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A fuzzy linear programming problem with equality constraints

by

STEFAN CHANAS

MICHAŁ KULEJ

Institute of Management
Technical University of Wroclaw
u. Smoluchowskiego 25
50-372 Wrocław, Poland

A fuzzy linear programming problem with equality constraints is analysed. It is shown that for obtaining the solution of the problem the method of parametric programming for linear programs with upper bounded variables can be used. This method makes it possible to obtain not only a maximizing solution according to Bellman and Zadeh's criterion but also other alternatives close to an optimal solution.

1. Introduction

The pionieering paper by Bellman and Zadeh [2] on decision making in a fuzzy environment has created a methodological basis for the development of fuzzy mathematical programming methods. A formulation of the fuzzy linear programming problem [5, 7, 9] is also a result of the above mentioned paper. A quite new element in Bellman and Zadeh's approach is a fuzzy decision (a fuzzy set in the space of alternatives) resulting from a confluence of fuzzy goals and constraints in the decision problem. The membership function of a fuzzy decision orders the alternatives according to the degree of simultaneous satisfaction of fuzzy constraints and goals. The problem of determination of a maximizing solution (an alternative maximizing the membership function of the fuzzy decision) can be reduced to a mathematical programming problem [7]. In particular, in the case of fuzzy linear programming the problem reduces to a classical linear programming one [4, 8]. However, it is, a simplification if one restricts the analysis of a maximizing alternative to solve the problem. The solution in the form of a fuzzy decision gives the decision-maker a more complete information, as well as about other "possible" alternatives. It has been shown in [3] that using the parametric linear programming method to solve a fuzzy linear programming problem one can get more than only a maximizing solution — it is namely possible to obtain an analytically expressed information about other alternatives "close" to the maximizing one.

In [5, 7, 9] and in the majority of papers developed till now in the field of fuzzy linear programming, the problem with inequality constraints is considered. Here we analyse the problem with constraints given in the form of fuzzy equalities. As a matter of fact, the problem can be transformed into the one with inequality constraints by replacing each equality constraint with two inequality constraints, but this operation leads to a considerable enlargement of the size of the problem.

In section 3 we show that to solve the problem without increasing its size one can use, similarly as in [3], the method of parametric programming — but this time the method adapted to programs with upper bounded variables [8].

2. Formulation of the problem

Let us consider the following formulation of the fuzzy linear programming problem

$$\begin{gathered} f(x)=\sum_{j=1}^{n} c_j x_j \to \widetilde{\max} \\ \sum_{j=1}^{n} a_{ij} x_j \cong \tilde{b}_i \quad i=1,2,\ldots,m \\ x_j \geqslant 0 \quad j=1,2,\ldots,n \end{gathered} \tag{1}$$

where $\tilde{b}_i$ $(i=1,2,\ldots,m)$ are fuzzy numbers of trapezoidal form (see Fig. 1), $\tilde{b}_i = (b_i^1, \underline{b}_i^1, b_i^2, \bar{b}_i^2)$ [4]. In a particular case there may be $b_i^1=b_i^2$ (a triangular form of the number $\tilde{b}_i$), $\underline{b}_i^1=0$ or $\bar{b}_i^2=0$ (one-sided linear tolerances) and also $\underline{b}_i^1=\bar{b}_i^2=0$

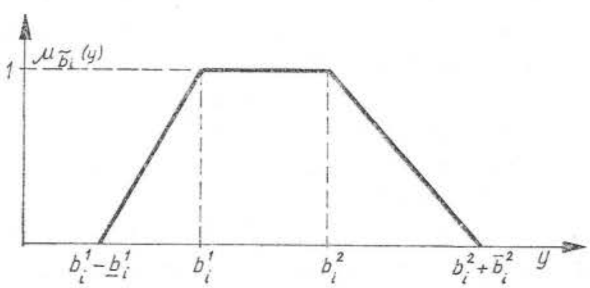

Fig. 1. The trapezoidal form of a fuzzy number $\tilde{b}_i$

(crisp intervals). The value of the membership functions $\mu_{\tilde{b}_i}$ at $\sum_{j=1}^{n} a_{ij} x_j$, $\mu_{\tilde{b}_i}(\sum_{j=1}^{n} a_{ij} x_j)$, is interpreted, similarly as in [5], as a feasibility degree of the solution $x=(x_1, \ldots, x_n)$ with respect to the i-th constraint. With the objective function of (1) we associate also a fuzzy number $\tilde{G}$ determining a "required" value of this function. The membership function $\mu_{\tilde G}$ of the fuzzy number $\tilde G$ is assumed to be

$$\mu_{\tilde G}(y)=\begin{cases} 0 & \text{for} \quad y\le c_1, \\ g(y) & \text{for} \quad y\in[c_1, c_0], \\ 1 & \text{for} \quad y\ge c_0, \end{cases} \tag{2}$$

where $g(y)$ is a continuous function, increasing from zero at c_1 to one at c_0 (see Fig. 2). In a particular case $g(y)$ may be a linear function. $\mu_{\tilde G}(f(x))$ determines the degree of the decision-maker's satisfaction with the achieved value of $f(x)$.

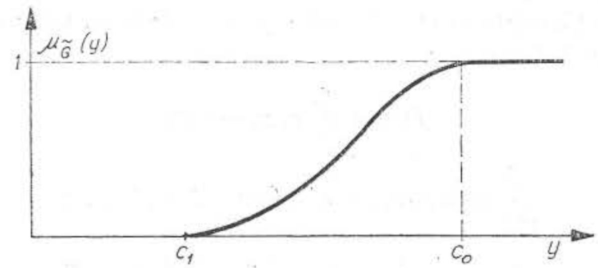

Fig. 2. The membership function of a fuzzy goal $\tilde G$

The problem of finding a maximizing alternative in the fuzzy decision given by (1), in accordance with Bellman and Zadeh's approach, reduces to the following problem

$$\mu_{\tilde D}(x)=\mu_{\tilde G}(f(x))\wedge\mu_{\tilde C}(x)\to\max \tag{3}$$

where

$$\mu_{\tilde C}(x)=\bigwedge_{1\le i\le m}\mu_{\tilde b_i}\left(\sum_{j=1}^{n}a_{ij}x_j\right) \tag{4}$$

and "$\wedge$" stands for the minimum operation. $\mu_{\tilde D}(x)$ means the degree to which x simultaneously satisfies the fuzzy constraints and fuzzy goal. Later we shall notice that the "$\wedge$" operation in (3) (but only in (3)) could be replaced by any other operation, nondecreasing with respect to its components. All operations from the class of t-norms (see [1, 6]) have this property.

3. Parametric approach to the problem

In this section we show that using the technique of parametric programming for linear programs with upper bounded variables (see [8]) we can analytically describe a set of solutions incorporating the whole range of possible values of the membership function $\mu_{\tilde D}$, i.e. we can identify practically the complete fuzzy decision $\tilde D$, not only a maximizing alternative. Let us go into details.

The $(1-r)$ — cuts, $r\in(0, 1]$, of the fuzzy numbers $\tilde b_i$ $(i=1, \dots, m)$ by the earlier accepted assumptions about these numbers are intervals of the form

$$\tilde b_i^{1-r}=\{y\mid\mu_{\tilde b_i}(y)\ge 1-r\}=[b_i^1-r\underline{b}_i^1,\; b_i^2+r\bar b_i^2] \tag{5}$$

Let us consider the following problem

$$
\begin{gathered}
f(x)=\sum_{j=1}^{n} c_j x_j \to \max \\
\sum_{j=1}^{n} a_{ij} x_j \in \tilde{b}_i^{1-r} \qquad i=1,2,\dots,m \\
x_j \geqslant 0 \qquad j=1,2,\dots,n
\end{gathered}
\tag{6}
$$

where $r\in[0,1]$ is a parameter.

The problem (6) reduces to the following parametric linear programming problem with upper bounded variables

$$
\begin{gathered}
f(x)=\sum_{j=1}^{n} c_j x_j \to \max \\
\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i^2 + \bar{b}_i^2 r \qquad i=1,2,\dots,m \\
x_{n+i} \leqslant b_i^2 - b_i^1 + (\underline{b}_i^1 + \bar{b}_i^2) r \qquad i=1,2,\dots,m \\
x_j \geqslant 0 \qquad j=1,2,\dots,n+m
\end{gathered}
\tag{7}
$$

where $r\in[0,1]$ is a parameter of variation. By solving the problem (7), for instance using the method presented in [8], we obtain an analytically expressed set of solutions $\{x(r)\}$, $r\in[0,1]$, explicitly depending on r. The solution $x(r)$ for a given r fulfils the constraints of (1) at least to the degree of $1-r$ $\big(\mu_{\tilde{C}}(x(r))\geqslant 1-r\big)$ and simultaneously maximizes the objective function $f(x)$ $\big($thereby $\mu_{\tilde{G}}(f(x))\big)$ by this condition.

REMARK. If $\underline{b}_i^1, \bar{b}_i^2>0$ for all $i=1,\dots,m$ then $\mu_{\tilde{C}}(x(r))=1-r$, except for the particular case when $x(r)$ is the zero vector. It results from the fact that $x(r)$ appears to be the basic solution to (7). The $y=f(x(r))$ substituted into (2) gives in effect the membership function $\mu_{\tilde{G}}$ which depends on the parameter r. By finding the value of parameter r for which

$$
\mu_{\tilde{D}}(x(r))=\mu_{\tilde{G}}\big(f(x(r))\big)\wedge(1-r)\to\max \tag{8}
$$

we identify a maximizing solution. $\mu_{\tilde{D}}(x(r))$ provides us also with information about a solution "close" to the maximizing alternative.

The operation "$\wedge$" in (8), as it was mentioned before, may be replaced by any other operation, which is nondecreasing with respect to its components.

Let us illustrate the proposed method with a numerical example. Consider the following fuzzy linear programming problem:

$$
\begin{gathered}
f(x)=2x_1-x_2+x_3\to \widetilde{\max} \\
x_1-x_2+2x_3\cong\tilde{b}_1 \\
2x_1+2x_2-x_3\cong\tilde{b}_2 \\
-x_1+x_2+x_3\cong\tilde{b}_3 \\
x_1,x_2,x_3\geqslant 0
\end{gathered}
\tag{9}
$$

where $\tilde{b}_1=(6, 1, 6, 4)$, $\tilde{b}_2=(4, 1, 4, 3)$, $\tilde{b}_3=(1, 4, 1, 2)$ are fuzzy numbers of the triangular form. The membership function of the fuzzy goal is linear:

$$\mu_{\tilde{G}}(y)=\begin{cases} 0 & \text{for } y\leqslant 6 \\ \frac{1}{6}y-1 & \text{for } 6\leqslant y\leqslant 12 \\ 1 & \text{for } y\geqslant 12 \end{cases} \tag{10}$$

The problem (9) reduces to the following parametric linear programming problem with upper bounded variables:

$$\begin{aligned} f(x)=2x_1-\ x_2+\ x_3\to\max \\ x_1-\ x_2+2x_3+x_4 \qquad\qquad &=6+4r \\ 2x_1+2x_2-\ x_3 \qquad +x_5 \qquad &=4+3r \\ -x_1+\ x_2+\ x_3 \qquad\qquad +x_6&=1+2r \\ x_4\leqslant 5r,\quad x_5\leqslant 4r,\quad x_6\leqslant 6r \\ x_i\geqslant 0 \quad i=1, \dots, 6. \end{aligned} \tag{11}$$

Using Panwalkar's method [8] to solve (11) we obtain the solution of (9) in the form

$$x_1=\frac{9}{4}+\frac{11}{4}r,\quad x_2=\frac{11}{2}-\frac{5}{4}r,\quad x_3=\frac{7}{3}\quad \text{for } r\in\left[0, \frac{11}{15}\right]$$

and

$$x_1=\frac{14}{5}+2r,\quad x_2=0,\quad x_3=\frac{8}{5}+r\quad \text{for } r\in\left[\frac{11}{15}, 1\right].$$

The membership function $\mu_{\tilde{G}}(f(x(r))$ takes on the form:

$$\mu_{\tilde{G}}\left(f(x(r))\right)=\begin{cases} 0 & \text{for } 0\leqslant r\leqslant\frac{1}{81} \\ -\frac{1}{72}+\frac{9}{8}r & \text{for } \frac{1}{81}\leqslant r\leqslant\frac{11}{15} \\ \frac{1}{5}+\frac{5}{6}r & \text{for } \frac{11}{15}\leqslant r\leqslant\frac{24}{25} \\ 1 & \text{for } r\geqslant\frac{24}{25} \end{cases}$$

The maximizing solution is obtained for $r=\frac{73}{153}\approx 0.477$ and equals $x(0.477)=$ $=(3.56, 0.32, 2.33)$ (see Fig. 3).

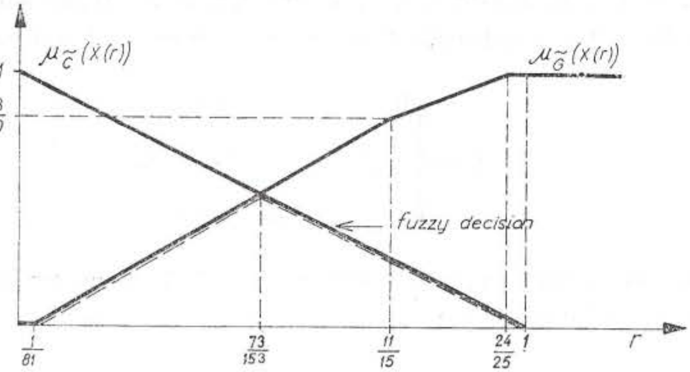

Fig. 3. Fuzzy decision for problem (9)

4. Concluding remarks

It seems natural that the solution of a fuzzy linear program should be fuzzy, too. The approach proposed in this paper fulfils this postulate. A fuzzy solution obtained by the proposed method provides an information (analytically expressed) not only about a maximizing alternative but also about other possible alternatives and their membership degrees in the fuzzy decision.

It is worth underlining that an extension of the size of the problem is not needed here contrarily to the earlier approaches. Transformation of the fuzzy linear program in to the respective parametric linear program does not enlarge the number of main constraints which must be considered in the simplex table.

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Rozmyte zadanie programowania liniowego z ograniczeniami równościowymi

Rozważa się zadanie rozmytego programowania liniowego z ograniczeniami równościowymi. Pokazano, że do otrzymania rozwiązań tego zadania można użyć metody programowania parametrycznego dla programów liniowych z ograniczeniami od góry na wartości zmiennych. Zaproponowana procedura umożliwia nie tylko otrzymanie rozwiązania maksymalizującego w myśl kryterium Bellmana-Zadeha, ale również i otrzymanie alternatywnych bliskich optymalnemu.

Нечеткая задача линейного программирования с ограничениями в виде равенств

Рассматривается задача нечеткого линейного программирования с ограничениями в виде равенств. Показано, что для получения решений этой задачи можно использовать методы параметрического программирования для линейных программ с ограничениями сверху значений переменных. Предлагаемая процедура позволяет не только получить максимизирующее решение согласно критерию Беллмана-Заде, но также получить альтернативные решения, близкие оптимальному.