lambda))(b-\beta L^{-1}(\lambda)),\ \forall z\leqslant ab \tag{53}$$

$$z=(a+\alpha' R^{-1}(\lambda))(b+\beta' R^{-1}(\lambda)),\ \forall z\geqslant ab \tag{54}$$

where $\lambda=\mu_{A(\,)B}(z)$. Bearing this in mind, the equation $S(\,)A=B$ can be solved by considering the following equations induced from (53) and (54):

$$b-\beta L^{-1}(\lambda)=z\left(a-\alpha L^{-1}(\lambda)\right), \quad z\leqslant\frac{b}{a} \tag{55}$$

$$b+\beta' R^{-1}(\lambda)=z\left(a+\alpha' R^{-1}(\lambda)\right), \quad z\geqslant\frac{b}{a} \tag{56}$$

where $\mu_{\hat{S}}(z)=\lambda$, when $\hat{S}$ exists. It is easily checked that $\hat{S}$ exists if and only if $\frac{a}{b}\geqslant \geqslant\max\left(\frac{\alpha}{\beta}, \frac{\alpha'}{\beta'}\right)$. This condition is equivalent to (46). Then $\hat{S}=B)(A^{-1}$ such that

$$\forall z\leqslant\frac{b}{a}, \quad \mu_{\hat{S}}(z)=L\left(\frac{b-az}{\beta-\alpha z}\right) \tag{57}$$

$$\begin{gathered}\forall z, \quad \frac{\beta'}{\alpha'}\geqslant z\geqslant\frac{b}{a}, \quad \mu_{\hat{S}}(z)=R\left(\frac{b-az}{\alpha' z-\beta'}\right)\\ \forall z\geqslant\frac{\beta'}{\alpha'}, \quad \mu_S(z)=0\,.\end{gathered} \tag{58}$$

Hence if A is of the $L-R$ type and so is B^{-1}, the optimistic product $A)(B$ is easily obtained by a suitable modification of (57) and (58). The latter equations express an optimistic quotient $B):(A$. Similarly, the solution of equation $S\oplus A=B$ is an optimistic difference $\hat{S}=B)-(A$.

2.4. Comparison of optimistic and pessimistic operations

To illustrate the difference in meaning between pessimistic and optimistic operation, let us consider a simple example. Suppose somebody has to figure out at what time he must wake up one morning in order not to miss a plane. Between getting up and catching the plane, he has a sequence of n tasks (washing, eating breakfast, etc ...) to accomplish. Let t_W be the wake up time, d_i the duration of task i, t_A the arrival time at the airport.

The duration of each task is subjectively assessed by a fuzzy number D_i so that the total duration is $D_1(+)D_2(+)\dots(+)D_n \triangleq D$. D encompasses both the upper and lower bounds on the total time $\sum_{i=1}^{n} d_i$.

Suppose the man decides to get up by T_w (a fuzzy value for t_W). Now, what can be known about his arrival time at the airport ? The answer requires the computation of $T_W(+)D$ (pessimistic sum) to be performed. $T_W(+)D$ is the set of possible values of the variable t_A.

Now if the man wants to make sure he arrives at the airport by time T_A (expressed as a fuzzy number) in spite of the imprecision on the duration D of his preceding activities, then he can obtain a range of wake up times under the form $T_A)-(D$. If he manages to wake up at t_w in this time range, he is then sure of reaching the airport on time independently of the actual durations of his various tasks. There are two reasons why he may fail to find a solution to his problem:

a) the required arrival time is too early, and $T_A)-(D$ overlaps too much a sleeping period. Such information could be obtained without using fuzzy arithmetics.

b) The knowledge on the duration of tasks is too imprecise, so that $T_A)-(D$ simply does not exist. Only optimistic fuzzy arithmetics can provide such information. Then the man must either decide to be more precise about D_i, $\forall i$, or to relax the precision of his demands about the arrival time T_A.

This example demonstrates that the choice between optimistic and pessimistic operations on fuzzy numbers is dictated by the way the problem under consideration is stated.

3. Concluding remarks

Results pertaining to equations involving fuzzy numbers are still rather scarce. This paper is only a first step towards a proper understanding of such equations. For instance, we can notice that the general fuzzy equation is of the form

$$A(\cdot)S=B(\cdot)S$$

which cannot be reduced to (41). The corresponding fuzzy relational equation is of the form $S\circ RA=S\circ RB$, which has never been considered in the past. Another interesting development would be the analysis of systems of fuzzy linear equations (see Dubois and Prade [4], and Pedrycz [14] for some discussions). Note that the particular feature of our problem is that we look for fuzzy solutions; usual fuzzy linear programming techniques (Zimmermann [23]) only consider optimal crisp solutions of imprecisely specified problems. Finding fuzzy solutions looks more attractive (because of providing ranges of flexibility) but is far more difficult.

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Operacje różnicy i zawierania zbiorów rozmytych i ich zastosowanie w analizie równań rozmytych

Dokonano przeglądu operacji na zbiorach rozmytych ze szczególnym uwzględnieniem różnic zbiorów i ich zawierania. Ma to przede wszystkim duże znaczenie dla właściwego formułowania i analizowania równań zawierających liczby rozmyte. Pokazano, że dla równań tych można otrzymać nowy rodzaj arytmetyki rozmytej, w której błędy czy nieścisłości wzajemnie się kompensują. Takie arytmetyki „optymistyczne" oparte są na zawieraniu, a więc implikacji, w przeciwieństwie do zazwyczaj stosowanych „pesymistycznych" arytmetyk rozmytych, opartych na przecięciu zbiorów.

Операции разности и включения нечетких множеств и их применение в анализе нечетких уравнений

Проведен обзор операций на нечетких множествах с особым учетом операций разности и включения. Описаны общие процедуры систематического генерирования определений таких операций. Это имеет существенное значение прежде всего для корректной формулировки и анализа уравнений, содержащих нечеткие числа. Показано, что для этих уравнений можно получить новый вид нечеткой арифметики, в которой ошибки или неточности взаимно компенсируются. Такие „оптимистические" арифметики основаны на включении, а значит импликации, в противоположность к обычно используемым „пессимистическим" нечетким арифметикам, основанным на пересечении множеств.