Control
and Cybernetics

VOL. 13 (1984) No. 3

Linguistic quantifiers and belief qualification in fuzzy multicriteria and multistage decision making

by

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The introduction of some natural language elements into basic formulations of fuzzy multicriteria and multistage decision making is discussed. First, we introduce fuzzy linguistic quantifiers exemplified by most, almost all, etc. For multicriteria decision making, instead of seeking an optimal decision which best satisfies all the fuzzy objectives as it has been traditionally assumed, we seek an optimal decision which best satisfies most, almost all, etc. (a linguistic quantifier, in general) fuzzy objectives. For multistage decision making we seek an optimal sequence of controls which best satisfies the fuzzy constraints and fuzzy goals at most, almost all, etc. control stages. A calculus of linguistically quantified propositions is employed based upon fuzzy sets and possibility theory. Second, we employ belief qualification to reflect a possibly varying belief (or certainty or confidence) as to the pieces of evidence concerning the (degree of) fulfilment of fuzzy objectives, constraints, and goals. The approach presented is a further step in bringing decision making models closer to a real practical perception of the essence of decision making.

1. Introduction

Omnipresence and vital importance of decision making in virtually all human activities triggered many attempts to formalize and solve it using some mathematical means. They were often successful, although much more so in "hard" problems arising in, e.g., technology than in "soft" problems commonly encountered in practically relevant problems of systems analysis which must account for both considerable uncertainties and many subjective, intangible, vague, etc. aspects. The former may be dealt with in probabilistic and statistical terms but for the latter, which are inherently imprecise, the conventional "precise" mathematics seems not to be the best suited.

Fuzzy sets and possibility theory (Zadeh [19]) provide tools meant to formally represent and manipulate imprecise and vague (fuzzy) concepts and relationships; they are relatively simple and intuitively appealing. Since their inception they have been applied in a variety of fields, with decision making as a notable example. The use of fuzzy sets for dealing with "soft" decision making problems was triggered by Bellman and Zadeh [1] who introduced a general framework involving as the main elements: fuzzy goals, fuzzy constraints, and a fuzzy decision; an optimal decision is sought which best satisfies the fuzzy goals and fuzzy constraints. This framework is a point of departure for virtually all fuzzy decision making models developed later on, including that in this paper.

Since real decisions are (or are intended to be) made by people for people, problem statements and solutions should be as consistent as possible with human perceptions of their underlying real problems. Since natural language is the only fully natural human communication means, the above consistency requirement may be viewed as suggesting that "softening" of decision making models should involve introduction of some elements of natural language.

A rich source of such elements is the so-called commonsense knowledge (Zadeh, [22]). One of important elements used for representing the commonsense knowledge is a disposition as, e.g., "winter days are cold". A disposition implicitly involves a linquistic quantifier (e.g., most), i.e. what the quoted disposition really says is exemplified by "most winter days are cold". The linguistic quantifiers play therefore a crucial role in the representation and handling of commonsense knowledge.

The above direction to use the linguistic quantifiers for "softening" decision models has been pursued by the authors for some time, involving: multicriteria decision making (e.g., Yager [14, 17], Yager and Kacprzyk [18], Kacprzyk and Yager [11]), multistage decision making and control (e.g., Kacprzyk [4, 6]), and group decision making (e.g., Kacprzyk [7, 8]); an application for evidence aggregation in knowledge engineering is Kacprzyk and Yager [9]. The purpose of this paper is to further explore this topic.

In conventional approaches to fuzzy multicriteria decision making (e.g., Blin [2], Kaufmann [12], Yager [13]) the problem is to find an optimal decision which best satisfies all the fuzzy objectives (goals and/or constraints), while for fuzzy multistage decision making (control) (e.g., Kacprzyk [4, 6]) the problem is to find an optimal sequence of controls which best satisfies the fuzzy constraints and fuzzy goals at all the control stages. The above "all" may often be viewed too restrictive and counter-intuitive; in many practical cases it may well be replaced by some milder requirement, e.g., "most", "almost all", etc., specified by a linguistic quantifier. Thus, we may seek in the multicriteria case an optimal decision which best satisfies most, almost all, etc. fuzzy objectives, and in the multistage case — an optimal sequence of controls which best satisfies the fuzzy constraints and fuzzy goals at most, almost all, etc. control stages.

The above idea which was proposed in the author's [4, 6, 11, 14, 17, 18] proved to be very fruitful and promising. It will be further explored here. We will also introduce a degree of importance of a particular objective or control stage, hence making it possible to seek, respectively, an optimal decision which best satisfies, e.g., most of the important objectives, or an optimal sequence of controls which best satisfies the fuzzy constraints and fuzzy goals at most of the important (earlier) control stages. Moreover, we will provide a belief qualification mechanism to account for a possibly varying belief (or certainty or confidence) as to a piece of evidence concerning the (degree of) fulfilment of a fuzzy objective (fuzzy constraint or goal).

First, we briefly sketch a calculus of linguistically quantified propositions providing means for handling fuzzy linguistic quantifiers. Then we consecutively show the application of these quantifiers in multicriteria and multistage decision making.

Some notational remarks:

The notation used will be standard. A fuzzy set A in X, $A \subsetneqq X$, will be characterized by (and often equated with) its membership function $\mu_A: X \to [0, 1]$. For a finite $X = \{x_1, ..., x_n\}$ A will be written as $A = \mu_A(x_1)/x_1 + ... \mu_A(x_n)/x_n$.

As to a more specific notation, we will also use the following binary operations:

(1) A t-norm, (t): $[0, 1] \times [0, 1] \to [0, 1]$, such that:

(a) $a\,(t)\,1 = a$

(b) $a\,(t)\,b = b\,(t)\,a$

(c) $a\,(t)\,b \geqslant c\,(t)\,d$ if $a \geqslant c$ and $b \geqslant d$

(d) $a\,(t)\,b\,(t)\,c = a\,(t)\,(b\,(t)\,c)) = (a\,(t)\,b)\,(t)\,c$

Some examples of t-norms are:

$a \wedge b = \min(a, b)$

$a \cdot b$

$1 - (1 \wedge ((1-a)^p + (1-b)^p)^{1/p} \quad p \geqslant 1$

We will also denote:

$$\mathop{(t)}_{i=1}^{n} a_i = a_1\,(t) \dots (t)\,a_n$$

and in particular

$$\bigwedge_{i=1}^{n} a_i = a_i \wedge \dots \wedge a_n$$

A t-norm generalizes the "and" operation in multivalued logic. The intersection of two fuzzy sets may be in general represented by a t-norm.

(2) An s-norm (t-conorm), (s): $[0, 1] \times [0, 1] \to [0, 1]$, such that:

(a) $a\,(s)\,0 = a$

(b) — (d) as for a t-norm

Some examples of s-norms are:

$$a \vee b = \max(a, b)$$

$$a + b - a \cdot b$$

$$1 \wedge (a^p + b^p)^{1/p} \quad p \geqslant 1$$

An s-norm generalizes the "or" operation in multivalued logic. The union of two fuzzy sets may be in general represented by an s-norm.

(3) A generalized implication or ply operation $(\rightarrow)$: $[0, 1]\times[0, 1]\rightarrow[0, 1]$, such that:

(a) $a(\rightarrow)1=1;\quad 0(\rightarrow)a=1;\quad 1(\rightarrow)a=a$

(b) $a(\rightarrow)b\geqslant a(\rightarrow)c$ if $b\geqslant c$

(c) $a(\rightarrow)b\geqslant c(\rightarrow)b$ if $a\leqslant c$

Some examples of $(\rightarrow)$ are:

b^a

$(1-a)(s)b$ where s is any s-norm ("$\vee$" in particular)

$1(t)(1-a+b)$ where t is any t-norm ("$\wedge$" in particular)

2. A calculus of linguistically quantified propositions

A linguistically quantified proposition is examplified by "most experts are convinced". In general, it may be written as

$$QY\text{'s are } F \tag{1}$$

where Q is a linguistic quantifier (e.g., most), $Y=\{y\}$ is a set of objects (e.g., experts) and F is a property (e.g., convinced).

Importance B may also be introduced into (1) yielding

$$QBY\text{'s are } F \tag{2}$$

i.e. "most of the important experts are convinced".

Basically, the problem is to find truth (QY's are F) knowing all truth (y_i is F), $y_i\in Y$, or — in case of (2) — to find truth (QBY's are F).

Moreover, with each "y_i is F" a degree of belief (or confidence or certainty) about this piece of evidence may be associated. The problem becomes then to find truth (QY's are F) knowing all truth (y_i is F|bel), $y_i\in Y$, where bel is a degree of belief. And analogously for (2).

Since the conventional two-valued predicate calculus makes it possible to find the above truths for the quantifiers "all" and "at least one", we will sketch now a fuzzy-logic-based calculus to account for more general quantifiers commonly used in practice as, e.g., most, almost all, much more than 50%, etc. Two methods of this calculus will be presented.

2.1. The algebraic or consensory method

In this classical method proposed by Zadeh [20, 21], the quantifier Q is assumed to be a fuzzy set in $[0, 1]$, $Q\subsetneq[0, 1]$. For instance, $Q=$"most" may be given as

$$\mu_{\text{"most"}}(x)=\begin{cases}1 & \text{for } x\geqslant 0.8\\ 2x-0.6 & \text{for } 0.3<x<0.8\\ 0 & \text{for } x\leqslant 0.3\end{cases} \tag{3}$$

Throughout the paper we will use only the so-called proportional fuzzy quantifiers exemplified by "most", "almost all", etc. For the so-called absolute quantifiers, e.g., "about 5", "much more than 10", etc. the reasoning is similar, but they should be defined as fuzzy sets in R, the real line.

Property F is defined as a fuzzy set in Y, $F \subsetneq Y$. If $Y=\{y_1, ..., y_p\}$, then truth (y_i is F)$=\mu_F(y_i)$, $i=1, ..., p$.

The calculation of truth (QY's are F) is based on the (nonfuzzy) cardinalities, Σ Counts, of the respective fuzzy sets (see, e.g., Zadeh [21]) and proceeds as follows:

1. Calculate

$$r=\sum \text{Count}(F)/\sum \text{Count}(Y)=\frac{1}{p}\sum_{i=1}^{p}\mu_F(y_i) \tag{4}$$

2. Calculate

$$\text{truth}(QY\text{'s are }F)=\mu_Q(r) \tag{5}$$

Importance is introduced into the above as follows. B="important" is defined as a fuzzy set in Y, $B \subsetneq Y$, such that $\mu_B(y_i)\in[0, 1]$ is a degree of importance of y_i: the higher its value the more important y_i.

We rewrite first "QBY's are F" as "Q (B and F) Y's are F" which leads to the following counterparts of (4) and (5):

1. Calculate

$$r'=\sum \text{Count}(B\text{ and }F)/\sum \text{Count}(F)=$$

$$=\sum_{i=1}^{p}(\mu_B(y_i)\,(t)\,\mu_F(y_i))/\sum_{i=1}^{p}\mu_F(y_i) \tag{6}$$

In the most common case, (t) is "$\wedge$" and (6) becomes

$$r'=\sum_{i=1}^{p}(\mu_B(y_i)\wedge\mu_F(y_i))/\sum_{i=1}^{p}\mu_F(y_i) \tag{7}$$

2. Calculate

$$\text{truth}(QBY\text{'s are }F)=\mu_Q(r') \tag{8}$$

EXAMPLE 1. Let Y="experts"$=\{Mr.\ X,\ Mr.\ Y,\ Mr.\ Z\}$; F="convinced"$=$ $=0.1/MR.\ X+0.6/MR.\ Y+0.8/MR.\ Z$; Q="most" is given by (3); B="important"$=0.2/MR.\ X+0.5/MR.\ Y+0.6/MR.Z$; (t) is "$\wedge$".

Then, on the one hand

$$r=\tfrac{1}{3}(0.1+0.6+0.8)=0.5$$

and

$$\text{truth}(\text{"most experts are convinced"})=2\cdot 0.5-0.6=0.4$$

On the other hand

$$r'=((0.1\wedge 0.2)+(0.6\wedge 0.5)+(0.8\wedge 0.6))/1.5=1.2/1.5=0.8$$

and

truth ("most of the important experts are convinced")=1

Instead of the nonfuzzy cordinalities of the respective fuzzy sets, Σ Counts, the fuzzy cardinalities, F Counts, may also be used. For details, see Zadeh [21].

Consideration of belief (or confidence or certainty) as to the piece of evidence "y_i is F" may be done through a belief qualification procedure by Yager [15]. Namely, we introduce a degree of belief $\text{bel}_i \in [0, 1]$ and "y_i is $F|\text{bel}_i$" denotes "y_i is F" with belief equal to bel_i. Now, "y_i is $F|\text{bel}_i$" is translated to "y_i is F^*" with implied $\text{bel}_i=1$, such that

$$\mu_{F*}(y_i)=(\mu_F(y_i)\,(t)\,\text{bel}_i)\,(s)\,(1-\text{bel}_i) \tag{9}$$

Its particular, widely used form is

$$\mu_{F*}(y_i)=(\mu_F(y_i)\wedge\text{bel}_i)+(1-\text{bel}_i) \tag{10}$$

Notice that $\text{bel}_i=1$ means that we are certain that "y_i" is F", while $\text{bel}_i=0$ means that we know nothing, hence each y_i is possible, i.e. $\mu_F(y_i)=1$ for each $y_i \in Y$.

The method presented may be viewed as yielding a consensory like aggregation of the pieces of evidence "y_i is F", with or without importance and belief (for details, see Kacprzyk [7] or Yager [16]).

2.2. The substitution or competitive method

This alternative method is proposed by Yager [14, 16]. As in Section 2.1 we have here $Y=\{y_1, ..., y_p\}$ and $F \subsetneq Y$. A proposition "y_i is F" is denoted by P_i, $i=1, ..., p$, and

$$\text{truth } P_i=\text{truth}\,(y_i \text{ is } F)=\mu_F(y_i) \tag{11}$$

We introduce the set

$$V=\{v\}=2^{\{P_1, ..., P_p\}} \tag{12}$$

whose generic element $\{v\}$ is $\{P_{k1}, ..., P_{km}\}$ meant as the proposition P_{k1} and ... and P_{km}, and written as $\{P_{k1}$ and ... and $P_{km}\}$. In the sequel v and $\{P_{k1}$ and ... and $P_{km}\}$ will be used interchangeably.

Each v, or its corresponding proposition P_{k1} and ... and P_{km}, is seen to have the truth

$$\mu_T(v)=\text{truth}\,(P_{k1} \text{ and ... and } P_{km})=\text{truth}\,P_{k1}\,(t)\,...\,(t)\,\text{truth}\,P_{km}=$$
$$=\underset{i=1}{\overset{m}{(t)}}\,\text{truth}\,P_{ki}=\mu_F(k1)\,(t)\,...\,(t)\,\mu_F(km)=\underset{i=1}{\overset{m}{(t)}}\,\mu_F(ki) \tag{13}$$

Its particular, widely used form for (t) being "$\wedge$" is

$$\mu_T(v)=\mu_F(k1)\wedge...\wedge\mu_F(km)=\bigwedge_{i=1}^{m}\mu_F(ki) \tag{14}$$

We obtain therefore a fuzzy set $T \subsetneq V$.

The fuzzy linguistic quantifier Q is defined as a fuzzy set in V, $Q \subseteq V$.

The truth of "QY's are F" is now seen as the possibility that some $v \in V$ both satisfies the meaning of Q and is true (T), i.e. (Zadeh [19])

$$\text{truth}\,(QY\text{'s are } F) = \max_{v \in V} \left(\mu_Q(v)\ (\mathbf{t})\ \mu_T(v) \right) \tag{15}$$

and for the most widely used $(\mathbf{t})$ — being "$\wedge$"

$$\text{truth}\,(QY\text{'s are } F) = \max_{v \in V} \left(\mu_Q(v) \wedge \mu_T(v) \right) \tag{16}$$

Assume now that $B \subseteq Y$ denotes importance. Then

$$\text{truth}\,(QBY\text{'s are } F) = \max_{v \in V} \left(\mu_Q(v)\ (\mathbf{t1}) \left(\overset{km}{\underset{i=k1}{(\mathbf{t2})}} \left(\mu_B(y_i)\ (\rightarrow)\ \mu_F(y_i) \right) \right) \right) \tag{17}$$

where $(\mathbf{t1})$ and $(\mathbf{t2})$ are t-norms and $(\rightarrow)$ is an implication operator (for a derivation of (17), see Yager [17]).

For instance, if $(\mathbf{t1})$ and $(\mathbf{t2})$ are "$\wedge$" and $(\rightarrow)$ is "$(1-a) \vee b$", then

$$\text{truth}\,(QBY\text{'s are } F) = \max_{v \in V} \left(\mu_Q(v) \wedge \bigwedge_{i=k1}^{km} \left(1 - \mu_B(y_i) \right) \vee \mu_F(y_i) \right) \tag{18}$$

Evidently, in (17) and (18) $\{v\}$ is equivalent to $\{P_{k1}$ and ... and $P_{km}\}$.

EXAMPLE 2. For the same data as in Example 1, let $(\mathbf{t1})$ and $(\mathbf{t2})$ be "$\wedge$", $(\rightarrow)$ be "$(1-a) \vee b$", and Q="most" be

$$\mu_{\text{"most"}}(v) = \begin{cases} 1 & \text{for} \quad v \in \{P_1 \text{ and } P_2 \text{ and } P_3\} \\ 0.7 & \text{for} \quad v \in \{P_1 \text{ and } P_2, P_1 \text{ and } P_3, P_2 \text{ and } P_3\} \\ 0.3 & \text{for} \quad v \in \{P_1, P_2, P_3\} \end{cases} \tag{19}$$

Since (14) yields: $\mu_T(P_1$ and P_2 and $P_3)=0.1$. $\mu_T(P_1$ and $P_2)=0.1$, $\mu_T(P_1$ and $P_3)=0.1$, $\mu_T(P_2$ and $P_3)=0.6$, $\mu_T(P_1)=0.1$, $\mu_T(P_2)=0.6$, $\mu_T(P_3)=0.8$), then due to (16)

$$\text{truth ("most experts are convinced")} = (1 \wedge 0.1) \vee (0.7 \wedge 0.1) \vee (0.7 \wedge$$
$$\wedge 0.1) \vee (0.7 \wedge 0.6) \vee (0.2 \wedge 0.1) \vee (0.2 \wedge 0.6) \vee (0.2 \wedge 0.8) = 0.6$$

and due to (18), with B="important" as in Example 1

$$\text{truth ("most of the important experts are convinced")} =$$
$$= (1 \wedge ((0.8 \vee 0.1) \wedge (0.5 \vee 0.6) \wedge (0.4 \vee 0.8))) \vee (0.7 \wedge$$
$$\wedge ((0.8 \vee 0.1) \wedge (0.5 \vee 0.6))) \vee (0.7 \vee ((0.8 \vee 0.1) \wedge$$
$$\wedge (0.4 \vee 0.8))) \vee (0.7 \wedge ((0.5 \vee 0.6) \wedge (0.4 \vee 0.8))) \vee$$
$$\vee (0.3 \wedge ((0.0 \vee 0.1))) \vee (0.3 \wedge ((0.5 \vee 0.6))) \vee (0.3 \wedge$$
$$\wedge ((0.4 \vee 0.8))) = 0.8 \vee 0.6 \vee 0.7 \vee 0.6 \vee 0.3 \vee 0.3 \vee 0.3 = 0.8$$

Notice that the substitution method may yield different results than the algebraic method.

The substitution method may be viewed to provide a competitive like aggregation of pieces of evidence (for details, see Kacprzyk [7] or Yager [16]).

Belief qualification may evidently proceed analogously as in Section 2.1.

We will now apply both methods of dealing with linguistically quantified propositions to multicriteria and multistage decision making.

3. Multicriteria decision making under fuzziness

As outlined in Section 1, multicriteria decision making under fuzziness may be formalized as: if $A=\{a\}=\{a_1, ..., a_q\}$ is a set of possible alternative decisions (options) and $o=\{o_1, ..., o_p\}$ is a set of fuzzy objectives (fuzzy constraints and/or goals) to be satisfied, then the degree to which $a \in A$ satisfies $o_i \in o$ is given by truth (o_i is satisfied (by a))$=\mu_{o_i}(a)$.

Traditionally it is postulated (e.g., Bellman and Zadeh [1]) that $a \in A$ satisfy "o_1 and ... and o_p", i.e. all the fuzzy objectives and hence the degree of that satisfaction is given by the fuzzy decision

$$\mu_D(a|\text{"al"l})=\text{truth}(o_1 \text{ and} ... \text{and } o_p \text{ are satisfied (by } a))= \\ =\text{truth}(\text{"all" } o\text{'s are satisfied})=\text{truth}(o_1 \text{ is satisfied}) \ (t) ... \\ ... (t) \ \text{truth}(o_p \text{ is satisfied})) = \mu_{o_1}(a) \ (t) ... (t) \ \mu_{o_p}(a) \tag{20}$$

and the problem is to find an optimal decision $a^* \in A$, such that

$$a^* = \arg\max_{a \in A} \mu_D(a|\text{"all"}) \tag{21}$$

It is easy to see that the requirement of satisfying "all" the fuzzy objectives exexpressed by (20) may be viewed too rigid and restrictive for practical purposes. An idea to replace that "all" by a milder requirement as e.g., "most" appeared in Yager [14], Yager and Kacprzyk [18], Kacprzyk and Yager [11]. Basically, it consists in requiring the fulfilment of Q (e.g., "most") fuzzy objectives, where Q is a linguistic quantifier, i.e. we seek an $a^* \in A$, such that

$$a^* = \arg\max_{a \in A} \mu_D(a \mid Q) \tag{22}$$

The purpose of this paper is to further analyze this type of problems.

3.1. Solution by the algebraic method

Following Section 2.1, we introduce first the fuzzy set $S=$"satisfied"$\subsetneq o$, such that $\mu_s(o_i)=$truth (o_i is satisfied (by a))$=\mu_{o_i}(a)$, $i=1, ..., p$, and a fuzzy set $B=$ $=$"important"$\subsetneq o$, such that $\mu_B(o_i) \in [0, 1]$ is the degree of importance of objective o_i. A linguistic quantifier is $Q \subsetneq [0, 1]$.

The fuzzy decision is

$$\mu_D(a|QB)=\mu_D(a|\text{"most" "important"})=\text{truth}(\text{"most" of the} \\ \text{"important" } o\text{'s are satisfied})=\text{truth}(QBo\text{'s are satisfied}) \tag{23}$$

and the problem is to find an optimal $a^* \in A$, such that

$$a^* = \arg\max_{a \in A} \mu_D(a \mid QB) \tag{24}$$

Using the algebraic method, we calculate first

$$r(a) = \sum_{i=1}^{p} (\mu_B(o_i)(t)\, \mu_{o_i}(a)) / \sum_{i=1}^{p} \mu_{o_i}(a) \tag{25}$$

and then

$$\mu_D(a \mid QB) = \mu_Q(r(a)) \tag{26}$$

Thus

$$a^* = \arg\max_{a \in A} \mu_Q(r(a)) \tag{27}$$

Notice that if we do not wish to account for importance, we set $\mu_B(o_i) = 1$ for each $o_i \in o$, and (25) becomes

$$r(a) = \frac{1}{p} \sum_{i=1}^{p} \mu_{o_i}(a) \tag{28}$$

while (26) and (27) remain the same.

It is easy to see that it is difficult to say something about the solution of (27) for a general Q. We introduce therefore the so-called nondecreasing quantifiers defined as follows

$$r' > r'' \Rightarrow \mu_Q(r') \geqslant \mu_Q(r'') \quad \text{for each} \quad r', r'' \in [0, 1] \tag{29}$$

i.e. the more objectives are satisfied the better. "Most" given by (19) is such a quantifier. Let us notice that the nondecreasing quantifiers are the only practically relevant in our topic.

A particularty important nondecreasing quantifier is the linear quantifier L defined as $\mu_L(r) = r$ for each $r \in R$.

As shown in, e.g., Yager [14, 17] or Yager and Kacprzyk [18], any optimal solution of (27) with the linear quantifier is also an optimal solution for an arbitrary nondecreasing quantifier, i.e. (27) becomes for a nondecreasing Q

$$a^* = \arg\max_{a \in A} \mu_D(a \mid LB) = \arg\max_{a \in A} \left(\sum_{i=1}^{p} (\mu_B(0_i)(t)\, \mu_{o_i}(a)) / \sum_{i=1}^{p} \mu_{o_i}(a) \right) \tag{30}$$

EXAMPLE 3. Let: $A = \{a_1, a_2, a_3\}$, $o_1 = 1/a_1 + 0.8/a_2 + 0.3/a_3$, $o_2 = 0.6/a_1 + 1/a_2 + 0.5/a_3$, $o_3 = 0.5/a_1 + 0.8/a_2 + 1/a_3$, $Q =$ "most" be given by (19), (t) be "$\wedge$", and $B =$ "important" $= 0.3/o_1 + 0.8/o_2 + 0.5/o_3$.

Then:

$\mu_D(a_1 \mid$ "most" "important"$) = 1.4/2.1 = 2/3$

$\mu_D(a_2 \mid$ "most" "important"$) = 2.4/2.6 = 12/13$

$\mu_D(a_3 \mid$ "most" "important"$) = 1.3/1.8 = 13/18$

i.e. $a^* = a_2$.

3.2. Solution by the substitution method

Following Section 2.2, we introduce the propositions P_i: "o_i is satisfied (by a)" and a fuzzy set S="satisfied" such that

$$\mu_S(o_i)=\mu_{o_i}(a)=\text{truth } P_i \quad i=1, \ldots, p \tag{31}$$

The set V is

$$V=\{v(a)\}=2^{\{p_1, \ldots, p_p\}} \tag{32}$$

The truth of $v(a)$, or its corresponding proposition P_{k1} and ... and P_{km} is

$$\mu_T(v(a))=\text{truth}(P_{k1} \text{ and} \ldots \text{and } P_{km})=\mu_{o_{k1}}(a)\,(t) \ldots (t)\, \mu_{o_{km}}(a) \tag{33}$$

If $B \subsetneq o$ denotes importance, and $Q \subsetneq V$, then

$$\text{truth}(QBo\text{'s are satisfied (by } a))=$$

$$=\max_{v(a) \in V}\left(\mu_Q(v(a))\,(\boldsymbol{t1})\,\left(\mathop{(\boldsymbol{t2})}_{i=k1}^{km}\left(\mu_B(o_i)\,(\to)\,\mu_{o_i}(a)\right)\right)\right) \tag{34}$$

and the problem is to find $a^* \in A$ such that

$$a^*=\arg\max_{a \in A}\max_{v(a) \in V}\left(\mu_Q(v(a))\,(\boldsymbol{t1})\,\left(\mathop{(\boldsymbol{t2})}_{i=k1}^{km}\left(\mu_B(o_i)\,(\to)\,\mu_{o_i}(a)\right)\right)\right) \tag{35}$$

where $(\boldsymbol{t1})$ and $(\boldsymbol{t2})$ are arbitrary t-norms.

EXAMPLE 4. Let us assume the same data as in Example 3 with Q="most" given by (19), and: $(\boldsymbol{t1})=(\boldsymbol{t2})$="$\wedge$", $a\,(\to)\,b=(1-a) \vee b$. Then, (35) becomes

$$a^*=\arg\max_{a \in A} \text{truth}(QBO\text{'s are satisfied (by } a))=$$

$$=\arg\max_{u \in A}\max_{v(a) \in V}\left(\mu_Q(v(a)) \wedge \bigwedge_{i=k1}^{km}\left((1-\mu_B(o_i))\, v \mu_o\ (a)\right)\right) \tag{36}$$

Thus, (36) yields:
truth $(QBO$'s are satisfied (by $a_1))=0.6$
truth $(QBO$'s are satisfied (by $a_2))=0.8$
truth $(QBO$'s are satisfied (by $a_3))=0.7$

Thus, $a^*=a_2$, i.e. the same as in Example 3 although it need not be so, in general.

The solution of (35) is analytically difficult; it may be transformed into an equivalent 0—1 programming problem (Yager [17]).

In practice, some simplified form of (38) is often used — with (t) being "$\wedge$" and all the objectives of the same importance, i.e. $\mu_B(o_i)=1$ for each $o_i \in o$ — which leads to the search of $a^* \in A$ such that

$$a^*=\arg\max_{a \in A}\max_{v(a) \in V}\left(\mu_Q(v(a)) \wedge \bigwedge_{i=k1}^{km} \mu_{o_i}(a)\right) \tag{37}$$

which is relatively easy to solve analytically.

Namely (see, e.g., Yager and Kacprzyk [18] for proofs), if:

(1) the linguistic quantifier $Q \subsetneq V$ is monotonic, i.e.

$$\mu_Q(v_3) \geqslant \mu_Q(v_1) \vee \mu_Q(v_2) \tag{38}$$

for any $v_1, v_2, v_3 \in V$ such that $v_3 = v_1$ and v_2 (for instance "most" given by (19) is monotonic). This monotonicity is close in spirit to the nondecreasingness (29) but not the same;

(2) there is a finite number of distinct membership grades of $\mu_Q(v)$, say $b_1 \leqslant b_2 \leqslant \ldots \ldots \leqslant b_3$;

(3) d_i is the i-th largest element of the set $\{\mu_{o_2}(a), \ldots, \mu_{o_p}(a)\}$ containing for a fixed a the grades of membership of a in the particular fuzzy objectives; then (37) is equivalent to seeking $a^* \in A$ such that

$$a^* = \arg\max_{a \in A} \max_{i=1,\ldots,s} (d_1 \wedge b_i) \tag{39}$$

EXAMPLE 5. For the same data as in Example 3 (but without importance, i.e. $B = 1/0_1 + 1/0_2 + 1/0_3$), we have:

— for a_1: $d_1 = 1$, $d_2 = 0.6$, $d_3 = 0.5$; and $b_1 = 0.2$, $b_2 = 0.7$, $b_3 = 1$. Hence
$$\max_{i=1,2,3} (d_i \wedge b_i) = 0.2 \vee 0.6 \vee 0.5 = 0.6$$

— for a_2: $d_1 = 1$, $d_2 = 0.8$; and $b_1 = 0.2$, $b_2 = 0.7$. Hence
$$\max_{i=1,2} (d_i \wedge b_i) = 0.2 \vee 0.7 = 0.7$$

— for a_3: $d_1 = 1$, $d_2 = 0.5$, $d_3 = 0.3$; and $b_1 = 0.2$, $b_2 = 0.7$, $b_3 = 1$. Hence
$$\max_{i=1,2,3} (d_i \wedge b_i) = 0.2 \vee 0.5 \vee 0.3 = 0.5$$

Therefore, $a^* = a_2$.

Belief qualification as to a piece of evidence "o_i is satisfied (by a)" may be easily introduced into (26), (30), (35) or (36). Namely we should first calculate $\mu_{o_i^*}(a)$ using $\mu_{o_i}(a)$ and bel_i, $i = 1, \ldots, p$, due to (9). Then, we should replace $\mu_{o_i}(a)$'s with $\mu_{o_i^*}(a)$'s in the respective formulas.

4. Multistage decision making under fuzziness

The essence of multistage decision making (control) under fuzziness may be stated as follows. At each time (control stage) t, the control $u_t \in U = \{c_1, \ldots, c_m\}$ is subjected to a fuzzy constraint $\mu_{C^t}(u_t)$, and on the state attained $x_{t+1} \in X = \{s_1, \ldots, s_n\}$ a fuzzy goal $\mu_{G^{t+1}} x_{t+1})$ is imposed; the state transitions are governed by $x_{t+1} = f(x_t, u_t)$; $x_t, x_{t+1} \in X$, $u_t \in U$, $t = 0, 1, \ldots, N$; N is some termination time. For extensions of the above basic statement, see Kacprzyk [3, 5].

It is commonly postulated (e.g., Bellman and Zadeh [1] or Kacprzyk [3, 5]) that at each t u_t satisfy the fuzzy constraint C^t and the fuzzy goal G^{t+1}, to be written as P_{t+1}: "C^t and G^{t+1} are satisfied (by u_t)". This satisfaction is evidently to the degree equal to truth

$$P_{t+1} = \text{truth}(\text{"}C^t \text{ and } G^{t+1} \text{ are satisfied"}) = \mu_{c^t}(u_t)\,(t)\,\mu_{G^{h+1}}(x_{t+1}) \tag{40}$$

Moreover, traditionally we require a sequence of controls to satisfy the fuzzy constraints and fuzzy goals at all the subsequent control stages, hence the fuzzy decision expressing the degree of that satisfaction is

$$\mu_D(u_0, \ldots, u_{N-1}|x_0, \text{“all”}) = \text{truth}\,(P_1 \text{ and} \ldots \text{and } P_N|\text{“all”}) = \\ = \overset{N-1}{\underset{t=0}{\textcircled{t}}}|\text{“all”})\ \text{truth}\, P_{t+1} = \overset{N-1}{\underset{t=0}{\textcircled{t}}}|\text{“all”})\left(\mu_{C^t}(u_t)\,\textcircled{t}\,\mu_{G^{t+1}}(x_{t+1})\right) \tag{41}$$

and the problem is to find an optimal sequence of controls $u^*, \ldots, u^*_{N-1}$, such that

$$u^*_0, \ldots, u^*_{N-1} = \arg \max_{u_0, \ldots, u_{N-1}} \mu_D(u_0, \ldots, u_{N-1}\,|x_0, \text{“all”}) \tag{42}$$

It is easy to see that the above requirement to satisfy the fuzzy constraints and fuzzy goals at all the control stages may be viewed as too rigid and counter-intuitive in practice. An approach to replace that “all” by a milder requirement expressed by a fuzzy linguistic quantifier Q, say “most”, “almost all”, etc., was proposed by Kacprzyk [4, 6]. Basically, $u_0, \ldots, u_{N-1}$ is required to fulfill the fuzzy constraints and fuzzy goals at Q control stages, hence (41) and (42) become, respectively:

$$\mu_D(u_0, \ldots, u_{N-1}\, u_0, Q) = \overset{N-1}{\underset{t=0}{\textcircled{t}}}\ |Q)\ \text{truth}\ P_{t+1} = \\ = \overset{N-1}{\underset{t=1}{\textcircled{t}}}\ |Q)\left(\mu_{C^t}(u_t)\,\textcircled{t}\,\mu_{G^{t+1}}(x_{t+1})\right) \tag{43}$$

and

$$u^*_0, \ldots, u^*_{N-1} = \arg \max_{u_0, \ldots, u_{N-1}} \mu_D(u_0, \ldots, u_{N-1}|Q) \tag{44}$$

We will now present the solution of this basic formulation using both the algebraic and substitution method. Importances will not be accounted for in order to be able to efficiently solve the resulting problems. Then, we will comment upon the role of importances and belief qualification with respect to discounting.

4.1. Solution by the algebraic method

To solve (44) by the algebraic method (see Section 2.1), we first calculate

$$r(u_0, \ldots, u_{N-1}|x_0) = (1/N) \sum_{t=0}^{N-1} \text{truth}\ P_{t+1} = \\ = (1/N) \sum_{t=0}^{N-1} \left(\mu_{C^t}(u_t)\,\textcircled{t}\,\mu_{G^{t+1}}(x_{t+1})\right) \tag{45}$$

then

$$\mu_D(u_0, \ldots, u_{N-1}|x_0, Q) = \mu_Q\left(r(u_0, \ldots, u_{N-1}|x_0)\right) = \\ = \mu_Q\left((1/N) \sum_{t=0}^{N-1} \left(\mu_{C^t}(u^t)\,\textcircled{t}\,\mu_{G^{t+1}}(x_{t+1})\right)\right) \tag{46}$$

and the problem is to find

$$u_0^*, \ldots, u_{N-1} = \arg \max_{u_0, \ldots, u_{N-1}} \mu_Q \left((1/N) \sum_{t=0}^{N-1} \left(\mu_{C^t}(u_t) \,(t)\, \mu_{G^{t+1}}(x_{t+1}) \right) \right) \tag{47}$$

As for (27), it is difficult to say something about the solution of (47) for an arbitrary Q. However, by an analogous line of reasoning as in Section 3.1 it may be shown (Kacprzyk [4]) that for a nondecreasing Q (in the sense of (29)) an optimal solution to (47) may be obtained by solving it for the linear quantifier, i.e. (47) becomes to find

$$u_0^*, \ldots, u_{N-1}^* = \arg \max_{u_0, \ldots, u_{N-1}} (1/N) \sum_{t=0}^{N-1} \left(\mu_{C^t}(u_t) \,(t)\, \mu_{G^{t+1}}(x_{t+1}) \right) \tag{48}$$

This may be solved, e.g., by dynamic programming. The set of recurrence equations yielding the solution is

$$\mu_{G_a^{N-i}}(x_{N-i}) = \max_{u_{N-i}} \left((1/N) \left(\mu_{c^{N-i}}(u_{N-i}) \,(t)\, \mu_{G^{N-i+1}}(x_{N-i+1}) \right) + \mu_{G_a^{N-i+1}}(x_{N-i+1}) \right) \quad i = 1, \ldots, N \tag{49}$$

$$x_{N-i+1} = f(x_{N-i}, u_{N-i})$$

where $\mu_{G^N}(x_N) = 0$ for each x_N.

Evidently, from (49) we obtain optimal policies $a_t^* : X \to U$, such that $u_t^* = a_t^*(x_t)$, $t = 0, 1, \ldots, N-1$.

EXAMPLE 6. Let $N=3$, the fuzzy constraints and fuzzy goals be

$$C^0 = 0.5/c_1 + 1/c_2 \qquad G^1 = 0.1/s_1 + 0.6/s_2 + 1/s_3$$
$$C^1 = 1/c_1 + 0.7/c_2 \qquad G^2 = 0.6/s_1 + 1/s_2 + 0.5/s_3$$
$$C^2 = 1/c_1 + 0.6/c_2 \qquad G^3 = 1/s_1 + 0.8/s_2 + 0.3/s_3$$

(t) be "$\wedge$", and the state transitions be governed by

	x_t		
$x_{t+1} = u_t$	s_1	s_2	s_3
c_1	s_3	s_3	s_3
c_2	s_2	s_1	s_2

Solving (49) consecutively for $i = 1, 2, 3$ we obtain:

$$a_2^*(s_1) = c_2, \quad a_2^*(s_2) = c_2, \quad a_2^*(s_3) = c_2;$$

next

$$a_2^*(s_1) = c_2, \quad a_2^*(s_2) = c_2, \quad a_2^*(s_3) = c_2;$$

and finally

$$a_0^*(s_1) = c_1 \text{ or } c_2, \quad a_0^*(s_2) = c_1, \quad a_0^*(s_3) = c_1 \text{ or } c_2.$$

4.2. Solution by the substitution method

Following the reasoning applied in Sections 2.2 and 3.2, we denote P_{t+1}: "C^t and G^{t+1} are satisfied (by u_t)" and

$$\text{truth } P_{t+1} = \mu_{C^t}(u_t) \, (t) \, \mu_{G^{t+1}}(x_{t+1}) \tag{50}$$

Next

$$V = \{v\} = 2^{\{P_1, \ldots, P_N\}} \tag{51}$$

and $Q \subseteq V$; $\{v\}$ is equivalent to $\{P_{k1}$ and ... and $P_{km}\}$.

The fuzzy decision (46) is now

$$\mu_D(u_0, \ldots, u_{N-1} | x_0, Q) = \text{truth}(P_1 \text{ and } \ldots \text{ and } P_N | Q) =$$

$$= (\underset{t=0}{\overset{N-1}{(t)}} | Q) \text{ truth } P_{t+1} = \text{Poss}(Q \cap T) = \max_{v \in V} (\mu_Q(v) \, (t) \, \mu_T(v)) \tag{52}$$

where $\mu_T(v) = \mu_T(P_{k1} \text{ and } \ldots \text{ and } P_{km}) = \underset{t=k1-1}{\overset{km-1}{(t)}} (\mu_{C^t}(u_t) \, (t) \, \mu_{G^{t+1}}(x_{t+1}))$.

The problem (47) becomes to find

$$u_0^*, \ldots, u_{N-1}^* = \arg \max_{u_0, \ldots, u_{N-1}} \max_{v \in V} (\mu_Q(v) \, (t) \, \mu_T(v)) \tag{53}$$

Evidently, $\mu_T(v)$ is a function of $u_0, \ldots, u_{N-1}$.

It may readily be seen that (53) virtually consists of the following two problems:

(a) the calculation of

$$\mu_D(u_0, \ldots, u_{N-1} \, x_0, Q) = \max_{v \in V} (\mu_Q(v) \, (t) \, \mu_T(v))$$

for fixed $u_0, \ldots, u_{N-1}$ and $x_1, \ldots, x_N$;

(b) the optimization of $u_0, \ldots, u_{N-1}$, i.e. the determination of $u_0^*, \ldots, u_{N-1}^*$ for which (53) holds.

Problem (a) is in fact equivalent to that of multicriteria decision making discussed in Section 3.1, hence will not be dealt with here. We will consider below the problem (b), i.e. optimization of the sequence of controls.

Since we assume that the control space U is finite, the multistage decision making process may be portrayed as a decision tree of the type shown in Fig. 1. Its nodes correspond to the successive states attained (x_0 in the root), and with its arcs there are associated the controls applied and the values of $\mu_{C^t}(u_t) \, (t) \, \mu_{G^{t+1}}(x_{t+1})$. The determination of $u_0^*, \ldots, u_{N-1}^*$ is therefore equivalent to finding a path in that decision tree for which (53) holds. An implicit enumeration algorithm given below may be used.

The point of departure is the solution of the conventional (i.e. for Q="all") multistage decision making problem: find

$$u_0^*, \ldots, u_{N-1}^* = \arg \max_{u_0, \ldots, u_{N-1}} \mu_D(u_0, \ldots, u_{N-1} | x_0, \text{"all"}) =$$

$$= \arg \max_{u_0, \ldots, u_{N-1}} \underset{t=0}{\overset{N-1}{(t)}} (\mu_{C^t}(u_t) \, (t) \, \mu_{G^{t+1}}(x_{t+1})) \tag{54}$$

which may be efficiently solved by dynamic programming or branch-and-bound (Kacprzyk [5]).

We have now an important property.

PROPOSITION 1. For any monotomic Q and any $u_0, \ldots, u_{N-1}$:

$$\mu_D(u_0, \ldots, u_{N-1}|x_0, Q) \geqslant \mu_D(u_0, \ldots, u_{N-1}|x_0, \text{“all”}) \tag{55}$$

and in particular

$$\mu_D(u_0^*, \ldots, u_{N-1}^*|x_0, Q) \geqslant \mu_D(u_0^*, \ldots, u_{N-1}^*|x_0, \text{“all”}) \tag{56}$$

The proof is analogous to that in Kacprzyk [4].

The solution of the generalized problem (53) is therefore never worse than that of the conventional one (54).

And the necessary condition for a path (its corresponding $u_0, \ldots, u_{N-1}$) to be an optimal solution of the generalized problem (53) is given below.

PROPOSITION 2. If $u_0^*, \ldots, u_{N-1}^*$ is an optimal solution to (53) for some monotonic Q, then in its corresponding path there exists at least one arc such that for u_t^* associated with this arc there holds

$$\mu_{C^t}(u_0^*)\,(t)\,\mu_{G^{t+1}}(f(x_t, u_t^*)) \geqslant \mu_D(u_0^*, \ldots, u_{N-1}^*|x_0, \text{“all”}) \tag{57}$$

The proof is analogous to that in Kacprzyk [4].

The above properties lead to the following algorithm for solving the problem considered (54):

1. Construct the decision tree of the type mentioned before (see, e.g., Fig. 1).
2. Solve the conventional problem (53), i.e. find

$$u_0^*, \ldots, u_{N-1}^* = \arg \max_{u_0, \ldots, u_{N-1}} \mu_0(u_0, \ldots, u_{N-1}|x_0, \text{“all”})$$

3. Find such arcs in the decision tree for which

$$\mu_{C^t}(u_t)\,(t)\,\mu_{G^{t+1}}(x_{t+1}) > \mu_D(u_0^*, \ldots, u_{N-1}^*|x_0, \text{“all”}) \tag{58}$$

   Notice that we have here only “$>$”, but we already have some arcs for which “$=$” holds, i.e. those corresponding to a solution found in Step 2. Here we seek a better solution.

4. Determine all the paths from x_0 to x_N containing the arcs found in Step 3.
5. For each path (its corresponding $u_0, \ldots, u_{N-1}$) found in Step 4, determine

$$\mu_D(u_0, \ldots, u_{N-1}|x_0, Q) = \max_{v \in V}(\mu_Q(v)\,(t)\,\mu_T(v)) \tag{59}$$

   and take as the optimal solution(*s*) the sequence(*s*) for which (59) takes on the maximum.

EXAMPLE 7. Let the problem specifications be the same as in Example 6. Assume that $x_0=s$, $N=3$, (t) is "$\wedge$" and $Q=$"most" is given by (19). We will seek

$$u_0^*, u_1^*, u_2^* = \arg\max_{u_0, u_1, u_2} \mu_D(u_0, u_1, u_2 | s_1, \text{"most"}) =$$

$$= \arg\max_{u_0, u_1, u_2} \Big(\bigwedge_{t=0}^{2} |\text{"most"}\Big) \big(\mu_{C^t}(u_t) \wedge \mu_{G^{t+1}}(x_{t+1})\big)$$

The consecutive steps of the algorithm are:

1. The decision tree is as in Fig. 1.

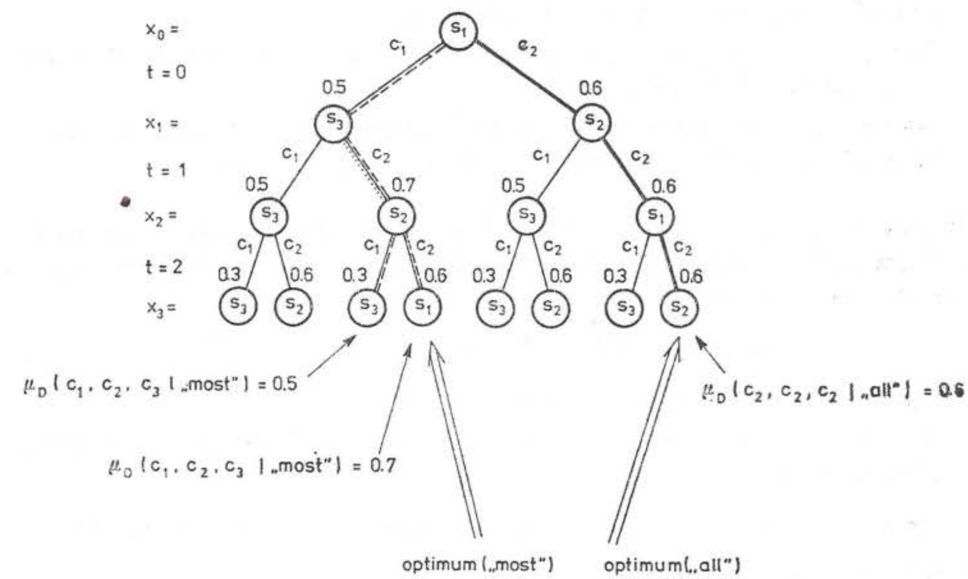

Fig. 1. Decision tree with the optimal solutions

2. The solution of the conventional problem (54) is $u_0^*=c_2$, $u_1^*=c_2$, $u_2^*=c_2$, for which $\mu_D(\cdot|s_1, \text{"all"})=0.6$, and corresponds to the path "—".
3. The arcs for which (58) holds are shown by "· · ·".
4. The arcs found in Step 3 are contained in the two paths denoted by "— — —" which correspond to

$$c_1 \quad c_2 \quad c_1 \qquad c_1 \quad c_2 \quad c_2$$

5. For the paths found in Step 4 we calculate using (59):

$$\mu_D(c_1, c_2, c_1 | s_1, \text{"most"})=0.5$$

$$\mu_D(c_1, c_2, c_2 | s_1, \text{"most"})=0.7$$

as given below the respective x_3' s.

The optimal solution to our problem, i.e. the sequence of controls u_0^*, u_1^*, u_2^* which best satisfies the fuzzy constraints and fuzzy goals at most of the control stages, is

$$u_0^*=c_1 \qquad u_1^*=c_2 \qquad u_2^*=c_2$$

4.3. Some remarks on importances and belief qualification and discounting

It is clear that if we introduced importances of the particular control stages following the reasoning presented in Sections 3.1 and 3.2, the counterparts of the problems to be solved would stand for "find an optimal sequence of controls which best satisfies the fuzzy constraints and fuzzy goals at, e.g., most of the important (earlier) control stages". We obtain therefore a mechanism for introducing discounting which is an important element of many multistage decision making models, reflecting a natural tendency to put more emphasis on what happens in the short term rather than in the distant future.

A different mechanism for introducing some sort of discounting is through belief qualification. Namely, if the degree of belief is decreasing as t increases, the consecutive pieces of evidence with implied bel=1, due to (9), have "flatter and flatter" membership functions, i.e. more and more values of the (degree of) fulfilment of the fuzzy constraints and goals are considered possible. Hence, they have a diminishing influence on the fuzzy decision, and as a consequence on the results. The above virtually reflects the fact that we know less about the later control stages, i.e. those in the distant future.

The above two views on discounting imply some interesting properties and will be discussed in a later paper.

5. Concluding remarks

In ther paper we discussed the introduction of fuzzy linguistic quantifiers and belief qualification into multicriteria and multistage decision making models to further "soften" them. The proposed problem formulations seem to be very intuitively appealing by better reflecting how the problems considered are really perceived by humans. On the other hand, in most cases the solutions may be efficiently obtained.

The approach discussed seems to be a further step in introducing elements of a natural language into decision making models to bring them closer to reality, and hence to make them easier implementable.

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Received, May 1984.

Kwantyfikatory lingwistyczne i kwalifikacja ufności w wielokryterialnym i wieloetapowym podejmowaniu decyzji w warunkach rozmytości

Rozpatruje się wprowadzenie pewnych elementów języka naturalnego do podstawowego sformułowania wielokryterialnego i wieloetapowego podejmowania decyzji w warunkach rozmytości. Wprowadzając najpierw kwantyfikatory lingwistyczne, zaproponowano modele umożliwiające: po pierwsze, poszukiwanie optymalnej decyzji najlepiej spełniającej większość, prawie wszystkie itp. rozmyte cele i ograniczenia, a po drugie, optymalnego ciągu sterowań najlepiej spełniających rozmyte cele i ograniczenia na większości, prawie wszystkich itp. etapach sterowania. Ponadto, w modelach umożliwiono przypisanie każdemu rozmytemu celowi i ograniczeniu ważności i ufności. Zastosowano rachunek zdań z kwantyfikatorami lingwistycznymi oparty na logice rozmytej.

Лингвистические кванторы и квалификация доверенности в многокритериальном и многоэтапном процессе принятия решений в условиях нечеткости

Рассматривается введение некоторых элементов естественного языка в основную формулировку многокритериального и многоэтапного процесса принятия решений в условиях нечеткости. Вводя вначале лингвистические кванторы предлагаются модели, позволяющие во-первых проводить поиск оптимального решения, наилучшим образом удовлетворяющего большинство, почти все и т.п. нечеткие цели и ограничения, а во-вторых, проводить поиск оптимальной последовательности управлений, наилучшими образом удовлетворяющих нечеткие цели и ограничения в большинстве, почти всех и т.п. этапах управления. Кроме этого в моделях имеется возможность приписания каждой нечеткой цели и ограничению важности и доверенности. Используется исчисление высказываний с лингвистическими кванторами, основанное не нечеткой логике.