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Analysis of a fuzzy dynamic system and synthesis of its optimal controller

by

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A mathematical model and a new development of a systematic synthesis technique for a fuzzy dynamic system with a fuzzy controller will be presented. In a fuzzy dynamic system, a fuzzy relational matrix will be introduced as a counterpart of a differential equation in conventional control theory. Then, as an application of the concepts of fuzzy inverse problems, a new approach to the design of a fuzzy controller for a given dynamic system will also be discussed.

A mathematical description of a fuzzy dynamic system will be developed: a systematic method to derive a fuzzy controller strategy from an underlying fuzzy system model will also be established. The theoretical development presented here enables the suboptimal control of the fuzzy system. The analysis proposed here may not be exhaustive but it does provide some insight into the basic operations and properties of fuzzy dynamic systems.

As an illustrative example, a fuzzy controller problem with a unit delay will be discussed in detail and simulation results will be presented.

Key words: Fuzzy dynamic system, fuzzy controller, fuzzy inverse problem, fuzzy feedback control system

1. Introduction

The fuzzy methodology is most effective in decision-making processes where available sources of information are inaccurate, subjectively interpreted, or uncertain. In this paper, as an application of the concepts of fuzzy inverse problems, a novel approach leading to the design of a fuzzy controller will be discussed.

A mathematical description of a fuzzy dynamic system will be developed; a systematic method to derive a fuzzy controller strategy from an underlying fuzzy system model will also be established. The theoretical development presented here enables the automatic control of a fuzzy system.

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As an illustrative example, a fuzzy controller problem with a unit delay will be discussed in detail and the simulation results will be presented.

The concept of "fuzzy controller" as a viable control system component has been successfully and convincingly demostrated to be effective in solving practical problems (Mamdani 1976, Kickert and Van Nauta Lemke 1976, Mamdani, Østergaard, and Lembessis 1983). In the past, the heuristic controller strategies were obtained through the observation of the human controller's responses to the process.

In this paper, however, we propose a systematic method as a way to synthesize the controller strategies based upon the underlying fuzzy dynamic system behavior. First, the mathematical model for a fuzzy dynamic system with a fuzzy controller is discussed. In a fuzzy system, a fuzzy relational matrix will be introduced to model the system in lieu of a differential equation as in the conventional control theory. Then, a fuzzy controller, also modeled by a fuzzy relational matrix, will be synthesized from the underlying theory. This would be a novel approach in handling the fuzzy control problem for such a system.

A unit-delay model, first introduced by Tong (1980) and consecutively refined by Sugeno and Takagi (1983), makes it possible for fuzzy set theory to exercise a far reaching impact in solving dynamic system problems. We will develop a mathematical model for this dynamic system, then apply the techniques capable of solving fuzzy inverse problems as a vehicle to achieve the controller strategy synthesis from the system response descriptions.

The analysis presented in this paper may not be exhaustive but it is certainly capable of providing some insights so far as the basic operations and properties of fuzzy dynamic systems are concerned. As an illustrative example, a fuzzy controller problem shall be discussed in detail and the simulation results shall be shown.

2. Fuzzy Dynamic System Model

Consider a classical feedback control system as shown in Fig. 1. The command, error and output signals are all functions of non-fuzzy variables. If any of these values becomes fuzzy, then we have a fuzzy control system and the question of how to describe it properly becomes critical.

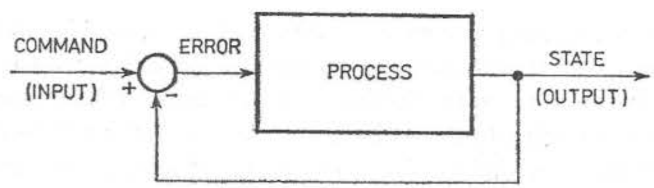

Fig. 1.

To deal with the fuzzy concept in the system, we have to adapt a conventional algebraic approach suitable for using the fuzzy set theory. In doing this, it is necessary to replace the comparison of command input and current state with a more general error function which maps two fuzzy variables to a fuzzy error variable.

Assuming that U is the universe of discourse of fuzzy state values and V is the universe of discourse of fuzzy command for the process and fuzzy error values, the error function f_e is then defined as

$$f_e\colon U\times V\to V$$

The process itself, on the other hand, is described by the function f_p:

$$f_p\colon U\times V\to U.$$

These two functions give a complete description of the process.

Assuming that the command and the state space are made finite and discrete, then the functions f_e and f_p are defined by the finite ternary functional matrices on the Cartesian product space $U\times V\times V$ and $U\times V\times U$, respectively. Further assume that the state Y is a fuzzy subset of U and that the command (input) X and the error E are fuzzy subsets of V; then, along with ternary relational matrices $R_e\subset U\times V\times V$ and $R_p\subset U\times V\times U$, the behavior of the system is governed by the following discrete time equations:

$$E_t=(Y_t\cap X_t)\circ R_e \tag{1}$$

$$Y_{t+1}=(Y_t\cap E_t)\circ R_p \tag{2}$$

where $\circ$ denotes the compositional rule of inference, $\cap$ denotes minimum, and t is an integer time index so that Y_t denotes the current state of the process and Y_{t+1} denotes the next state of the process. Both $(Y_t\cap X_t)$ and $(Y_t\cap E_t)$ denote binary fuzzy subsets on the Cartesian product space $U\times V$. The block diagram of this system is shown in Fig. 2. Note that this diagram represents a drastic departure from a dynamic system represented by the conventional block diagram.

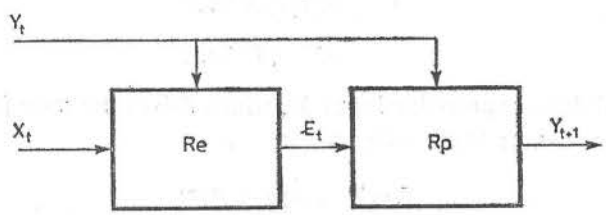

Fig. 2.

For the further investigation of the behavior of a fuzzy dynamic system, the following relational equation is required.

PROPOSITION. If X and Y are fuzzy subsets on U and V, respectively, and if R is a ternary relation on $V\times U\times V$, then

$$(Y\cap X)\circ R=Y\circ(X\circ R)=X\circ(Y\circ R) \tag{3}$$

where $\circ$ denotes the max-min operation (i.e., the Zadehian fuzzy inference). The proposition given above is essentially the same as that of Tong (1980).

Combining the error relation (1) and the process relation (2), we have

$$Y_{t+1}=[Y_t \cap ((Y_t \cap X_t) \circ R_e)] \circ R_p \tag{4a}$$

$$=(Y_t \cap X_t) \circ (R_e \circ R_p)$$

$$=(Y_t \cap X_t) \circ R \tag{4b}$$

where $R=R_e \circ R_p$.

In the above equation $(Y_t \cap X_t)$ denotes the binary fuzzy subset over the Cartesian product space $U\times V$; the finite ternary discrete relation R is defined over the Cartesian product space $U\times V\times U$. The overall fuzzy relation R is thus the transition relation for the closed loop system with a first order delay unit. In other words, in conjunction with Zadehian fuzzy inference, R gives the next state of the system in terms of the current state and current command input. Equation (4b) may also be expressed in block diagram form as shown in Fig. 3.

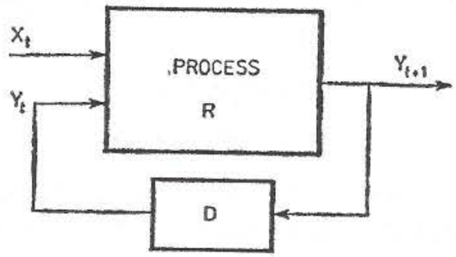

Fig. 3.

Based upon (4b) and Proposition, we have

$$Y_{t+1}=(Y_t \cap X_t) \circ R \tag{5}$$

$$=X_t \circ (Y_t \circ R).$$

Then it is possible to express the input X_t, which drives the state from Y_t to Y_{t+1}, in terms of the inverse fuzzy relation as:

$$X_t=Y_{t+1} * (Y_t \circ R)^{-1} \tag{6}$$

where $*$ denotes an inverse operator and the superscript -1 represents the inverse relation of $Y_t \circ R$. The essential characteristics and properties of the $*$ operator as well as the inverse relation can be found in the next section.

3. A Fuzzy Inverse Relation and some Inverse Compositions

A fuzzy inverse problem was first recognized and studied by Sanchez (1976). Tsukamoto (1977) and Togai (1982) extended Sanchez's work and proposed various new operators in order to provide a solution for the problem. The statement of the problem may be summarized as "given a fuzzy relation R over $U\times V$ and a fuzzy subset B of V, find a fuzzy subset A of U such that $A \circ R = B$". An inverse operator $*$ may be used to find such a fuzzy subset A through the composition

$$A = B * R^{-1}. \tag{7}$$

Definition of the fuzzy relation R^{-1} and the inverse operators proposed by the various researchers will now be presented.

Definition 1. (Inverse relation)
Let $R \subset U \times V$ be a fuzzy relation and u and v be generic elements of U and V, respectively. The inverse of a fuzzy relation R, denoted by R^{-1}, is the relation on $V \times U$ defined by

$$\mu R^{-1}(v, u) = \mu_R(u, v). \tag{8}$$

For the definitions to be presented, we will use p and q to present the real values over [0, 1].

Definition 2. ($\circ$-composition)
This composition is denoted $A = B \circ R^{-1}$ and defined by

$$\mu_A(u) = \bigwedge_u [\mu_B(v) \wedge \mu R^{-1}(v, u)]. \tag{9}$$

Note that this composition is the so-called "max-min" composition.

Definition 3. (α-composition)
The α-composition is denoted $A = B \alpha R^{-1}$ and defined as follows

$$\mu_A(u) = \bigwedge_v [\mu_B(v)\ \alpha \mu R^{-1}(v, u)] \tag{10}$$

where

$$p \alpha q = \begin{cases} q \text{ for } p > q \\ 1 \text{ for } p = q \\ \emptyset \text{ for } p < q, \end{cases} \tag{11}$$

and $\emptyset$ stands for an empty set.
Note that α-operator gives the maximum value of $x \in [0, 1]$ such that $p \wedge x = q$. (Sanchez, 1976).

Definition 4. (π-composition)
The π-composition is denoted by $A = B \pi R^{-1}$ and defined by

$$\mu_A(v) = \bigwedge_v [\mu_B(v)\ \pi\ \mu R^{-1}(v, u)] \tag{12}$$

where

$$p \pi q = \begin{cases} q & \text{for } p > q \\ [q, 1] & \text{for } p = q \\ \emptyset & \text{for } p < q. \end{cases} \tag{13}$$

Note that π-composition gives all possible $x \in [0, 1]$ such that $p \wedge x = q$. It can be interpreted as a a generalized form of α-composition (Tsukamoto, 1979).

DEFINITION 5. (ω-composition)
The ω-composition is denoted by $A=B(\omega)R^{-1}$ and defined by

$$\mu_A(u)=\bigcap_v [\mu_B(v)\,\omega\mu R^{-1}(v,u)] \tag{14}$$

where

$$p\omega q=\begin{cases}[q,1] & \text{for } p\geqslant q\\ [0,p] & \text{for } p<q.\end{cases} \tag{15}$$

The symbol $\bigcap_v$ denotes the operation of finding the common interval (or intersection) among $[q, 1]$'s and $[0, p]$'s (Togai, 1982).

4. An Illustrative Example

Assume that a process performance of a given fuzzy controlled system shown in Fig. 4 is characterized by a set of rules obtained through learning or experimental procedure such that

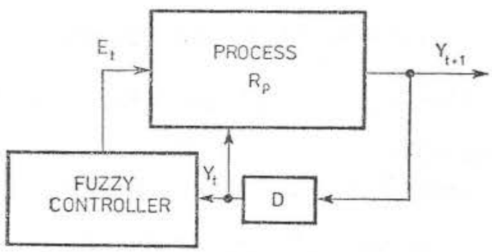

Fig. 4.

Rule 1: (Y_t is PM, E_t is PM)$\rightarrow Y_{t+1}$ is PB
Rule 2: (Y_t is PM, E_t is NM)$\rightarrow Y_{t+1}$ is PS
Rule 3: (Y_t is PS, E_t is NS)$\rightarrow Y_{t+1}$ is ZO
Rule 4: (Y_t is NS, E_t is PS)$\rightarrow Y_{t+1}$ is ZO
Rule 5: (Y_t is NM, E_t is PM)$\rightarrow Y_{t+1}$ is NS
Rule 6: (Y_t is NM, E_t is NM)$\rightarrow Y_{t+1}$ is NB

Here P and N are fuzzy variables implying "positive" and "negative" respectively, and B, M, and S are fuzzy variables implying "Big", "Medium" and "Small", respectively. Thus we have seven possible quantized levels described by linguistic labels for X's and Y's as listed below:

(1) PB =Positive Big,
(2) PM =Positive Medium,
(3) PS =Positive Small,
(4) ZO =Zero,
(5) NS =Negative Small,
(6) NM=Negative Medium,
(7) NB =Negative Big.

The grade of membership values are assigned subjectively to define the meaning of the labels of the fuzzy sets. Take "Positive Big" as an example, "Positive Big" is defined explicitly by the membership function as shown in the following (cf. Fig. 5):

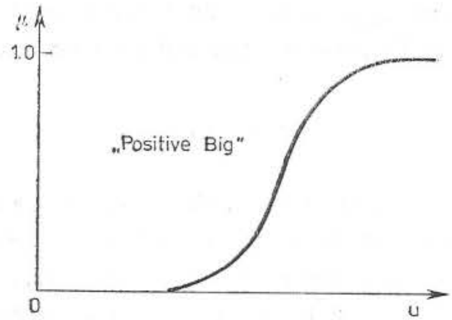

Fig. 5.

$$\text{Positive Big}=[0, 0, 0, 0, 0, 0, 0, 0.1, 0.4, 0.8, 1.0]$$

with the universe of discourse being

$$U=\{-5, -4, -3, -2, 1, 0, 1, 2, 3, 4, 5,\}.$$

All values of membership functions used for the linguistic labels are shown in Table 1

Table 1. Table for linguistic values.

	−5	−4	−3	−2	−1	0	+1	+2	+3	+4	+5
PB	0	0	0	0	0	0	0	0.1	0.4	0.8	1.0
PM	0	0	0	0	0	0.1	0.3	0.7	1.0	0.7	0.3
PS	0	0	0	0.1	0.4	0.8	1.0	0.8	0.4	0.1	0
ZO	0	0	0.1	0.3	0.7	1.0	0.7	0.3	0.1	0	0
NS	0	0.1	0.4	0.8	1.0	0.8	0.4	0.1	0	0	0
NM	0.3	0.7	1.0	0.7	0.3	0.1	0	0	0	0	0
NB	1.0	0.8	0.4	0.1	0	0	0	0	0	0	0

The ternary process relation R_p is defined by the given rules. The relational matrix R_i defined by rule i is

$$R_i=(Y_{t,i}\cap E_{t,i})\rightarrow Y_{t+1,i}$$

where → denotes implication function. There are a number of ways that the implication function can be defined (Togai, 1982). In our simulation we chose min operation i.e.,

$$R_i=(Y_{t,i}\cap E_{t\,i})\cap Y_{t+1\,i}.$$

The transition relation R_p will then be defined by using a proper choice of connective f_{ELSE} i.e.,

$$R_p = f_{ELSE}(R_i).$$

The choice of connective f_{ELSE} depends upon the type of implication function being used (Togai, 1982). In our simulation OR (or union) connective is used, i.e.

$$R_p = \bigcup_{i=1}^{6} R_i.$$

Now suppose that our objective is to design a suboptimal controller to maintain the output Y_{t+1} at "Zero" in the steady state. The problem is to find a control strategy for the error E_t from the given state Y_t and Y_{t+1}.

For example, suppose that Y_t is observed as "Positive Big" at time t, then the design problem is to find the proper input E_t capable of decreasing the output at time $t+1$ to, say, "Positive Medium". Evaluating (6) and using o-composition, for instance, a set of control strategy will be uniquely obtained as follows:

Rule 1: (Y_t is PB, (Y_{t+1} is PM))$\rightarrow E_t$ is PS
Rule 2: (Y_t is PM, (Y_{t+1} is PS))$\rightarrow E_t$ is ZO
Rule 3: (Y_t is PS, (Y_{t+1} is ZO))$\rightarrow E_t$ is NM
Rule 4: (Y_t is NS, (Y_{t+1} is ZO))$\rightarrow E_t$ is PM
Rule 5: (Y_t is NM, (Y_{t+1} is NS))$\rightarrow E_t$ is ZO
Rule 6: (Y_t is NB, (Y_{t+1} is NM))$\rightarrow E_t$ is NS

These rules define the characteristics of the fuzzy controller shown in Fig. 4. Note that in the controller rule 1, for example, either "Y_{t+1} is PM" or "Y_{t+1} is PS" gives the same result, i.e., "E_t is PS". This shows that the fuzzy process is relatively

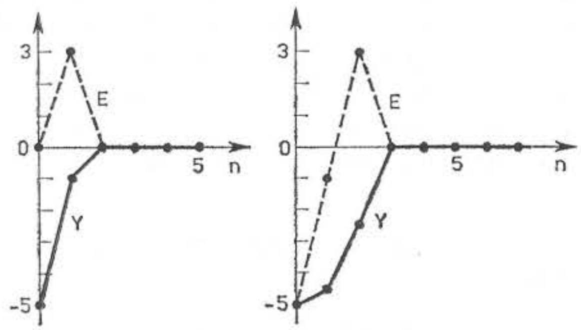

Fig. 6.

insensitive to its inputs. Let us apply this controller to the process. The simulated results for Y_t and E_t are illustrated in Fig. 6. The output response for a given fuzzy system turned to be analogous to that of a well-damped system.

5. Concluding Remarks

A mathematical description of a fuzzy dynamic system has been developed in this paper. A systematic method to derive a fuzzy control strategy from an underlying fuzzy system model has also been established. The theoretical development presented here enables the fuzzy system to be automatically controlled. Since both nonlinear and linear systems may be reasonably and accurately described by a set of fuzzy rules (or description of system's behavior), it is advantageous that the same method can be applied to solve the control problem of these systems.

The approach taken here for a fuzzy control problem is different from that of the conventional control theory. But it is not different so far as the control objectives is concerned. The difference, however, stems from the fact that conventional control theory uses differential equations to model a dynamic system, while the fuzzy control theory uses fuzzy relations to model a dynamic system. From a mathematical point of view, the fuzzy relation describes a much looser structure of the system, while the differential equation describes precise behavior of the system. This is the reason why fuzzy set theory can solve some difficult problems of a very complex system.

Thus we have obtained a novel tool to handle the automatic control problem of some dynamic systems even though their precise mathematical models are not available.

In the illustrative example we investigated the system behavior based on the fuzzy process defined by (2). A further study of the whole system response including the error function will be necessary. The performance criteria for a fuzzy dynamic system would also be a future object of study. It is especially urgent for us to establish a definition of stability for a fuzzy dynamic system.

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Analiza rozmytego układu dynamicznego i synteza jego optymalnego regulatora

Przedstawiono model matematyczny i nową metodę syntezy rozmytego regulatora dla rozmytego układu dynamicznego. Do opisu rozmytego układu dynamicznego wprowadzono macierz relacji rozmytej. W nowej metodzie regulatora rozmytego wykorzystano ideę rozmytej macierzy odwrotnej. Otrzymana metoda pozwala na realizację regulacji suboptymalnej układu rozmytego. Jako przykład ilustrujący proponowany opis i metodę rozważono syntezę regulatora rozmytego w zadaniu z jednostkowym opóźnieniem, a także przedstawiono odpowiednie wyniki symulacji.

Анализ нечеткой динамической системы и синтез ее оптимального регулятора

Представлена математическая модель и новый метод синтеза нечеткого регулятора для нечеткой динамической системы. Для описания нечеткой динамической системы вводится матрица нечетких отношений. В новом методе проектирования нечеткого регулятора используется идея нечеткой обратной матрицы. Полученный метод позволяет реализовать субоптимальное регулирование нечеткой системы. В качестве примера, иллюстрирующего предлагаемые описание и метод, рассматривается синтез нечеткого регулятора в задаче с единичным запаздыванием, а также представлены соответствующие результаты моделирования.