Control
and Cybernetics
VOL. 13 (1984) No. 3

Decision-making based on fuzzy stochastic dominance

by

NICHOLAS SLYADZ

ARKADY BORISOV

Department of Automatized Control Systems
Riga Polytechnical Institute
1, Lenin Street
Riga, Latvian SSR, USSR

An accurate assessment of the decision-maker's utility function is an important problem. Sometimes we are faced with the situation of stochastically dominating alternatives when we do need an exact utility function. The paper considers stochastic dominance tests based on fuzzy initial information.

1. Introduction

Decision analysis, as presented in [1], supposes some alternative a_p to be strictly preferred to another alternative a_q if and only if it has a greater expected utility value:

$$a_p > a_q \Leftrightarrow EU_p > EU_q, \tag{1}$$

where $>$ is the strict preference relation on the set of alternatives A; EU_p, EU_q are expected utility values of the alternatives $a_p, a_q \in A$, respectively.

The assessment of utility functions and probability distributions for the evaluation of EU_p, EU_q is an important problem in using the expected utility theory. So, from the practical point of view, it is useful to examine situations when a best alternative can be chosen according to (1) without the evaluation of expected utility value.

2. The concept of stochastic dominance

Some alternative may be preferred in terms of the criterion of maximum expected utility even when the information about the utility function is sufficient only to draw limited conclusions about its shape (e.g., non-decreasing, concave, etc.). It is the so-called stochastic dominance (*SD*) situation. Some alternative stochastically dominates the others if it has such stochastic characteristics (i.e. probability distribution features) that we can assert the exceeding value of its expected utility when the decision-maker's (*DM*'s) utility function is not precisely known [2]. Note that an analogous question of utility dominance arises if we have no sufficient information about probability distributions [3].

In the case of *SD* the incompleteness of information means that what is known about the *DM*'s utility function is that it is contained in a certain class of real-valued functions. According to the specific type of this class, both the strict and non-strict *SD* may be defined [2]. The strict first-degree *SD* is considered in the paper. Instead of the preference relation "$>$" on the set A, the strict first-degree *SD* relation "$>_D$" is defined as

$$a_p >_D a_q \Leftrightarrow EU_p > EU_q,$$

which requires, in contrast to (1), the *DM*'s utility function to be determined only in terms of being contained in the class of strictly increasing functions.

The necessary and sufficient conditions of the strict first-degree *SD* are given by the following theorem [2].

THEOREM *An alternative a_p stochastically dominates an alternative a_q if and only if*

a) *the DM's utility function lies in the class of strictly increasing ones (assertion* α*);*

b) $Q(x) \geqslant P(x)$ *for any possible consequence* x;

c) *there exists such a consequence* x *that* $Q(x) > P(x)$.

Here $P(x)$ and $Q(x)$ are the cumulative probability distribution functions under the alternatives a_p and a_q, respectively.

It is easy to show that the conditions (b) and (c) taken together are equivalent to the assertion:

β: "Alternatives a_p and a_q satisfy the relation R^s",

where R^s, the strict preference relation on the set A, is defined as follows. Consider R, the stochastic preference relation on the set A, i.e.

$$a_p R a_q \Leftrightarrow Q(x) \geqslant P(x) \text{ for any consequence } x, \tag{2}$$

which is equivalent to the condition (b). The characteristic function of the relation R has the form:

$$\mu_R(a_p, a_q) = \begin{cases} 1 & \text{iff } Q(x) \geqslant P(x) \text{ for any } x, \\ 0 & \text{otherwise}. \end{cases}$$

The corresponding relation R^s may be defined as the difference

$$R^s = R - R^{-1}$$

with the characteristic function

$$\mu_R^s(a_p, a_q) = \max[0, \mu_R(a_p, a_q) - \mu_R(a_q, a_p)] \tag{3}$$

The difference $R-R^{-1}$ is a subset of such elements of the Cartesian product $A\times A$ which satisfy the relation R and do not satisfy its inverse R^{-1}.
Therefore

$$a_p >_D a_q \Leftrightarrow \alpha \wedge \beta,$$

where "$\wedge$" stands for the conjunction of the assertions α and β. The latter may be written in terms of the characteristic function $\mu_D(a_p, a_q)$ of the strict first-degree *SD* relation "$>_D$" and the truth value $T(\alpha\wedge\beta)$ of the conjunction $\alpha\wedge\beta$:

$$\mu_D(a_p, a_q)=T(\alpha\wedge\beta)$$

which means

$$\mu_D(a_p, a_q)=\begin{cases}1 \text{ iff both } \alpha \text{ and } \beta \text{ are true,}\\ 0 \text{ otherwise.}\end{cases}$$

The strict *SD* relation of any degree has the following main properties [2]:
1) antireflexivity

$$\mu_D(a_p, a_p)=0 \quad \text{for any} \quad a_p\in A,$$

2) asymmetry

$$\mu_D(a_p, a_q)>0\Rightarrow u_D(a_q, a_p)=0 \quad \text{for any} \quad a_p, a_q\in A,$$

3) transitivity

$$a_p >_D a_q \wedge a_q >_D a_s \Rightarrow a_p >_D a_s \quad \text{for any} \quad a_p, a_q, a_s \in A.$$

This means that the strict *SD* relation "$>_D$" is a strict partial order on the set A of alternatives.

3. The concept of the fuzzy SD relation

Consider the fuzzy relation $>_{FD}$ corresponding to the strict first-degree *SD* relation $>_D$ which takes place when the necessary and sufficient conditions of *SD* (i.e. the assertions α and β) are true not only with the absolute degree but with any (possibly fuzzy) value from [0, 1]. The fuzzy *SD* relation is defined as the fuzzy subset of the Cartesian product $A\times A$ with the membership function

$$\mu_{FD}(a_p, a_q)=T(\alpha\wedge\beta),$$

where $T(\alpha\wedge\beta)$ is the truth value of the conjunction of assertions α and β as before. Let $T(\alpha\wedge\beta)$ be an ordinary (i.e. non-fuzzy) number in [0, 1]. According to the max-min interpretation of the set-theoretic operations [4], we have

$$\mu_{FD}(a_p, a_q)=\min[T(\alpha), T(\beta)], \tag{4}$$

where $T(\alpha)$, $T(\beta)$ are the truth values of assertions α and β, respectively.

The assertions α and β and the whole *SD* relation are fuzzy when such are the initial data for assessing the utility functions and probability distributions. More specifically, the fuzzy initial data may be represented by a finite collection of fuzzy propositions [5, 6]. For assessing a utility function such a proposition has the general form:

$$g_i = \text{"The utility of a consequence } G_i \text{ is } \nu_i\text{"} \quad i=\overline{1, N}, \tag{5}$$

where G_i is usually a fuzzy number [4] in the feasible set X of consequences with the membership function $\mu_{G_i}(x)$, $x \in X$; ν_i is the corresponding fuzzy utility value, a fuzzy number in [0, 1] with the membership function $\mu_{\nu_i}(u)$, $u \in [0, 1]$. While assessing the probability distribution, say for an alternative a_p, the respective fuzzy proposition is

$$g_{pj} = \text{"The probability of the consequence } G_j \text{ under } a_p \text{ is } \lambda_{pj}\text{"}, \quad j=\overline{1, M_p}, \tag{6}$$

where λ_{pj} is the fuzzy probability value with the membership function $\mu_{\lambda_{pj}}(p)$, $p \in [0, 1]$, which is also a fuzzy number. Let us consider the evaluation of $T(\alpha)$ and $T(\beta)$ when initial data are represented by fuzzy proposition of the form (5) and (6).

Suppose that the propositions of the form (5) are ranked in the increasing order of G_i, i.e.

$$G_i < G_{i+1}, \quad i=\overline{1, N-1}.$$

Possible ways of ranking of fuzzy numbers are discussed in [7]. In this case the assertion α is the conjunction:

$$\alpha = \bigwedge_{i=1}^{N-1} \alpha_i,$$

where

$$\alpha_i = \alpha_i' \leftrightarrow \alpha_i'',$$

$$\alpha_i' \Leftrightarrow G_i < G_{i+1},$$

$$\alpha_i'' \Leftrightarrow \nu_i < \nu_{i+1}.$$

According to [4]:

$$T(\alpha) = \min_{i=\overline{1, N-1}} T(\alpha_i),$$

$$T(\alpha_i) = \max\{\min[T(\alpha_i'), T(\alpha_i'')], \min[1-T(\alpha_i'), 1-T(\alpha_i'')]\}.$$

Dubois and Prade have proposed in [7] two possible interpretations of $T(\alpha_i')$ and $T(\alpha_i'')$ in terms of comparison indices. Firstly, as the degree of possibility:

$$T(\alpha_i') = \sup_{y \in X} \inf_{\substack{y \geq x \\ y \in X}} \min[1-\mu_{G_i}(y), \mu_{G_{i+1}}(x)],$$

$$T(\alpha_i'') = \sup_{u \in [0, 1]} \inf_{\substack{v \geq u \\ v \in [0, 1]}} \min[1-\mu_i(v), \mu_{i+1}(u)].$$

Secondly, as the degree of necessity:

$$T(\alpha_i') = 1 - \sup_{\substack{y \geq x \\ x, y \in X}} \min[\mu_{G_i}(y), \mu_{G_{i+1}}(x)],$$

$$T(\alpha_i'') = 1 - \sup_{\substack{y \geq x \\ x, y \in X \geq u}} \min[\mu_{\nu_i}(v), \mu_{\nu_{i+1}}(u)].$$

Suppose that the propositions of the form (6) for the alternatives a_p, a_q are uniform, i.e.

$$M_p = M_q = M,$$

$$G_{pj} = G_{qj} = G_j, \quad j = \overline{1, M}.$$

Then the fuzzy stochastic preference relation R corresponding to (2) may be defined as the conjunction:

$$a_p\, R a_q \Leftrightarrow \bigwedge_{j=1}^{M} \beta j',$$

where

$$\beta_j' \Leftrightarrow Q\,(G_j) \geqslant P\,(G_j).$$

Respectively:

$$a_q\, R a_p \Leftrightarrow \bigwedge_{j=1}^{M} \beta_j'',$$

where

$$\beta_j'' \Leftrightarrow P\,(G_j) \geqslant Q\,(G_j).$$

According to (3):

$$T\,(\beta) = \max\,[0,\ \mu_R\,(a_p, a_q) - \mu_R\,(a_q, a_p)],$$

where

$$\mu_R\,(a_p, a_q) = \min_{j=\overline{1, M}}\ T\,(\beta_j'),$$

$$\mu_R\,(a_q, a_p) = \min_{j=\overline{1, M}}\ T\,(\beta_j'').$$

Let us give the interpretation of $T\,(\beta_j')$ and $T\,(\beta_j'')$ in terms of comparison indices [7]. Firstly, as the degree of possibility:

$$T\,(\beta_j') = \sup_{\substack{q \geqslant p \\ p,\, l \in [0,\, 1]}}\ \min\,[\mu_{PG}\,(p),\ \mu_{QG}(q)], \tag{8}$$

$$T\,(\beta_j'') = \sup_{\substack{q \leqslant p \\ p,\, q \in [0,\, 1]}}\ \min\,[\mu_{PG}\,(p),\ \mu_{QG}\,(q)], \tag{9}$$

Secondly, as the degree of necessity:

$$T\,(\beta_j') = \inf_{q \in [0,\, 1]}\ \sup_{\substack{p \leqslant x \\ p \in [0,\, 1]}}\ \max\,[\mu_{PG}\,(p),\ 1 - \mu_{QG}\,(q)], \tag{10}$$

$$T\,(\beta_j'') = \inf_{p \in [0,\, 1]}\ \sup_{\substack{q \leqslant p \\ q \in [0,\, 1]}}\ \max\,[1 - \mu_{P\,(G_j)}\,(p),\ \mu_{Q\,(G_j)}\,(q)]. \tag{11}$$

As to $P\,(G_j)$ and $Q\,(G_j)$, fuzzy values of cumulative probability distribution functions for the fuzzy consequence G_j under the corresponding alternatives a_p and a_q, we may define them in the following way. $P\,(G_j)$ is the probability of a consequence x to belong to the set G_j^* of numbers not greater than the fuzzy number G_j while the alternative a_p is chosen. According to [7]:

$$\mu_{G_j^*}\,(x) = \sup_{\substack{y \geqslant x \\ y \in X}}\ \mu_{G_j}\,(y), \qquad x \in X.$$

Zadeh has shown in [5] that $P(G_j)$, the probability of a fuzzy event "$x \in G_j^*$", is *a* fuzzy number with the membership function

$$\mu_{P(G_j)}(p) = \sup_{f \in D} \min_{j=\overline{1,M}} \pi_j(f), \; p \in [0, 1],$$

where

$$D: \int_X \mu_{G_j^*}(x) f(x)\, dx = p, \qquad \pi_j(f) = \mu_{\lambda_{pJ}}\Big[\int_x \mu_{G_j}(x) f(x)\, dx\Big]. \tag{12}$$

Here $f(x)$ is some probability density associated with the set X of consequences.

4. Example

Let the *DM*'s utility function be strictly increasing, i.e. the assertion α is absolutely true, $T(\alpha)=1$.Then according to (4):

$$\mu_{FD}(a_p, a_q) = T(\beta) \quad \text{for any} \quad a_p, a_q \in A\,. \tag{13}$$

Suppose that $A=\{a_p, a_q, a_s\}$ and each alternative in A has the two corresponding propositions of the form (6) about the probability of some fuzzy consequences G_1 and G_2. After (12) we obtain the fuzzy values of corresponding comulative probability distribution functions for these consequences. Let them be as in Figure 1.

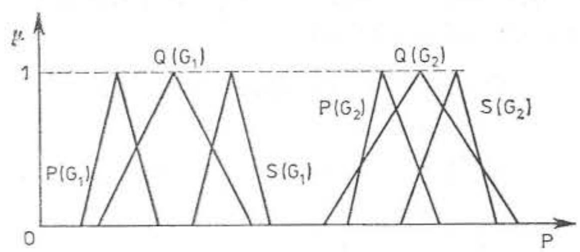

Fig 1.

According to (8), (9) and taking into account (7) and (13), we obtain the fuzzy *SD* relation represented in Table 1. On the other hand, due to (10), (11) we obtain another variant of the fuzzy *SD* relation represented in Table 2.

Table 1.

μ_{FD}	a_p	a_q	a_s
a_p	0.0	0.5	1.0
a_p	0.0	0.0	0.5
a_s	0.0	0.0	0.0

Table 2.

μ_{FD}	a_p	a_q	a_s
a_p	0.0	0.38	1.0
a_q	0.0	0.0	0.85
a_s	0.0	0.0	0.0

5. Concluding remarks

Note that the obtained fuzzy *SD* relations have the following properties:

1) antireflexivity

$$\mu_{FD}(a_p, a_p)=0 \quad \text{for every} \quad a_p \in A,$$

2) asymmetry

$$\mu_{FD}(a_p, a_q)>0 \Rightarrow \mu_{FD}(a_q, a_p)=0 \quad \text{for any} \quad a_p, a_q \in A,$$

3) max—min transitivity

$$\mu_{FD}(a_p, a_s) \geqslant \min [\mu_{FD}(a_p, a_q), \mu_{FD}(a_q, a_s)] \quad \text{for any} \quad a_p, a_q, a_s \in A.$$

According to [4] such a fuzzy *SD* relation is a fuzzy strict partial order on the set A of alternatives. This property of the fuzzy *SD* relation holds even when $T(\alpha)<1$.

References

[1] Raiffa H. Decision Analysis: Introductory Lectures on Choices under Uncertainty. Reading. Addison Wesley, 1968.

[2] Fishburn P. C., Vickson R. G. Theoretical fundations of stochastic dominance, in: Stochastic Dominance: An Approach to Decision Making under Risk, G. A. Whitmore, M. C. Findlay, Eds. Lexington, D. C. Heath and Co., 1977.

[3] Fishburn P. C. Analysis of decisions with incomplete knowledge of probabilities. *Operations Research*, **13** (1965), 217—237.

[4] Dubois D., Prade H. Fuzzy Sets and Systems: Theory and Applications. New York, Academic Press, 1980.

[5] Zadeh L. A. Fuzzy sets and information granularity, in: Advances in Fuzzy Set Theory and Applications, M. M. Gupta, R. K. Ragade, R. R. Yager, Eds. Amsterdam, North Holland, 1979.

[6] Dubois D., Prade H. Towards the analysis and the synthesis of fuzzy mappings, in: Fuzzy Set and Possibility Theory: Recent Developments, R. R. Yager, Ed. New York, Pergamon Press, 1982.

[7] Dubois D., Prade H. Ranking fuzzy numbers in the setting of possibility theory. *Information Sciences*, **30** (1983).

Received, May 1984.

Podejmowanie decyzji na podstawie rozmytej dominacji stochastycznej

Rozważono testy stochastycznej dominacji oparte na rozmytej informacji wyjściowej. Testy takie są stosowane w przypadkach, gdy nie jest konieczna dokładna znajomość własności funkcji użyteczności. W pracy analizowana jest koncepcja rozmytej relacji dominacji stochastycznej. Otrzymane własności są następnie zilustrowane na prostym przykładzie numerycznym.

Принятие решений на основе нечеткой стохастической доминантности

Рассматриваются критерии стохастической доминантности, основанные на нечеткой выходной информации. Такие критерии используются в случае, когда не является необходимым точное знание свойств функции полезности. В работе анализируется концепция нечеткого отношения стохастической доминантности. Полученные свойства иллюстрируются простым численным примером.