Control
and Cybernetics
VOL. 13 (1984) No. 3

Multiobjective programming problems with fuzzy parameters

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In this paper, two approaches to the analysis of multiobjective programming problems are presented based on a systematic extension of the traditional problem formulation for obtaining a statement applicable in processing information in the form of fuzzy sets. Solutions are based on trade-offs between achieving the greatest possible degree of nondominance and greatest possible degree of feasibility.

1. Introduction

Multiobjective (MO) programming problems form a subclass of decisionmaking problems in which preferences between alternatives are described by means of a number of objective functions defined on the set of alternatives in such a way that greater (or smaller) values of any of these functions correspond to more preferable alternatives with respect to the corresponding objective. Values of the objective functions describe the effects from choices of one or other alternative. In economic problems, as an example, these values may reflect profits obtained using various production means; in water management problems they may mean electric power production for various water yields from a reservoir. The set of feasible alternatives in MO problems is described by means of equations and/or inequalities representing relevant relationships between variables. In any case, results of the analysis using given formulation of an MO problem depend largely upon the degree of adequacy with which various factors of the real system or a process are reflected in the description of the objective function and of the constraints.

Descriptions of the objective functions and of the constraints in a MO problem include parameters, the values of these parameters being considered as data that should be supplied exogenously for the analysis. Clearly, the values of such parameters depend on multiple factors not included in the formulation of the problem.

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If when attempting to make the model more representative of the real system we include the corresponding complex dependences to it, then the model may become cumbersome and analytically intractable. Moreover, it can happen that such attempts to increase "the adequateness" of the model will be of no practical value due to the impossibility to measure, or, to measure the values of the newly introduced parameters to a sufficient accuracy. On the other hand, the model with some fixed values of its parameters may still be too crude, since these values are often chosen in a quite arbitrary way.

An intermediate and flexible approach may be based on the introduction of the means of a more adequate representation of experts' understanding of the nature of the parameters in the form of fuzzy sets of their possible values into the model. The resultant model, although not taking into account many details of the real system in question, is a more adequate representation of the reality than that with more or less arbitrarily fixed values of the parameters. In this way we obtain a new type of MO problems containing fuzzy parameters. Treating such problems requires application of fuzzy-set theoretic tools in a logically consistent manner.

MO programming problems with fuzzy information were extensively analyzed and many papers have been published displaying a variety of formulations and approaches to their analysis (see, for instance, Zimmerman, 1978; Yager, 1978; Takeda and Nishida, 1980; Hannan, 1981; Luhandjula, 1982; Feng, 1983; Buckley, 1983; Tong, 1982). Most of the approaches to fuzzy MO problems are based on the straightforward use of the intersection of fuzzy sets representing goals and constraints and on the subsequent maximization of the resultant membership function.

Here we present two approaches based on a systematic extension of the traditional formulation of MO problems with fuzzy parameters to obtain a formulation applicable for processing information in the form of fuzzy sets. This paper is based on results described in Orlovski 1978, 1980, 1981, 1983.

Two aspects of a fuzzy MO problem are of major importance. The first is that while in a traditional problem each objective function represents a linear ordering of alternatives, in a fuzzy MO problem we have only fuzzy preference relations between alternatives. Due to this, the concept of domination requires further definition and we can only speak about determining alternatives with various degrees of nondominance. The second aspect lies in the fact that in a fuzzy MO problem alternatives can be chosen only on the basis of trade-offs among two generally conflicting objectives: achieving greater possible degree of nondominance and greater possible degree of feasibility. Both aspects are considered in this paper.

2. Problem formulation

Here we assume that alternatives from a given set $X=\{x\}$ are pairwise compared with each other using n objective functions $J_i(x, \bar{q})$, $i=1, \ldots, n$ in such a way that greater values of each of these functions are considered to be more preferable. Each of these functions contains a vector of parameters $\bar{q}$ whose values are known fuzzily and described by means of the membership function $\mu; Q \to [0, 1]$. We also assume that the subset of feasible alternatives is described by means of a system of inequalities

$$\bar{\psi}_{\xi}(x, \bar{p}) \leqslant 0, \ j=1, \ldots, m$$

in which the vector of parameters $\bar{p}$ has a fuzzy value $v(\bar{p})$. Speaking informally, the problem lies in determining feasible alternatives giving greater possible values of the objective functions.

In a traditional MO problem, when the values of parameters $\bar{q}$ are specified unambiguously, the rational choices are Pareto-maximal alternatives which can be determined using well-known computational techniques. Here, with fuzzily specified values of parameters $\bar{q}$, we can have only fuzzy evaluations of the corresponding objective functions and, in addition, should therefore define the meaning of a rational choice more precisely.

First, we describe these approaches assuming that the set of feasible alternatives is nonfuzzy and coincides with the set X, and consider the fuzzy set of feasible alternatives later for both approaches.

3. First approach

3.1. Reformulation of the problem

This approach is based on the consideration of value levels of the objective functions which can be attained to a highest possible degree. More formally, we understand our problem in the following way:

$$(J_1^0, \ldots, J_n^0, \alpha) \to \overline{\max_{x, J_i^0, \alpha}} \tag{1a}$$

$$\text{degree}\, \{J_1(x, \bar{q}) \geqslant J_1^0, \ldots, J_n^0(x, \bar{q}) \geqslant J_n^0\} \geqslant \alpha \tag{2a}$$

$$x \in X$$

Maximization in (1a) is understood, of course, in the Pareto sense and to indicate this we use the symbol $\overline{\max}$. Constraint (2a) reflects the fact that with fuzzy values of functions J_i we can only consider satisfying the inequalities $J_i(x, \bar{q}) \geqslant J_i^0$ to a certain degree. An essential point in this formulation is that the multiobjective choice in this case should be based not on the trade-offs among the values of the objective functions, which are fuzzy due to the fuzzy nature of parameters $\bar{q}$, but among the lower estimates of these values obtainable to a certain degree α. This formulation also implies that when deciding upon the trade-offs among the lower estimates of the objectives, the decisionmaker should consider the possible degree α of these estimates.

In the following subsection we demonstrate that the above formulation can be reduced to a traditional form of a MO problem.

3.2. Analysis of the problem

For convenience, we shall consider functions $J_i(x, \bar{q})$, $i=1, \ldots, n$ as components of a vector function $\bar{J}(x, \bar{q})$ with values from the real vector space R^n. If we denote by $\bar{J}^0$ the vector of levels $(J_1^0, \ldots, J_n^0)$, then problem (1a)–(2a) can be written in the form:

$$(\bar{J}^0, \alpha) \to \overline{\max_{x, \bar{J}^0, \alpha}} \tag{1b}$$

$$\text{degree}\,\{\bar{J}(x, \bar{q}) \geqslant \bar{J}^0\} \geqslant \alpha \tag{2b}$$

$$x \in X$$

To formulate constraints (2b) more explicitly, we can directly use the extension principle (Zadeh, 1973) to obtain:

$$\text{degree}\,\{\bar{J}(x, \bar{q}) \geqslant \bar{J}^0\} = \sup_{\bar{q}:\, \bar{J}(x, \bar{q}) \geqslant \bar{J}^0} \mu(\bar{q}). \tag{3}$$

which, in fact, represents the extension of the "greater or equal" relation from the vector space R^n onto the class of fuzzy vector-values of the objective vector--function with fuzzy parameters $\bar{q}$.

Finally, using (3) we can formulate problem (1b)–(2b) as follows:

$$(\bar{J}^0, \alpha) \to \overline{\max_{x, \bar{J}^0, \alpha}}$$

$$\sup_{\bar{q}:\, \bar{J}(x, \bar{q}) \geqslant \bar{J}^0} \mu(\bar{q}) \geqslant \alpha \tag{4}$$

$$x \in X$$

If some tuple $(x^s, \bar{J}^{0s}, \alpha^s)$ is a solution to this problem, then the tuple $(J_1^{0s}, \ldots, J_n^{0s}, \alpha^s)$ is Pareto optimal which means that any other alternative x providing for better values of some of the components of $(\bar{J}^0, \alpha)$ gives worse values of some of the other components of this tuple.

Now it is a simple exercise to verify that for functions $J_i(x, \bar{q})$ continuous in $\bar{q}$ (for any $x \in X$), $i=1, \ldots, n$ and $\mu(\bar{q})$ problem (4) can equivalently be formulated as follows:

$$(\bar{J}^0, \alpha) \to \overline{\max_{x, \bar{q}, \bar{J}^0, \alpha}}$$

$$\mu(\bar{q}) \geqslant \alpha$$

$$\bar{J}(x, \bar{q}) \geqslant \bar{J}^0$$

$$x \in X, \bar{q} \in Q,$$

and finally, in the form:

$$(\bar{J}(x, \bar{q}), \alpha) \to \overline{\max_{x, \bar{q}, \alpha}} \tag{5}$$

$$\mu(\bar{q}) \geqslant \alpha$$

$$x \in X, \bar{q} \in Q.$$

4. Second approach

4.1. Reformulation of the problem

This approach is based on the extension of the natural order of values of the objective functions on the real line onto the class of fuzzy subsets of this line. In this, we obtain the preference relations which can be used for comparing the fuzzy values of the objective functions for various alternatives. Then, using these relations, we define a strict fuzzy preference relation on the set of alternatives and determine the corresponding fuzzy subset of nondominated alternatives.

As before, we consider n objective functions $J_i(x, \bar{q})$, with $\bar{q}$ being a fuzzily-valued vector of parameters described by membership function $\mu(\bar{q})$. Using the extension principle the corresponding fuzzy values of these functions can be obtained in the following form:

$$\varphi_i(x, r) = \sup_{\bar{q}: J_i(x, \bar{q}) = r} \mu(\bar{q}), \quad i = 1, ..., n. \tag{6}$$

Now we can obtain the following fuzzy nonstrict preference relations induced on the set of alternatives by φ_i:

$$\eta_i(x_1, x_2) = \sup_{z \geqslant y} \min \{\varphi_i(x_1, z), \varphi_i(x_2, y)\}, \quad i = 1, ..., n. \tag{7}$$

The next step is to define a way of comparing alternatives with each other using all these n preference relations. To do that we define strict dominance relation on X in the following way. Let $\eta_i^s(x_1, x_2)$ be the fuzzy strict preference relation corresponding to $\eta_i(x_1, x_2)$, defined as follows (see Orlovski, 1978):

$$\eta_i^s(x_1, x_2) = \begin{cases} \beta = \eta_i(x_1, x_2) - \eta_i(x_2, x_1), \text{ if } \beta > 0 \\ 0, \text{ otherwise} \end{cases}$$

Then we say that the degree $\eta^s(x_1, x_2)$ to which alternative x_1 is strictly prefered to alternative x_2 is as follows:

$$\eta^s(x_1, x_2) = \min_i \eta_i^s(x_1, x_2).$$

In a nonfuzzy formulation this would mean that x_1 is strictly preferable to x_2 if it is strictly better than x_2 with respect to each objective function. The respective nondominated alternatives are commonly referred to as semiefficient or weakly effective.

Having defined η^s we can describe the corresponding fuzzy subset η^{ND} of nondominated alternatives in the form (Orlovski, 1978):

$$\eta^{ND}(x) = 1 - \sup_{y \in X} \eta^s(y, x) = 1 - \sup_{y \in X} \min_i \eta_i^s(y, x),$$

and using the above formulation of η_i^s, we have:

$$\eta^{ND}(x) = 1 - \sup_{y \in X} \min_i [\eta_i(y, x) - \eta_i(x, y)]. \tag{8}$$

The value $\eta^{ND}(x)$ is the nondominance degree of the respective alternative. If $\eta^{ND}(x) \geqslant \alpha$ then alternative x may be strictly dominated by some other alternative to a degree smaller than $1-\alpha$.

4.2. Determining alternatives nondominated to a prespecified degree

Now consider the problem of determining alternatives satisfying:

$$\eta^{ND}(x) \geqslant \alpha, \tag{9}$$

where α is the desired degree of nondominance.

Let us formulate the following nonfuzzy multiobjective problem:

$$\bar{r}=(r_1, \ldots, r_n) \to \overline{\max_{\bar{r}, x}} \tag{10}$$

$$\varphi_i(x, r_i) \geqslant \alpha, \; i=1, \ldots, n.$$

$$x \in X, \; \bar{r} \in R^n.$$

The following theorem states that under some conditions any solution z to problem (10) satisfies (9).

THEOREM *If for any of the functions $\varphi_i(\cdot, \cdot)$, $i=1, \ldots, n$ and any $x \in X$ there exist $r_i \in R^1$, $i=1, \ldots, n$ such that $\varphi_i(x, r_i) \geqslant \alpha$, then for any solution to problem* (10) *we have $\eta^{ND}(x) \geqslant \alpha$.*

P r o o f. Let $(x^0, \bar{r}^0)$ be a solution to problem (10). Then, as follows from (8), to prove the theorem it suffices to show that

$$\sup_{y \in X} X \min_i [\eta_i(y, x^0) - \eta_i(x^0, y)] \leqslant 1-\alpha.$$

Assume the contrary, i.e. that $y' \in X$ and $\varepsilon > 0$ can be found, such that

$$\min_i [\eta_i(y', x^0) - \eta_i(x^0, y')] > 1-\alpha+\varepsilon,$$

or

$$\eta_i(y', x^0) - \eta_i(x^0, y') > 1-\alpha+\varepsilon, \; i=1, \ldots, n. \tag{11}$$

Using (7) we can write (11) in the form:

$$\sup_{r_i \geqslant z_i} \min \{\varphi_i(y', r_i), \varphi_i(x^0, z_i)\} - \sup_{r_i \geqslant z_i} \min \{\varphi_i(x^0, r_i), \varphi_i(y', z_i)\} > \\ > 1-\alpha+\varepsilon \quad i=1, \ldots, n. \tag{11a}$$

Let us choose z'_i, $i=1, \ldots, n$, such that $\varphi_i(y', z'_i) \geqslant \alpha$ for all $i=1, \ldots, n$ (the existence of z'_i, $i=1, \ldots, n$ follows from the assumptions about functions φ_i). Since $(x^0, \bar{r}^0)$ is a solution to problem (10), we have that $r^0_{i_0} \geqslant z'_i$ for at least one $i=i_0$ among $i=1, \ldots, n$. Thus we have

$$\varphi_{i_0}(x^0, r^0_{i_0}) \geqslant \alpha, \; \varphi_{i_0}(y', z'_{i_0}) \geqslant \alpha, \; r^0_{i_0} \geqslant z'_{i_0}.$$

Therefore, we have

$$\sup_{r_{i_0} \geqslant z_{i_3}} \min \{\varphi_{i_0}(x^0, r_{i_0}),\ \varphi_{i_0}(y', z_{i_0})\} \geqslant \alpha .$$

Hence, the inequality with index i_0 in (11a) does not hold, since its first additive term does not exceed 1. This contradiction proves the theorem. ■

Using (6) we can now write problem (11) as:

$$\bar{r} = (r_1, \ldots, r_n) \to \overline{\max_{\bar{r},\, x}}$$

$$\sup_{\bar{q}_0\ J_i(x, \bar{q}) = r_i} \mu(\bar{q}) \geqslant \alpha,\ i = 1, \ldots, n,$$

$$x \in X,$$

or equivalently:

$$\bar{J}(x, \bar{q}) \to \overline{\max_{\bar{q},\, x}} \tag{12}$$

$$\mu(\bar{q}) \geqslant \alpha,$$

$$x \in X,\ \bar{q} \in Q .$$

As can be seen this formulation is quite the same as the corresponding MO formulation (5) for the first approach (see Sect. 3.2) in the case of a fixed α. Therefore, both the approaches are equivalent to each other in the sense that both may lead to choices of the same alternatives.

5. Fuzzy set of feasible alternatives

Let us also assume that the set of feasible alternatives is described by the following system of inequalities:

$$\bar{\psi}(x, \bar{p}) \leqslant 0, \tag{13}$$

with $\bar{\psi}$ being a given real vector-valued function, and $\bar{p}$ being a vector of parameters with the membership function $v: P \to [0, 1]$ fuzzily describing its possible values.

To use this type of information, we first determine a clear description of the corresponding fuzzy subset of feasible alternatives in the form of a membership function $\omega(x)$. If we introduce the notation

$$P(x) = \{\bar{p} \mid \bar{p} \in P,\ \bar{\psi}(x, \bar{p}) \leqslant 0\}$$

then using the extension principle we can write this membership function in the form:

$$\omega(x) = \sup_{\bar{p} \in P(x)} v(\bar{p}) .$$

The value $\omega(x)$ of this function is understood as the feasibility degree of the corresponding alternative, and these values should also be taken into account when making choices of alternatives.

Alternatives in this case should be evaluated by two generally conflicting factors: their degree of nondominance $\eta^{ND}(x)$ (in the second approach) and their degree of feasibility $\omega(x)$. Let α be the desired degree of nondominance and β be the desired level of feasibility. Then an alternative having a degree of nondominance not smaller than α and feasible to a degree not smaller than β should satisfy the following inequalities:

$$\eta^{ND}(x) \geqslant \alpha, \quad \omega(x) \geqslant \beta.$$

Therefore, with the fuzzy set of feasible alternatives formulation (12) will have the following additional constraints:

$$\sup_{\bar{p} \in P(x)} v(\bar{p}) \geqslant \beta,$$

$$P(x) = \{\bar{p} \mid \bar{p} \in P, \ \bar{\psi}(x, \bar{p}) \leqslant 0\}.$$

And it can be easily verified that with this type of constraints problem (12) can be written as follows:

$$\bar{J}(x, \bar{q}) \to \overline{\max_{\bar{q}, \bar{p}, x}}$$

$$(x, \bar{p}) \leqslant 0,$$

$$\mu(\bar{q}) \geqslant \alpha, \quad v(\bar{p}) \geqslant \beta,$$

$$x \in X, \ \bar{q} \in Q, \ \bar{p} \in P.$$

By varying the values of α and β we can determine the alternatives with various trade-offs among the degrees of nondominance and feasibility.

6. Concluding remarks

Two approaches to MO problems with fuzzy parameters are suggested in this paper. Both are based on the systematic use of the extension principle as the means of processing fuzzy information about parameters. The rationality of choice is based on trade-offs among degrees of feasibility and nondominance. It is shown that rational alternatives in both approaches can be determined by solving similar MO problems in a traditional form.

The use of fuzzy sets for describing information about real systems is a relatively new area and further work is needed in order to find practically effective methods allowing to combine the fuzziness of human judgement with the powerful logics and tools of mathematical analysis. Successful development in this direction may help overcome one of the essential obstacles in application of mathematical modeling to the analyses of real systems, namely, the existing gap between the language used in mathematical models and the language used by potential users of those models.

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Received, May 1984.

Zadania programowania wielokryterialnego z rozmytymi parametrami

Rozważa się dwa podejścia do analizy zadań programowania wielokryterialnego, oparte na systematycznym rozszerzaniu standardowego sformułowania problemu otrzymania postaci zadań, które nadawałyby się do uwzględnienia informacji określonej w formie zbiorów rozmytych. Rozwiązania opierają się na kompromisach pomiędzy osiąganiem możliwie wysokiego stopnia niezdominowania, a osiąganiem możliwie dobrej dopuszczalności.

Задачи многокритериального программирования с нечеткими параметрами

Рассматриваются два подхода к анализу задач многокритериального программирования, основанные на систематическом расширении стандартной формулировки проблемы получения такого вида задач, который позволил бы учитывать информацию, определяемую в форме нечетких множеств. Решения основаны на компромиссе между достижением возможно высокой степени непреобладания и достижением достаточной допустимости.