

# A model problem of the theory of non electrolyte transfer through deformable semipermeable membranes

by

LEV RUBINSTEIN

School of Applied Science and Technology  
The Hebrew University of Jerusalem

The theory of the convective-diffusion mass transfer through deformable semipermeable membranes not resisting deformations, as it is presented in author's paper [9], [10], needs studying its mathematical aspects. The paper deals with a model problem of this theory formulated for its one-dimensional version. Fields of concentrations, velocities of the solution motion along compartments divided by the membrane, and the shape of the latter are subjects of the solution of a boundary value problem formulated for a system of two non-linear parabolic equations of the second order, one non-linear hyperbolic equation of the first order and two ordinary differential equations of the first order with relevant boundary and initial conditions. The local existence and uniqueness theorem is proved by means of reducing the problem to a system of Volterra integral equations with the use of certain contraction mapping arguments.

## 1. Introduction

In what follows we consider a plane cell

$$D = \{x, y: 0 < x < 1; 0 < y < A\} \quad (1.1)$$

divided by the line  $y = y(x, t)$  into two regions

$$D_1 = \{x, y: 0 < x < 1; y(x, t) < y < A\}; D_2 = D \setminus \bar{D}_1 \quad (1.2)$$

where  $A = \text{const} > 0$  is assumed to be much smaller than 1.

Boundaries  $(x=0; 0 < y < A)$  and  $(x=1; 0 < y < 1)$  are called below the basal and the apical membranes; the line  $y = y(x, t)$  — the lateral one. We assume that the cell  $D$  is filled with water solution of a certain non-electrolyte which cannot penetrate through all aforementioned membranes as well as through the boundaries  $(0 < x < 1; y=0)$  and  $(0 < x < 1; y=A)$ . These boundaries, as well as parts  $(x=0; 0 < y < y(x, t))$  and  $(x=0; y(x, t) < y < A)$  of the basal and apical membranes are impermeable for water too. Hence the lateral membrane is a main passway for the water transfer, originated by the immersion of the cell  $D$  into water solution of the same impermeant, hypertonic/hypotonic on the basal and hypotonic/hypertonic

on the apical side of the cell. The deformability of the lateral membrane leads to the appearance of the convective motion of water, accompanying the diffusion mass redistribution. Author's earlier numerical computations [2], [12] were based on the use of the pure diffusion approximation of the one-dimensional formalism of the theory, consisting in neglect of the convective mass transfer compared with that by diffusion. As it is emphasized in [10] such a neglect appreciably violates the mass conservation law, although the quantitative impact of this violation on the calculated membrane's shape remains unestimated. At the same time the very formulation of the problem when taking into account the convective mass redistribution requires a careful reconsideration. As it is emphasized in [10] the system of conditions derived there appears to be overdetermined. This is a result of applying an approach, named "The membrane approximation" [9]. This approach consists in the following. From the very beginning the deformable membrane is considered as a thick shell which separates two regions, filled with the same solutions of the same components, one of which cannot penetrate through this shell. The shell itself is a solution of all components, mentioned above, including water (except for their impermeant) in the shell's constituent. Conditions of the dynamical compatibility and of a local thermodynamic equilibrium are valid on the shell boundaries. The latter ones are taken in the approximate form of the Nerst distribution theorem with a constant coefficient of distribution. Conditions of the dynamical compatibility express laws of mass and momentum conservation on surfaces of a strong discontinuity. The important peculiarity of the theory consists in the use of so called "system of the average volume velocity" [5], [10] for describing the convective-diffusion mass transfer, and of the "system of the center of mass" [5], [10] for describing the momentum conservation. It is important that in the system of average volume velocity all solutions behave as incompressible liquids.

The aforementioned "membrane approximation" of the theory consists in the contraction of both the boundaries of the shell to the two-sided surface which becomes the representative of the membrane. Consideration of the membrane as a thick shell permits, in principle, taking into account its shrinking and swelling as well as separately evolving motion of two membrane interfaces. In contrast to this the membrane approximation does not allow such a freedom. As a result the system of equations of the membrane approximation becomes overdetermined. In particular this relates to the conditions of adhesion of solutions, bathing the lateral membrane to its interfaces. One has to omit one of them, or replace both of them by their linear combination. Using unidimensional formalism of the theory\*) one may choose such a combination of adhesion conditions which gives the expression of the total volume flux (that is in the system of average volume velocity) in terms of the total diffusion flux of the impermeant. As a result one obtains a condition, replacing two adhesion ones, the use of both of which makes the problem overdetermined. In what follows we deal with namely this approach.

\*) This formalism consists in replacing concentrations and velocities of motion in apartments  $D_1$  and  $D_2$  by their average, the averaging being performed over crosssections  $x = \text{const}$ .

One has to distinguish cases of thin and of thick membranes. In the first case the membrane's thickness is much smaller than sizes of both the compartments  $D_1$  and  $D_2$  (i.e.  $\min(y(x, t), A - y(x, t)) \gg 2\Delta \forall x \in [0, 1], \forall t \geq 0$ , where  $2\Delta$  is the membrane's thickness). In the second case one of the values  $y(x, t)$  or  $A - y(x, t)$  is of order, or even smaller, than  $2\Delta$ . In such a case the problem appears to be a free boundary problem, formulated for a system of parabolic equations, including one with reversed time. Using pure diffusion approximation we deal with such a problem in another paper [11]. Here we consider the case of a thin membrane.

Omitting some inessential non-linearities\*) we may describe the problem we deal with as the following one: Find  $T > 0$ ,  $u_i(x, t)$ ,  $v_i(x, t)$ ,  $i = 1, 2$ ;  $w(x, t)$ ,  $y(x, t)$  and  $f(t)$  such that  $\forall (x, t) \in D_T$

$$D((1-y)u_{1x})_x - ((1-y)u_1v_1)_x = ((1-y)u_1)_t, \quad (1.3_1^*)$$

$$(yu_{2x})_x - (yu_2v_2)_x = (yu_2)_t, \quad (1.3_2^*)$$

$$y_t + p_y(u_1 - u_2) = w, \quad (1.3_3)$$

$$\delta^2 w y_x - \lambda((1-y)Du_{1x} + yu_{2x}) + f(t) = 0, \quad (1.3_4)$$

$$((1-y)v_1)_x - w = 0, \quad (1.3_5)$$

$$(yv_2)_x + w = 0, \quad (1.3_6)$$

$$f(t) = (1-y)v_1 + yv_2, \quad (1.3_7)$$

$$Du_{1x} = u_1v_1; v_1 = p_x(u_1 - u^1); v_2 = 0; u_{2x} = 0 \text{ at } x = 0, \quad (1.3_8)$$

$$u_{1x} = 0; v_1 = 0; u_{2x} = u_2v_2; v_2 = p_y(u^2 - u_2) \text{ at } x = 1, \quad (1.3_9)$$

$$u_i(x, 0) = u_i^0(x), i = 1, 2; y(x, 0) = y^0(x); 0 < x < 1. \quad (1.3_{10})$$

Here

$$D_T = \{x, t: 0 < x < 1; 0 < t \leq T\} \quad (1.4)$$

In what follows we take, for simplification of writing,

$$D = 1; \delta = 1 \quad (1.5)$$

Functions to be determined have the following sense:

$u_1$  and  $u_2$  are molar concentrations of the impermeant in compartments  $D_1$  and  $D_2$ .

$v_1$  and  $v_2$  are velocities of the solution motion in  $D_1$  and  $D_2$ , averaged in  $y$ -direction and taken in the system of the average volume velocity.

$w$  — is the velocity of the solution transfer from  $D_2$  into  $D_1$  in the same system.

$f(t)$  is a total flux of solutions in the cell  $D$  in the system of the average volume velocity.

$y = y(x, t)$  is the equation of the lateral membrane.

\*) For comparison of the rigorous and simplified systems of equations see [10]

Boundary conditions (1.3<sub>8</sub>) and (1.3<sub>9</sub>) express the mass conservation law for water transfer through the basal and the apical membranes, expressed in terms of impermeant concentration and its gradient as well as those of velocities  $v_1$  and  $v_2$ .  $u^1(t)$  and  $u^2(t)$  are prescribed concentrations of the impermeant in solutions bathing the basal and the apical sides of the cell.  $p_\alpha$ ,  $p_\beta$  and  $p_\gamma$  are water permeabilities of the basal, lateral and apical membranes. We assume that these permeabilities are constants.

Using (1.3<sub>5</sub>), (1.3<sub>6</sub>) and (1.3<sub>3</sub>) we may replace equations (1.3<sub>1</sub><sup>\*</sup>) and (1.3<sub>2</sub><sup>\*</sup>) by the following ones:

$$u_{1xx} + F_1(x, t | u_1, u_{1x}, u_2, v_1, y, y_x) = u_{1t} \quad (1.3_1)$$

$$u_{2xx} + F_2(x, t | u_1, u_2, u_{2x}, v_2, y, y_x) = u_{2t} \quad (1.3_2)$$

where

$$\begin{aligned} F_1 &= -((1-y)^{-1} y_x + v_1) u_{1x} + p_\beta u_1 (u_1 - u_2) (1-y)^{-1} \\ F_2 &= (y^{-1} y_x - v_2) u_{2x} + p_\beta u_2 (u_1 - u_2) y^{-1} \end{aligned} \quad (1.6)$$

Equations (1.3<sub>1</sub>) and (1.3<sub>2</sub>) will be used in the algorithm described below (see section 3) rather than equations (1.3<sub>1</sub><sup>\*</sup>) and (1.3<sub>2</sub><sup>\*</sup>). The equivalency of problems (1.3<sub>1</sub><sup>\*</sup>), (1.3<sub>2</sub><sup>\*</sup>), (1.3<sub>i</sub>),  $i=3, \dots, 10$  and (1.3<sub>1</sub>) — (1.3<sub>10</sub>) is obvious.

As it is seen from (1.3<sub>1</sub>) — (1.6)  $w(x, t)$  may be excluded from the consideration, so that (1.3<sub>4</sub>) will be replaced by the equation

$$p + H(x, t | y, q | f, u_1, u_2, u_{1x}, u_{2x}) = 0 \quad (1.7)$$

where

$$\begin{aligned} H &= -\varphi | q + p_\beta (u_1 - u_2); \quad \varphi = -f(t) + \lambda ((1-y) u_{1x} + y u_{2x}) \\ p &= y_t; \quad q = y_x \end{aligned} \quad (1.8)$$

The method of the solution of the problem is as follows. Concentrations  $u_1$  and  $u_2$  are determined by their integral representations obtained with the use of the relevant Green functions. Further, considering  $H$  as a known function of  $t$ ,  $x$ ,  $y$  and  $q$ , we associate with (1.7) the system of characteristic equations. Let

$$L_z = \{x, t: x = X(t, z); 0 < t \leq T; 0 \leq z \leq 1\} \quad (1.9)$$

be characteristic lines outgoing from points  $x=z$  of the  $x$ -axis. We assume that

$$X_t(0, z) > 0 \quad \forall z \in [0, 1] \quad (1.10)$$

Hence we may assume  $T > 0$  to be so small that  $L_0$  divides the strip  $D_T$  into two subregions

$$D_{OXT} = \{x, t: 0 < x < X(t, 0); 0 < t \leq T\}; \quad D_{X1T} = D_T \setminus \bar{D}_{OXT} \quad (1.11)$$

whereas  $L_1$  does not belong to  $D_T$ . Correspondingly we consider for the aforementioned system of characteristic equations two Cauchy problems. The first one — in

$D_{x1T}$  with Cauchy data prescribed on the segment  $0 \leq x \leq 1$  of the  $x$ -axis (these data are determined by the initial conditions of the problem), and the second one in  $D_{oxt}$  with Cauchy data

$$t=\tau; y=y_0(\tau); q=q_0(\tau); p=p_0(\tau) \text{ at } x=0; 0 < t \leq T \quad (1.12)$$

Here  $y_0$ ,  $q_0$  and  $p_0$  are unknown functions which have to be determined in the course of the problem solution. Values of  $x$ ,  $y$ ,  $q$  and  $p$  are determined in  $D_{x1T}$  and  $D_{oxt}$  along characteristic lines from integral equations, equivalent to the respective characteristic differential equations. These characteristics depend on two parameters  $z$  and, respectively  $\tau$ -points of the carriers of Cauchy data. Being the implicit functions of  $x$  and  $t$  these parameters may be determined in functions of  $x$  and  $t$  as solutions of respective integral equations.

Thus we deal with a system of non-linear Volterra integral equations, the local solution of which is constructed by the use of the suitable contraction mapping arguments. The respective operator is constructed so that its action is an equivalent to a "diagonal" iterative procedure. The use of such a procedure allows to avoid any references to results of a general modern theory of parabolic and hyperbolic non-linear equations, but to be restricted with the use of absolutely elementary estimates. This, in its turn, permits us to restrict ourselves with no more than formulation of the main theorem and with a very conspicious sketch of its proof.

It is suitable to discuss here the following question. The algorithm of the problem solution includes determining velocities  $v_1$  and  $v_2$  as functions, each of them satisfies the ordinary differential equation (1.3<sub>5</sub>) or (1.3<sub>6</sub>) of the first order and two boundary conditions (1.3<sub>8</sub>) and (1.3<sub>9</sub>). Integration of these equations in  $D_{oxt}$  and in  $D_{x1T}$ , taking these conditions into account, gives

$$\begin{aligned} v_1(x, t) &= (1 - y(x, t))^{-1} \left\{ f(t) + \int_0^x w(s, t) ds \right\} \\ &\text{in } D_{oxt} \quad (1.13) \\ v_2(x, t) &= -y(x, t)^{-1} \int_0^x w(s, t) ds \end{aligned}$$

and

$$\begin{aligned} v_1(x, t) &= -(1 - y(x, t))^{-1} \int_x^1 w(s, t) ds \\ &\text{in } D_{x1T} \quad (1.14) \\ v_2(x, t) &= y(x, t)^{-1} \left\{ f(t) + \int_x^1 w(s, t) ds \right\} \end{aligned}$$

Assuming that

$$0 < y(x, t) < 1 \quad \forall x \in [0, 1], \quad \forall t \in [0, T] \quad (1.15)$$

and that  $y$ ,  $f$  and  $w$  are continuous functions everywhere in regions of their definition, we see that  $v_1$  and  $v_2$  are continuous if and only if

$$f(t) + \int_0^1 w(s, t) ds = 0 \quad \forall t \in [0, T] \quad (1.16)$$

Naturally the question arises whether one has to consider this equality as an independent, additional condition or as a corollary of all other ones. It is easy to see that the latter option is the case. Indeed, assume that there exists the solution to our problem such that  $u_i, u_{ix}, v_i, w, y$  and  $y_x$  are bounded and satisfy (1.15). Then  $F_1$  and  $F_2$  are also bounded. But this implies the continuity of  $u_i$  and  $u_{ix}$ ,  $i=1, 2$  [4]. On the other hand equations (1.3<sub>1</sub><sup>\*</sup>) and (1.3<sub>2</sub><sup>\*</sup>) may be rewritten in the form

$$U_{ixx} + F_i^* = U_{it}, \quad i=1,2 \quad (1.17)$$

where

$$U_1 = (1-y)u_1; \quad U_2 = yu_2 \quad (1.18)$$

and

$$F_1^* = u_{1x}y_x - ((1-y)u_1v_1)_x - u_1(w - p_\beta(u_1 - u_2)) \quad (1.19)$$

$$F_2^* = -u_{2x}y_x - (yu_2v_2)_x + u_2(w - p_\beta(u_1 - u_2))$$

so that  $F_1^*$  and  $F_2^*$  are also bounded. This means, in turn, that  $U_1, U_2, U_{1x}$  and  $U_{2x}$  are continuous too. Since  $u_i$  and  $u_{ix}$  are continuous the continuity of  $U_i$  and  $U_{ix}$  imply the continuity of  $y$  and  $y_x$ .

Take now an arbitrary small  $\varepsilon > 0$ ,  $t_1$  and  $t_2$  and denote

$$D_\varepsilon = \{x, t: X(t, 0) - \varepsilon < x < X(t, 0) + \varepsilon; 0 < t \leq T\}; \quad L_\varepsilon = \partial D_\varepsilon. \quad (1.20)$$

Equalities (1.3<sub>5</sub>), (1.3<sub>6</sub>) and (1.3<sub>3</sub>) imply

$$\begin{aligned} \int_{D_\varepsilon} [((1-y)v_1)_x - y_t] dx dt &= \int_{D_\varepsilon} p_\beta(u_1 - u_2) dx dt \\ \int_{D_\varepsilon} [(yv_2)_x + y_t] dx dt &= - \int_{D_\varepsilon} p_\beta(u_1 - u_2) dx dt \end{aligned} \quad (1.21)$$

so that

$$\begin{aligned} \int_{L_\varepsilon} (1-y)v_1 dt + y dx &= \int_{D_\varepsilon} p_\beta(u_1 - u_2) dx dt \\ \int_{L_\varepsilon} yv_2 dt - y dx &= \int_{D_\varepsilon} p_\beta(u_1 - u_2) dx dt \end{aligned} \quad (1.22)$$

Denoting  $[h]$  the jump of any function  $h$  on the line of discontinuity and using the continuity of  $y$ , boundedness of  $u_1$  and  $u_2$ , and arbitrariness of  $\varepsilon$ ,  $t_1 > 0$  and  $t_2 > 0$  we conclude that

$$[v_i]_{x=X(t,0)} = 0 \quad \forall t \in [0, T]; \quad i=1,2. \quad (1.23)$$

so that the continuity of  $y$  implies that of  $v_1$  and  $v_2$ . Hence the continuity of  $y$  implies the validity of the condition (1.16) Q.E.D.\*

\* Note that the continuity of fluxes  $J_1 = (1-y)(u_1v_1 - u_{1x})$  and  $J_2 = y(u_2v_2 - u_{2x})$  in compartments  $D_1$  and  $D_2$  is the necessary and sufficient condition for conservation of masses  $M_1$  and  $M_2$  of the

The problem (1.3<sub>1</sub>) — (1.3<sub>10</sub>) is formulated as a model one, not containing non-essential complications. However, from the point of view of biological applications, and first of all modelling the process of swelling and shrinking of fast muscle fibres immersed into hypotonic or hypertonic solutions [6], [2], [9] or modelling the process of water transfer through plane epithelial tissues [6], [2], this model is far from being an adequate one. Much more realistic problem must be formulated in the region

$$D_s = \{x, t: 0 < x < s(t); 0 < y < A\} \quad (1.24)$$

Besides, the basal and apical membranes bounding compartments  $D_2$  and respectively  $D_1$  have to be considered as permeable for water, but having water permeabilities different from those of the basal and apical membranes, bounding  $D_1$  and respectively  $D_2$ . Moreover, boundaries  $x=0$  and  $x=s(t)$  may be, in one of regions  $D_1$  or  $D_2$ , say in  $D_2$ , permeable not only for water but for that component of solutions which cannot penetrate through other membranes. All these alterations are essential for mathematical modelling of the aforementioned biological processes, i.e. for that purpose which has motivated the very appearance of papers [9], [10]. However these alterations do not affect the principal feature of the problem under consideration, consisting in the inseparable interaction of parabolic and hyperbolic aspects of the problem. One only have to refer to estimates obtained in [8] and to apply the method of the simultaneous use of two coordinate systems: the laboratory one and the second-linked to the free boundary  $x=s(t)^*$ . Therefore we leave such a generalized problem without any further consideration, but restrict ourselves with this hint.

The following notations are used below. Given functions  $f(x, t, \xi, \tau)$ ,  $R(x, \xi, t - \tau)$ ,  $X(t, z)$ ,  $T^0(t, \tau)$  and the number  $T > 0$  we denote  $D_{OXT}$  and  $D_{X1T}$  regions defined by (1.11) and define

$$\begin{aligned} D_{XT} &= D_{OXT} \vee D_{X1T} \\ D_{tz} &= \{x, t: 0 < t < T_1(z); 0 \leq z \leq 1\}; \\ T_1(z) &= T \text{ if } X(t, z) < 1 \quad \forall t \leq T; \\ T_1(z) &= t_1(z) \text{ if } X(t, z) < 1 \quad \forall t < t_1(z); X(t_1(z), z) = 1 \\ D_{t\tau} &= \{t, \tau: 0 < \tau \leq t \leq T\} \end{aligned} \quad (1.25)$$

impermeant in these compartments. Indeed, assume that  $J_i$  have jumps along a line  $L$ . Then equations (1.3<sub>1</sub><sup>\*</sup>) and (1.3<sub>2</sub><sup>\*</sup>), and boundary conditions (1.3<sub>8</sub>) and (1.3<sub>9</sub>) imply

$$M_{1t} = (\partial/\partial t) \int_0^1 (1-y) u_1 dx = [J_1]$$

and analogously for  $M_2$ .

\*) The use of this method allows to avoid all difficulties connected with differentiating the thermal potential of the double layer distributed along the free boundary and having unknown density [8].



$$I^0(x, t|f|R|a, b) = \int_a^b f(x, t, \xi, 0) R(x, \xi, t) d\xi \quad (1.26_1)$$

$$I_{km}^1(x, t|f|a|R) = \int_0^t f(x, t, a, \tau) (\delta^{k+m}|\delta\xi^k \delta\tau^m) R(x, a, t-\tau) d\tau \quad (1.26_2)$$

$$I_{km}^2(x, t|f|R|a, b) = \int_0^t d\tau \int_a^b f(x, t, \xi, \tau) (\delta^{k+m}|\delta\xi^k \delta\tau^m) R(x, \xi, t-\tau) d\xi \quad (1.26_3)$$

Further  $\forall f(x, t)$

$$f_{km}(x, t) = \delta^{k+m} f|\delta x^k \delta^m \quad \forall k, m \geq 0 \quad (1.27)$$

and  $\forall N = \text{const} > 0, \forall i, j, \dots, n$

$$C_{i, j, \dots, n}(N) \text{ and } C_{i, j, \dots, n}(N|t) \quad (1.28)$$

denote non-negative continuous increasing functions of their arguments such that

$$C(N|0) = 0 \quad (1.31)$$

Subscripts  $i, j, \dots, n$  may be omitted.

The following spaces are involved:

$$f(x, t) \in C^{k, m}(D_T); g(x, t) \in C_{k, m}^{1, 0}(D_T); h(x, t) \in C_{k, m}^{0, 1}(D_T) \quad (1.32)$$

if there exist and are continuous in  $D_T f_{ij}(x, t), g_{ij}(x, t)$  and  $h_{ij}(x, t) \forall i \leq k, \forall j \leq m$  and if there exist and are continuous in  $D_T t^{\frac{1}{2}} g_{k+1, m}$  and  $t^{\frac{1}{2}} h_{k, m+1}$ . Norms in  $C^{k, m}(D_T), C_{k, m}^{1, 0}(D_T)$  and in  $C_{k, m}^{0, 1}(D_T)$  are defined as

$$\begin{aligned} \|f\| &= \max_{i, j} \max_{(x, t) \in D_T} \{|f_{ij}(x, t)|\}; \quad i \leq k; j \leq m \\ \|g\| &= \max_{i, j} \max_{(x, t) \in D_T} \{|g_{ij}(x, t)|; t^{\frac{1}{2}} |g_{k+1, m}(x, t)|\} \\ \|h\| &= \max_{i, j} \max_{(x, t) \in D_T} \{|h_{ij}(x, t)|; t^{\frac{1}{2}} |h_{k, m+1}(x, t)|\} \end{aligned} \quad (1.33)$$

The space  $C_k^1([a, b])$  of functions  $f(x)$  defined in  $a \leq x \leq b$  is analogous to  $C_{k, m}^{1, 0}(D)$ .  $(m, n)$  means referring to the  $n$ -th formula of the  $m$ -th section.

Section 2 is devoted to the formulation of integral equations, serving for determining  $y$  and its derivatives,  $v_i, u_i, i=1, 2$ , and their derivatives. Section 3 contains definition of the operator, whose fix point is a solution of the problem (1.3<sub>1</sub>) — (1.3<sub>10</sub>), formulation of the main existence theorem and a sketch of its proof. Section 4 contains some conclusive remarks.

## 2. Integral equations determining $y, v_i, u_i$ and their derivatives.

Below we write

$$\varphi \equiv \varphi(x, t|y|f, u_{ik0}); \quad i=1, 2; k=0. \quad (2.1)$$

$$H \equiv H(x, t|y, q|f, u_{ik0}); \quad i=1, 2; k=0, 1.$$



Let us point out that (2.1) imply

$$\begin{aligned} H_q &= H_q(x, t|y, q|f, u_{ik0}); \quad H_y = H_y(x, t|q|f, u_{ik0}); \quad i=1, 2; \quad k=0, 1 \\ H_x &= H_x(x, t|y, q|f, u_{ik0}); \quad i=1, 2; \quad k=0, 1, 2 \\ H_t &= H_t(x, t|y, q|f, \dot{f}, u_{ikm}); \quad i=1, 2; \quad k=0, 1; \quad m=1 \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} dH/dx &= H_x(x, t|) + H_q(x, t|) q_{10} + H_y(x, t|) y_{10} = \\ &= H^{10}(x, t|y_{k0}, q_{k0}|f, u_{im0}); \quad i=1, 2; \quad k=0, 1; \quad m=0, 1, 2 \\ dH/dt &= H^{01}(x, t|y_{0k}, q_{0k}|f, \dot{f}, u_{im1}); \quad i=1, 2; \quad k=0, 1; \quad m=0, 1 \\ dH_q/dx &= H_q^{10}(x, t|y_{k0}, q_{k0}|f, u_{im0}); \quad i=1, 2; \quad k=0; \quad m=0, 1, 2 \\ dH_q/dt &= H_q^{01}(x, t|y_{0k}, q_{0k}|f, \dot{f}, u_{im1}); \quad i=1, 2; \quad k=0, 1; \quad m=0, 1 \\ dH_y/dx &= H_y^{10}(x, t|q_{k0}|f, u_{im0}); \quad i=1, 2; \quad k=0, 1; \quad m=0, 1, 2 \\ dH_y/dt &= H_y^{01}(x, t|q_{0k}|f, \dot{f}, u_{imn}); \quad i=1, 2; \quad k=0, 1; \quad m=0, 1; \quad n=0, 1 \\ dH_x/dx &= H_x^{10}(x, t|y_{k0}, q_{k0}|f, u_{im0}); \quad i=1, 2; \quad k=0; \quad m=0, 1, 2, 3 \\ dH_t/dt &= H_t^{01}(x, t|y_{0k}, q_{0k}|f^{(m)}, u_{iltm}); \quad i=1, 2; \quad k=0, 1, 2; \quad l=0, 1; \\ &\quad m=0, 1, 2; \quad n=0, 1, 2 \\ dH_x/dt &= dH_t/dx = H_x^{10}(x, t|y_{km}, q_{km}|f, \dot{f}, u_{iltm}); \quad i=1, 2; \quad k=0, 1; \\ &\quad m=0, 1; \quad l=0, 1, 2; \quad n=0, 1 \end{aligned} \quad (2.3)$$

Assume that the input data of the problem satisfy the following conditions:

$$u^i(t) \in C^2([0, t]); \quad u_i^0(x) \in C^4([0, 1]); \quad y^0(x) \in C^3([0, 1]); \quad i=1, 2 \quad (2.4)$$

$$f(0) = p_\alpha (1 - y^0(0)) (u_1^0(0) - u^1(0)) = p_\gamma y^0(1) (u^2(0) - u_2^0(1)) \quad (2.5)$$

$$|q^0(x)| = |\dot{y}^0(x)| \geq \eta > 0; \quad 0 < \rho \leq y^0(x) \leq 1 - \rho; \quad 0 \leq x \leq 1; \quad (2.6)$$

$$H_q(x, 0/y^0, q^0/f(0), u_i^0, \dot{u}_i^0) = \varphi(x, 0/)/q^{02}(x) \geq \eta; \quad 0 \leq x \leq 1 \quad (2.7)$$

$$\dot{u}_1^0(0) = p_\alpha u_1^0(0) (u_1^0(0) - u^1(0)); \quad \dot{u}_2(1) = p_\gamma u_2^0(1) (u^2(0) - u_2^0(1)) \quad (2.8)$$

In what follows we define

$$p^0(x) = -H(x, 0|); \quad w^0(x) = p^0(x) + p_\beta (u_1^0(x) - u_2^0(x)) \quad (2.9)$$

and assume that

$$f(0) + \int_0^1 w^0(x) dx = 0 \quad (2.10)$$

Finally, assuming that

$$y_0(t) = y(0, t); \quad \varphi_0(t) = \varphi(0, t|); \quad u_i(0, t); \quad i=1, 2 \quad (2.11)$$

are known, define

$$p_0(t) = \dot{y}_0(t); \quad q_0(t) = \varphi_0(t)/\{p_0(t) + p_\beta [u_1(0, t) - u_2(0, t)]\} \quad (2.12)$$

Let us now formulate all integral equations of the interest. Assume that  $u_i(x, t)$  are known. Then the system of characteristic equations, corresponding to the equation (1.7) is

$$\begin{aligned} dx/dt &= H_q(x, t | |); \quad dy/dt = p + qH_y(x, t | |); \\ dq/dt &= -(H_x(x, t | |) + qH_y(x, t | |)) \end{aligned} \quad (2.13)$$

The last equation  $dp/dt = -(H_t + pH_y)$  may be replaced by the equation

$$p = -H(x, t | |) \quad (2.14)$$

since  $p + H = \text{const}$  is the integral of the system of characteristic equations.

Let us associate with (2.13), (2.14) the Cauchy data

$$x = z; \quad y = y^0(z); \quad q = q^0(z); \quad p = p^0(z) \quad (2.15)$$

Denote

$$x = X(t, z); \quad y = Y(t, z); \quad q = Q(t, z); \quad p = P(t, z) \quad (2.16)$$

the solution of the Cauchy problem (2.13) — (2.15). By virtue of assumptions (2.5) — (2.8) the characteristic lines  $x = X(t, z)$  are directed inside the region  $D_T$ . Let

$$D_{x1T} = \{x, t: X(t, 0) < x < 1; 0 < t \leq T\} \quad (2.17)$$

For  $T > 0$  small enough this region is covered by those characteristic lines, whereas the region  $D_{OXT} = D_T \setminus \bar{D}_{x1T}$  remains not covered by them.

Consider now the Cauchy problem for the system (2.12) — (2.14) with Cauchy data

$$x = 0; \quad t = \tau; \quad y = y_0(\tau); \quad q = q_0(\tau); \quad p = p_0(\tau) = \dot{y}_0(\tau) \quad (2.18)$$

where  $q_0(\tau)$  is defined by (2.12). Here  $y_0(\tau)$  is not prescribed and has to be found in the course of the problem solution.

Denote

$$x = X_0(t, \tau); \quad y = Y_0(t, \tau); \quad q = Q_0(t, \tau); \quad p = P_0(t, \tau) \quad (2.19)$$

the solution of this Cauchy problem. Assume that

$$y_0(0) = y^0(0); \quad q_0(0) = q^0(0); \quad p_0(0) = p^0(0) \quad (2.20)$$

and that

$$H_q(0, \tau | |) > 0; \quad \tau \leq t \leq T; \quad p_0(0) + p_\beta(u_1^0(0) - u_2^0(0)) \neq 0 \quad (2.21)$$

Then the characteristic lines  $x = X_0(t, \tau)$  are directed inside  $D_{OXT}$  and cover it. Moreover, (2.20) imply

$$Y_0(t, 0) = Y(t, 0); \quad Q_0(t, 0) = Q(t, 0); \quad P_0(t, 0) = P(t, 0) \quad (2.22)$$

if  $f(t)$ ,  $u_i(x, t)$  and their derivatives, entering  $H$  and its relevant derivatives are continuous in  $D_T$ .

$X, Y, Q, P$  and  $X_0, Y_0, Q_0, P_0$  are solutions of the system of integral equations equivalent to the Cauchy problem (2.13) — (2.15) and respectively (2.13), (2.14), (2.19), i.e.

$$\begin{aligned} X(t, z) &= z + \int_0^t H_q(X(s, z), s | |) ds = X^*(t, z |), \\ Y(t, z) &= y^0(z) + \int_0^t [p(X(s, z), s) + q(X(s, z), s) H_y(X(s, z), s | |)] ds = \\ &= Y^*(t, z |) \quad (2.23) \\ Q(t, z) &= q^0(z) - \int_0^t [H_x(X(s, z), s | |) + \\ &\quad + q(X(s, z), s) H_y(X(s, z), s | |)] ds = Q^*(t, z |) \\ P(t, z) &= -H(X(t, z), t | |) = P^*(t, z |) \end{aligned}$$

and respectively

$$\begin{aligned} X_0(t, \tau) &= \int_{\tau}^t H_q(X_0(s, \tau), s | |) ds = X_0^*(t, \tau |) \\ Y_0(t, \tau) &= y_0(\tau) + \int_{\tau}^t [p(X_0(s, \tau), s) + q(X_0(s, \tau), s) H_q(X_0(s, \tau), s | |)] ds = \\ &= Y_0^*(t, \tau |) \quad (2.24) \\ Q_0(t, \tau) &= q_0(\tau) - \int_{\tau}^t [H_x(X_0(s, \tau), s | |) + \\ &\quad + q(X_0(s, \tau), s) H_y(X_0(s, \tau), s | |)] ds = Q_0^*(t, \tau |) \\ P_0(t, \tau) &= -H(X_0(t, \tau), t | |) = P_0^*(t, \tau |) \end{aligned}$$

Assertion (2.22) may be replaced by a stronger one. Assume that  $u_i(x, t)$  and their derivatives, entering the right hand sides of (2.24), as well as  $y(x, t)$ ,  $q(x, t)$ , entering there, are continuous in  $\bar{D}_T$ . Then

$$\begin{aligned} X_0^*(t, 0 |) &= X^*(t, 0 |); \quad Y_0^*(t, 0 |) = Y^*(t, 0 |); \\ Q_0^*(t, 0 |) &= Q^*(t, 0 |); \quad P_0^*(t, 0 |) = P^*(t, 0 |). \end{aligned} \quad (2.25)$$

even when  $(X^*, Y^*, \dots, P^*)$  and  $(X_0^*, Y_0^*, \dots, P_0^*)$  are not solutions of the Cauchy problems under consideration. This remark will be essentially used in the proof of the existence theorem.

Dealing with  $v_i(x, t)$  and  $u_i(x, t)$ ,  $i=1, 2$ , we need to take  $(x, t)$  as independent variables. Let

$$z = Z(x, t); \quad \tau = T^0(x, t) \quad (2.26)$$

Then  $(Z, T^0, y, q, p)$  may be considered as a solution of the system of integral equations

$$\begin{aligned} Z(x, t) &= x - \int_0^t H_q(X(s, Z(x, t), s) | |) ds = Z^*(x, t |) \\ y(x, t) &= Y^*(t, Z(x, t) |); \quad q(x, t) = Q^*(t, Z(x, t) |) \\ p(x, t) &= P^*(t, Z(x, t) |); \quad (x, t) \in D_{x1T} \end{aligned} \quad (2.27)$$

and respectively

$$\begin{aligned} x &= \int_{T^0(x, t)}^t H_q(X_0(s, T^0(x, t)), s | |) ds; \\ y(x, t) &= Y_0^*(t, T^0(s, t) |); \quad q(x, t) = Q_0^*(t, T^0(s, t) |); \\ p(x, t) &= P_0^*(t, T^0(s, t) |); \quad (x, t) \in D_{0XT} \end{aligned} \quad (2.28)$$

It is desirable to deal with an integral representation of  $T^0(x, t)$  having the form solved with respect to it. Differentiation of (2.28) gives

$$\begin{aligned} T_{10}^0(x, t) &= B^{-1}(T^0(x, t), t |); \quad T_{01}^0(x, t) = H_q(x, t | |) B^{-1}(T^0(x, t), t |) \\ B(T^0(x, t), t) &= -H_q(0, T^0(x, t) | |) + \\ &+ \int_{T^0(x, t)}^t (d/dX_0) H_q(X_0(s, T^0(x, t), s) | |) X_{0z}(s, T_0(x, t)) ds \end{aligned} \quad (2.29)$$

from where it follows that

$$T^0(x, t) = \int_{X(t, 0)}^x B^{-1}(T^0(s, t), t |) ds; \quad 0 < x < X(t, 0). \quad (2.30)$$

Simultaneously with (2.29) we have

$$\begin{aligned} Z_{10}(x, t) &= 1 - Z_{10}(x, t) \int_0^t (d/dx) H_q(X(s, Z(x, t), s) | |) X_z(s, Z(x, t)) ds; \\ Z_{01}(x, t) &= -H_q(x, t | |) - \\ &- Z_{01}(x, t) \int_0^t (d/dx) H_q(X(s, Z(x, t), s) | |) X_z(s, Z(x, t)) ds \end{aligned} \quad (2.31)$$

We also need integral representations of derivatives  $y_{km}, q_{km}, p_{km}$  for  $k+m \leq 2$ . Equalities (2.27) and (2.31) imply

$$\begin{aligned} y_{10}(x, t) &= Z_{10}(x, t) \{ \dot{y}^0(Z(x, t)) + \\ &+ \int_0^t X_{01}(s, Z(x, t)) (d/dx) [p(X(s, Z(x, t), s) + \\ &+ q(X(s, Z(x, t), s) H_y(X(s, Z(x, t), s) | |) ds] = Z_{10}(x, t) Y_{01}(t, Z(x, t) |); \\ y_{01}(x, t) &= p(x, t) + q(x, t) H_y(x, t | |) + Z_{01}(x, t) Y_{01}(t, Z(x, t)); \end{aligned}$$

$$\begin{aligned}
q_{10}(x, t) &= Z_{10}(x, t) \{ \dot{q}^0(Z(x, t)) - \\
&\quad - \int_0^t X_{01}(s, Z(x, t)) (d/dx) [H_x(X(s, Z(x, t), s | |)) + \\
&\quad + q(X(s, Z(x, t), s) H_y(X(s, Z(x, t), s | |))] ds \} = \\
&= Z_{10}(x, t) Q_{01}(t, Z(x, t)) \\
q_{01}(x, t) &= Z_{01}(x, t) Q_{01}(t, Z(x, t))
\end{aligned} \tag{2.32}$$

and

$$p_{01}(x, t) = -(d/dt) H(x, t | |); \quad p_{10} = -(d/dx) H(x, t | |) \tag{2.33}$$

Equalities (2.32) and (2.33) are valid in  $D_{X1T}$ . At the same time equalities (2.24) imply quite analogous relations in  $D_{OXT}$ .

Comparing (2.31) — (2.33) with (2.3) we see that

$$\begin{aligned}
&y_{10}, y_{01}, p, q || f, u_{ik0}; \quad k=0, 1, 2 \\
&q_{01}, q_{10}, p_{10} || f, u_{ik0}; \quad k=0, 1, 2, 3 \\
&p_{01} || f, \dot{f}, u_{ikm}; \quad k, m=0, 1.
\end{aligned} \tag{2.34}$$

Analogous calculations show that

$$\begin{aligned}
&y_{02}, y_{20}, y_{11} || f, \dot{f}, u_{ikm}; \quad k=0, 1, 2, 3 \text{ if } m=0; \quad k=0, 1 \text{ if } m=1 \\
&q_{11}, q_{02}, q_{20} || f, f, u_{ikm}; \quad k=0, 1, 2, 3, 4 \text{ if } m=0; \quad k=0, 1, 2 \text{ if } m=1; \\
&\quad k=0 \text{ if } m=2
\end{aligned} \tag{2.35}$$

Here  $f || a_1, \dots, a_n$  means that  $f$  depends on  $a_1, \dots, a_n$ .

Let us point out that in contrast to the continuity of  $Y^*, Q^*, P^*$  in  $D_T$  (so that  $Y_0^*, Q_0^*$  and  $P_0^*$  are the continuous prolongations of  $Y^*, Q^*$  and  $P^*$  from  $D_{X1T}$  into  $D_{OXT}$ )  $Y_{km}^*, Q_{km}^*$  and  $P_{km}^*$  have, for  $k+m \geq 1$ , jumps on the characteristic line  $x=X(t, 0)$ . This is a consequence of the existence of the angle point on the line

$$L = \{x=0 \text{ for } t \geq 0; \quad t=0 \text{ for } x > 0\} \tag{2.36}$$

Integral representations of  $v_i(x, t)$ ,  $i=1, 2$ , in  $D_{OXT}$  and  $D_{X1T}$  are given by (1.13) and (1.14). Let us recall that the continuity of  $y$  implies the continuity of  $v_i$  in  $D_T$  (see introduction, page 10).

The straightforward computations, taking into account (2.34) and (2.35) show that

$$v_{ikm}(x, t) || f_b, y_{cb}, u_{iab}, p_{ab} \tag{2.37}$$

where

$$0 \leq b \leq m; \quad 0 \leq a \leq k-1; \quad 0 \leq c \leq k; \quad i=1, 2 \tag{2.38}$$

Finally let us present integral representations of  $u_{ikm}$ . Denote by  $g(x, \xi, t-\tau)$  and  $G(x, \xi, t-\tau)$  Green functions of the first and of the second boundary value problems of heat conduction in  $D_T$ . Assume that  $u_1(x, t)$  and  $u_2(x, t)$  are solu-

tions of equations (1.3<sub>1</sub>) and (1.3<sub>2</sub>), satisfying boundary and initial conditions (1.3<sub>8</sub>) — (1.3<sub>10</sub>). One may exclude  $v_1$  and  $v_2$  from these conditions. Thus, conditions of regularity (2.5) and identities

$$g_x = -G_\tau; \quad g_\tau = -G_x; \quad g_{x\tau} = G_\tau; \quad g_\tau = G_{x\tau} \quad (2.39)$$

imply

$$u_{ikm}(x, t) = U_{ikm}(x, t); \quad k=0, 1, 2, 3, 4 \text{ if } m=0; \quad k=0, 1, 2 \text{ if } m=1; \\ k=0 \text{ if } m=2 \quad (2.40)$$

where

$$U_{100} = I_{00}^1(x, t | -p_\alpha u_{10}(\tau) (u_{10}(\tau) - u^1(\tau)) | 0 | G) + I^0(x, t | u_1^0(\xi) | G | 0, 1) + \\ + I_{00}^2(x, t | F_1(\xi, \tau) | G | 0, 1), \quad (2.41_1)$$

$$U_{110} = I_{10}^1(x, t | p_\alpha u_{10}(\tau) (u_{10}(\tau) - u^1(\tau)) | 0 | g) + I^0(x, t | \dot{u}_1^0(\xi) | g | 0, 1) + \\ + I_{10}^2(x, t | -F_{100}(\xi, \tau) | g | 0, 1) = I_{10}^1(x, t | p_\alpha u_{10}(\tau) (u_{10}(\tau) - \\ - u^1(\tau)) | 0 | g) + I_{00}^2(x, t | F_{110}(\xi, \tau) | g | 0, 1). \quad (2.41_2)$$

$$U_{101} = I_{00}^1(x, t | -p_\alpha (d/d\tau) (u_{10}(\tau) (u_{10}(\tau) - u^1(\tau))) | 0 | G) + \\ + I^0(x, t | \ddot{u}_1^0(\xi) + F_{100}(\xi, 0) | G | 0, 1) + I_{00}^2(x, t | F_{101}(\xi, \tau) | G | 0, 1) \quad (2.41_3)$$

$$U_{120} = U_{101} - F_{100}(x, t) \quad (2.41_4)$$

$$U_{111} = I_{10}^1(x, t | p_\alpha (d/d\tau) (u_{10}(\tau) (u_{10}(\tau) - u^1(\tau))) | 0 | g) + \\ + I^0(x, t | u_1^{0(3)}(\xi) + F_{110}(\xi, 0) | g | 0, 1) + I_{10}^2(x, t | F_{101}(\xi, \tau) | g | 0, 1) \quad (2.41_5)$$

$$U_{130} = U_{111} - F_{110}(x, t) \quad (2.41_6)$$

$$U_{102} = [u_1^{0(3)}(0) + F_{110}(0, 0) - p(d/d\tau) (u_{10}(0) (u_{10}(0) - u^1(0)))] G(x, 0, t) - \\ - [u_1^{0(3)}(1) + F_{110}(1, 0)] G(x, 1, t) + F_{101}(x, t) + \\ + I_{00}^1(x, t | -p(d^2/dt^2) (u_{10}(\tau) (u_{10}(\tau) - u^1(\tau))) | 0 | G) + \\ + I^0(x, t | u_1^{0(4)}(\xi) + F_{120}(\xi, 0) | G | 0, 1) + I_{10}^1(x, t | -F_{101}(0, \tau) | 0 | G) + \\ + I_{10}^1(x, t | F_{101}(1, \tau) | 1 | G) + I_{10}^2(x, t | F_{111}(\xi, \tau) | G | 0, 1) \quad (2.41_7)$$

$$U_{121} = U_{102} - F_{101}(x, t) \quad (2.41_8)$$

$$U_{140} = U_{102} - F_{120}(x, t) - F_{101}(x, t) \quad (2.41_9)$$

### 3. Algorithm of the problem solution. Theorem of existence and uniqueness.

Consider the space  $\mathfrak{M}_T$  of vector functions

$$r = \{u_1(x, t), u_2(x, t), y(x, t), q(x, t), p(x, t), X(t, z), X_0(t, \tau), \\ Z(x, t), T^0(x, t)\} \quad (3.1)$$

such that

$$\begin{aligned} u_i(x, t) \in C^{1,0}(\bar{D}_T); \quad y, q, p \in C(\bar{D}_T), \quad X(t, z) \in C(\bar{D}_{tz}), \quad X_0(t, \tau) \in C(\bar{D}_{t\tau}) \\ Z(x, t) \in C(\bar{D}_{x1T}); \quad T^0(x, t) \in C(\bar{D}_{oXT}) \end{aligned} \quad (3.2)$$

and also

$$\begin{aligned} Z(x, 0) = x, \quad T^0(t, 0) = t; \quad X(0, z) = z; \quad X_0(t, t) = 0; \\ 0 \leq X(t, z) \leq 1; \quad 0 \leq X_0(t, \tau) \leq X(t, 0); \quad X(t, 0) = X_0(t, 0). \end{aligned} \quad (3.3)$$

Let the norm  $\|r\|_{\mathfrak{M}}$  be defined as the maximum of the respective norms of all components of  $r$ , as they are defined in introduction.

Let now  $\mathfrak{M}_T(N)$  be the subset of  $\mathfrak{M}_T$  defined as follows. Let  $r \in \mathfrak{M}_T(N)$ .

Then

$$\begin{aligned} u_i(x, t) \in C_{3,0}^{1,0}(\bar{D}_{XT}) \cap C^{1,1}(\bar{D}_{XT}) \cap C_{0,1}^{0,1}(\bar{D}_{XT}) \cap C^{2,0}(\bar{D}_T), \quad i=1, 2; \\ y(x, t), q(x, t), p(x, t) \in C^{1,0}(\bar{D}_{XT}) \cap C^{0,1}(\bar{D}_{XT}); \\ X(t, z) \in C_{1,0}^{1,0}(\bar{D}_{tz}) \cap C_{0,1}^{0,1}(\bar{D}_{tz}); \quad X_0(t, \tau) \in C_{1,0}^{0,1}(\bar{D}_{t\tau}) \cap C_{0,1}^{0,1}(\bar{D}_{t\tau}) \\ Z(x, t) \in C^{1,0}(\bar{D}_{x1T}) \cap C^{0,1}(\bar{D}_{x1T}); \quad T^0(x, t) \in C^{1,0}(\bar{D}_{oXT}) \cap C^{0,1}(\bar{D}_{oXT}) \end{aligned} \quad (3.4)$$

Further  $r \in \mathfrak{M}_T(N)$  implies

$$\|r\| \in N \quad (3.5)$$

$$u_i(x, 0) = u_i^0(x); \quad y(x, 0) = y^0(x); \quad q(x, 0) = q^0(x); \quad p(x, 0) = y_i(x, 0) \quad (3.6)$$

and also  $p(x, t)$  is connected with  $u_i, y$  and  $q$  by the equality (2.14), and  $u_i(x, t)$ ,  $i=1, 2$ , satisfy boundary conditions (1.3<sub>8</sub>), (1.3<sub>9</sub>) from where  $v_1$  and  $v_2$  are excluded. The norm  $\|r\|$  in  $\mathfrak{M}_T(N)$  is defined as a maximum maximum of norms of all components of  $r$  in relevant spaces, entering (3.4) (see (1.33)).

Define now an operator

$$r' = \{u'_1, u'_2, \dots, T^{0'}\} = R(r); \quad r \in \mathfrak{M}_T(N) \quad (3.7)$$

by the following set of conditions. Let

$$f(t) = p, y(1, t) (u^2(t) - u_{21}(t)) \quad (3.8)$$

and  $H(x, t | \cdot)$ ,  $w(x, t)$ ,  $v_i(x, t)$  and  $F_i(x, t | \cdot)$  be defined by the equalities (1.8), (1.3<sub>3</sub>), (1.13), (1.14) and (1.6) where  $y_t$  and  $y_x$  are replaced by  $p$  and  $q$  respectively. Define further

$$u'_{i0}(t) = U_{i00}(0, t | u_{i, t-1}, F_{i00}); \quad u'_{i1} = U_{i00}(1, t | u_{i, t-1}, F_{i00}); \quad i=1, 2 \quad (3.9)$$

and  $X'(t, z)$ ,  $Y(t, z)$ ,  $Q(t, z)$ ,  $P(t, z)$  equal to the right hand sides of (2.23) where  $Y(t, z)$ ,  $Q(t, z)$  and  $P(t, z)$  are replaced by  $y(X(t, z), t)$ ,  $q(X(t, z), t)$  and respectively by  $p(X(t, z), t)$ . Let further  $Z'(x, t)$ ,  $y'(x, t)$ ,  $q'(x, t)$  and  $p'(x, t)$  be defined in  $D_{X'1T}$  by the equalities

$$\begin{aligned} Z' = Z^*(x, t | X, Z, u_{ikm}, f); \quad y' = Y^*(t | Z'); \quad q' = Q^*(t | Z'); \\ p' = P^*(t | Z'); \quad (x, t) \in D_{X'1T} \end{aligned} \quad (3.10)$$



where  $Z^*, \dots, P^*$  are defined by (2.27). After this define

$$\begin{aligned} f'(t) &= p_{\gamma} y'(1, t) (u^2(t) - u'_{21}(t)); \\ y'(0, t) &= 1 - f'(t) p_x^{-1} (u'_{10}(t) - u^1(t))^{-1}. \end{aligned} \quad (3.11)$$

The definition of  $Z'(x, t)$  admits computation of  $Z'_{01}(x, t)$  and  $Z'_{10}(x, t)$  for  $t > 0$  on the strength of conditions (3.4) and (2.6). In particular there exists  $y'_{01}(1, t)$ , and, by virtue of (3.11),  $y'_{01}(0, t)$ . This allows to determine

$$p'_0(t) = (d/dt) y^*(0, t) \quad (3.12)$$

Let now  $q^1_0$  be defined by the equality

$$q'_0(t) = \varphi'(0, t) [p'_0(t) + p_{\beta} (u'_{10}(t) - u'_{20}(t))]^{-1} \quad (3.13)$$

where

$$\varphi'(0, t) = f'(t) (1 - u'_{10}(t)) \quad (3.14)$$

Thus we now have got the initial data for evaluation of  $X'_0(t, \tau)$ ,  $Y'_0(t, \tau)$ ,  $Q'_0(t, \tau)$  and  $P'_0(t, \tau)$  by the equalities (2.24), into the right hand sides of which  $q'_0$ ,  $X_0(t, \tau)$ ,  $u_i(x, t)$ ,  $y(x, t)$  and  $q(x, t)$  are inserted. After this we determine  $T^{0'}(x, t)$  as equal to the right hand side of (2.30).

Let us point out that

$$T^{0'}(X'(t, 0), t) = 0 \quad (3.15)$$

Indeed

$$X'(t, 0) = \int_0^t H_q(X(s, 0), s|Y(s, 0), Q(s, 0)|f(s), u_{ik}(X(s, 0), s)) ds \quad (3.16)$$

so that

$$X'(0, 0) = 0 \quad (3.17)$$

At the same time

$$X'_0(t, 0) = \int_0^t H_q(X_0(s, 0), s|Y_0(s, 0), Q_0(s, 0)|f(s), u_{ik}(X_0(s, 0), s)) ds \quad (3.18)$$

and, by virtue of (3.3) and definitions (2.23) and (2.24)

$$X_0(s, 0) = X(s, 0); Y_0(s, 0) = Y(s, 0); Q_0(s, 0) = Q(s, 0) \quad (3.19)$$

Hence

$$X'_0(t, 0) = X'(t, 0) \quad (3.20)$$

Further, by virtue of (2.30),

$$T^{0'}(X'_0(t, 0), t) = \int_{X(t, 0)}^{X'_0(t, 0)} B^{-1}(T^0(s, t), t) ds \quad (3.21)$$

so that (3.15) is indeed true.

Define now  $y'(x, t)$ ,  $q'(x, t)$  and  $p'(x, t)$  in  $D_{Ox, T}$  as equal to the right hand sides of (2.28), where  $T^0$  is replaced by  $T^{0'}$ ,  $v'_i$  as functions of  $f'$ ,  $y'$ ,  $p'$  and  $u_i(x, t)$  by the equalities (1.13) and (1.14). Define finally

$$F'_i = F_i(x, t | u_{ik0}, u_{j00}, v'_i, y', q'); \quad i=1, 2; \quad k=0, 1; \quad j \neq i \quad (3.22)$$

where  $F_i$  are defined by (1.6). After this we may determine  $u'_i(x, t)$  by the equality

$$u'_i(x, t) = U_{i00}(x, t | u'_{i, t-1}, F'_i); \quad i=1, 2 \quad (3.23)$$

This accomplishes the definition of the operator  $R(r)$ .

The following theorem is valid:

**Theorem 1.** Assume that conditions (2.4) — (2.8) are valid. Then  $\exists N > 0$  so large and  $T > 0$  so small that the operator  $R(r)$  realizes the into mapping of  $\mathfrak{M}_T(N)$ , contraction in the norm of  $\mathfrak{M}_T$ . The fixed point

$$r = R(r) \quad (3.24)$$

of this mapping possesses the following properties:

$$q(x, t) = y_x(x, t); \quad p(x, t) = y_t(x, t) \quad (3.25)$$

so that

$$y(x, t) \in C^{1,0}(\bar{D}_T) \cap C^{0,1}(\bar{D}_T) \quad (3.26)$$

Let further  $v_i(x, t)$ ,  $i=1, 2$  be defined as above. Then

$$v_i(x, t) \in C^{1,0}(\bar{D}_T); \quad i=1, 2. \quad (3.27)$$

Simultaneously with (3.26) and (3.27)

$$u_i(x, t) \in C^{2,0}(\bar{D}_T) \cap C^{0,1}(\bar{D}_T); \quad i=1, 2 \quad (3.26)$$

Equalities (3.25) — (3.28) mean that  $u_i(x, t)$ ,  $v_i(x, t)$ ,  $y(x, t)$ ,  $f(t)$  and  $w(x, t)$ , defined by (1.3<sub>3</sub>) and respectively by (3.8), satisfy all the conditions of the original boundary value problem (1.3<sub>1</sub>) — (1.3<sub>10</sub>), so that its local solution exists and unique.

The proof of this theorem is quite trivial. Indeed, let  $M$  be equal to the maximum maximum of modules of all derivatives of all input data, entering the definition of the operator  $R(r)$ . Let us first subordinate  $T > 0$  to the conditions

$$\begin{aligned} \min y^0(x) - NT \geq a > 0; \quad \max y^0(x) + NT \leq 1 - a; \quad 0 \leq x \leq 1; \\ \min |y^0(x)| - NT \geq a > 0; \end{aligned} \quad (3.29)$$

$$u_1^1(0) - u_1^0(0) - 2NT > 0; \quad u_2^0(1) - u^2(0) - 2NT > 0$$

This is possible by virtue of conditions (2.4) — (2.8).

Since  $u_i(x, t) \in C^{0,1}_{0,1}(\bar{D}_{xT})$  we see from (3.11) and (2.35) that  $\exists \dot{p}'_0(t)$  such that

$$|t^{\frac{1}{2}} \dot{p}'_0(t)| < C(N/t)$$

This means that  $T > 0$  may also be subordinated to the requirement

$$|p'_0(t) + p_\alpha[u_{10}(t) - u_{20}(t)]| \geq \rho > 0; \quad t \in [0, T] \quad (3.30)$$

where  $\rho$  depends only on initial data. After this it becomes clear that whatever the component  $S'$  of  $r' = R(r)$ ,  $r \in \mathfrak{M}_T(N)$ , is, all its derivatives  $S'_{km}$ , entering the definition of the space to which  $S'$  belongs\*) exist and satisfy the inequalities

$$|S'_{km}| < C_{km}(N) + C_{km}(N/t) \quad (3.11)$$

This means that indeed  $\exists N > 0$  so large and  $T > 0$  so small that  $r \in \mathfrak{M}_T(N)$  implies

$$\|r'\| < N \quad (3.32)$$

so that the operator  $R(r)$  realizes the into mapping of  $\mathfrak{M}_T(N)$ .

Let

$$r' = R(r); r^{**} = R(r^*), r \in \mathfrak{M}_T(N); r^* \in \mathfrak{M}_T(N) \quad (3.33)$$

Then, it is emphasized in section 2  $y', q', p'$  and  $y^{**}, q^{**}$  and  $p^{**}$  are functions continuous in  $\bar{D}_T$ .

Note that if  $F$  is a differentiable function of a vector argument  $r(x, t)$ , and  $X(t, z)$ ,  $Z(x, t)$  are differentiable functions of their arguments, then

$$\begin{aligned} |F(r(X(s, Z(x, t)), s)) - F(r^*(X^*(s, Z^*(x, t)), s))| &< C(N) \{ \max |Z(x, t) - \\ &- Z^*(x, t)| + \max |X(s, Z^*(x, t)) - X^*(s, Z^*(x, t))| + \\ &+ \max |r(s, X^*(s, Z^*(x, t))) - r^*(s, X^*(s, Z^*(x, t)))| \} \end{aligned} \quad (3.34)$$

where  $N$  is the maximum maximum of modules of derivatives of  $F(r)$ ,  $X(t, z)$  and  $r(x, t)$ .

Using this remark, continuity of  $y', \dots, p^{**}$  in  $\bar{D}_T$ , definition of the operator  $R(r)$  and well known properties of heat potentials and their derivatives [4]\*) we easily conclude that

$$\|r' - r^{**}\|_{\text{in}} < C(N|t) \|r - r^*\|_{\text{in}} \quad (C(N|t) < 1 \quad \forall t \leq T) \quad (3.35)$$

Inequality (3.35) means that there exists the fixed point of the mapping. It is a limit of a uniformly convergent sequence of vector functions  $r$  belonging to  $\mathfrak{M}_T(N)$ :

$$\begin{aligned} u_i(x, t) &= \text{Lim } u_{in}(x, t); y(x, t) = \text{Lim } y_n(x, t); \\ q(x, t) &= \text{Lim } q_n(x, t); p(x, t) = \text{Lim } p_n(x, t) \\ X(t, z) &= \text{Lim } X'_n(t, z); X_0(t, \tau) = \text{Lim } X_{0n}(t, \tau) \\ Z(x, t) &= \text{Lim } Z_n(x, t); T^0(x, t) = \text{Lim } T_n^0(x, t) \end{aligned} \quad n \rightarrow \infty \quad (3.36)$$

It follows from the definition of  $\mathfrak{M}_T(N)$  that sequences

$$\begin{aligned} \{ \partial^{k+m} u_{in} / \partial x^k \partial t^m \}; k=0, 1, 2, 3 \text{ if } m=0; k=0, 1 \text{ if } m=1 \\ \{ \partial^{k+m} S_n / \partial x^k \partial t^m \}; S=y, q \text{ or } p; k=0, 1 \text{ if } m=0; k=0 \text{ if } m=1 \end{aligned} \quad (3.37)$$

\*) Let  $E(x - \xi, t - \tau)$  be the fundamental solution of the heat conduction equation:  $E(x, t) = \exp(-x^2/2\sqrt{\pi t})$ . Then  $\forall k \geq 0, \forall m \geq 0 \exists C_{km}$  such that  $|\partial^{k+m} R(x, \xi, t - \tau) / \partial x^k \partial t^m| < C_{km} |\partial^{k+m} E(x - \xi, t - \tau) / \partial x^k \partial t^m|$  where  $R(x, \xi, t) = g(x, \xi, t)$  or  $G(x, \xi, t)$ .

are uniformly bounded and equicontinuous in  $\bar{D}_{XT}$ . This means that not only sequences  $\{u_{in}\}$ ,  $\{y_n\}$ ,  $\{q_n\}$  and  $\{p_n\}$  converge uniformly, but the same is valid for all sequences (3.37). Hence

$$u_i(x, t) \in C^{3,0}(\bar{D}_{XT}) \cap C^{1,1}(\bar{D}_{XT}); \quad i=1, 2; \quad (3.38)$$

$$y(x, t), q(x, t), p(x, t) \in C^{1,0}(\bar{D}_{XT}) \cap C^{0,1}(\bar{D}_{XT})$$

Together with this

$$v_i(x, t) = \lim_{n \rightarrow \infty} v_{in}(x, t) \in C^{1,0}(\bar{D}_{XT}) \cap C^{0,1}(\bar{D}_{XT}) \cap C(\bar{D}_T) \quad (3.39)$$

$$f(t) = \lim_{n \rightarrow \infty} f_n(t) \in C^1(0, T) \quad i=1, 2$$

It is now obvious that

$$q(x, t) = y_x(x, t); \quad p(x, t) = y_t(x, t) \quad (3.40)$$

Indeed, let us insert  $u_i(x, t)$  and  $f(t)$  into the right hand side of equations (2.23). The solution of this system exists and is unique. The general theory of the Cauchy problem for the non-linear partial differential equations of the first order [1] shows that the solution of this system is the solution of the equation (1.7) satisfying the initial conditions

$$y(x, 0) = y^0(x); \quad q(x, 0) = q^0(x); \quad p(x, 0) = p^0(x) \quad (3.41)$$

so that (3.40) is indeed valid in  $D_{X1T}$ . Quite analogously inserting  $u_i(x, t)$  into the right hand side of (2.24) together with values of  $y_0(\tau)$  and  $q_0(\tau)$ , determined by  $u_i(x, t)$  and their derivatives as in section 2, we conclude that (3.40) is true not only in  $D_{X1T}$  but in  $D_{OXT}$  either.

Finally it is obvious that

$$u_i(x, t) \in C^{2,0}(\bar{D}_T) \cap C^{0,1}(\bar{D}_T)$$

since it follows from the boundedness of  $F_{i10}$  in  $D_{XT}$  if  $F_i$  are determined by equalities (1.6). This means that  $u_i(x, t)$ ,  $y(x, t)$ ,  $f(t)$ ,  $v_i(x, t)$  and  $w(x, t)$  indeed satisfy all the conditions of the original boundary value problem.

#### 4. Conclusive remarks

The existence and uniqueness theorem, proved above, is essentially local. In fact, one might apply the method of prolongation. However the very possibility of prolongation is strongly restricted by the requirement

$$q(x, t) \neq 0 \quad (x, t) \in \bar{D}_T \quad (5.1)$$

This requirement is not natural from the applied point of view. As it is demonstrated in [2], [6] the non-monotonicity of the change of the shape of the lateral membrane is a characteristic peculiarity of the behaviour of such biological objects as tubulus of a *T*-system (tubular system) or fast muscle fibres or a plane epithelial tissue im-

mersed into a not isotonic solutions. Therefore it is desirable to reject the restriction (5.1). At the same time this restriction is essential for the theory above being far not an artificial one. It becomes obvious if we replace the non-linear equation (1.7) by the quasilinear hyperbolic equation of the second order, which may be obtained by means of differentiating (1.7). The characteristic equation will be

$$y_t = p_B(u_1 - u_2) (dt/dx)^2 - y_x (dt/dx) = 0 \quad (5.2)$$

so that lines  $t = \text{const.}$  are characteristic ones, and the line  $y_x = 0$  is the line of the parabolic degeneration. This means that the non-monotonicity of  $y$  leads to the necessity to deal with the problem of an essentially different nature than that studied above. At the same time the most natural initial condition  $y^0(x) = \text{const.}$  is excluded from our consideration.

We also recall that the problem under consideration was formulated in [9], [10] under the assumption that the lateral membrane does not resist deformations. Only this assumption led to the very possibility of separating the kinematic and diffusion parts of the problem from the dynamical one. Therefore the use of the system of the average volume velocity, lying in the basis of the consideration above, loses, perhaps, its advantage in more general cases. As far as we know a general consideration of the problem remains untackled.

## References

- [1] COURANT R. Methods of Mathematical Physics by R. Courant and D. Hilbert. Vol. II. Partial Differential Equations. N.Y.-London, 1960.
- [2] GEIMAN H., RUBINSTEIN L. Passive transfer of low-molecular non-electrolytes across deformable semipermeable membranes. II. Dynamics of a single muscle fibre shrinking and swelling and related changes of the T-system tubule form. *Bull. Math. Biol.* 36 (1974) 4, 379-401.
- [3] GELFAND I.M. Some problems of the theory of quasilinear equations. *Uspehi Mat. Nauk.* 14 (1959) 2.
- [4] GEVREY M. Sur les equations aux derivees partielles du type parabolique. *J. de Math. Pure et Appl.* 13 (1913), 305-475.
- [5] HAASE R. Thermodynamik of Irreversible Prozesses. Darmstadt, 1963.
- [6] KROLENKO S.A. The T-system of Muscle Fibres; Structure and Functions. Inst. of Cytology of A.N., USSR; Nauka, 1975 (Russian).
- [7] RUBINSTEIN L. On the solution of Verigin's problem. *Dokl. Akad. Nauk SSSR*, 113 (1957), 50-53.
- [8] RUBINSTEIN L. The Stefan Problem. Transl. Math. Monogr. 27, AMS, 1971.
- [9] RUBINSTEIN L. Passive transfer of low-molecular non-electrolytes across deformable semipermeable membranes. I. Equations of convective diffusion transfer of non-electrolytes across deformable membranes of a large curvature. *Bull. Math. Biol.* 36 (1974) 4, 365-377.
- [10] RUBINSTEIN L. On the equations of convective diffusion transfer of low-molecular non-electrolytes across deformable membranes of a large curvature. In E. Magenes (ed) Free boundary problems. Proc. of a seminar held in Pavia, Sept.-Octob. 1979. Roma, 2, 1980.
- [11] RUBINSTEIN L. Free boundary problem for a non-linear system of parabolic equations including one with reserved time. *Annali di Matematica Pura ed Applicata*, (IV), Vol. CXXXV (1983) 29-72.

- (The short version of the paper see in G. Hammerlin and K.-H. Hoffmann (ed) *Improperly Posed Problems and Their Numerical Treatment*. Intern. Series of Numerical Math. ISNM 63, 1983, 205–226.
- [12] RUBINSTEIN L., GEIMAN H. On the water transfer through epithelial tissues. *Cytology* 1 (1974) 8, 970–976 (Russian).

*Received, February 1984.*

### **Zagadnienie modelowe teorii transportu nieelektrolitów przez deformowalne błony półprzepuszczalne**

Teoria konwekcyjno-dyfuzyjnego transportu masy przez deformowalne błony półprzepuszczalne nie przeciwdziałające zniekształceniom, będąca przedmiotem prac autora [9], [10], wymaga analizy jej aspektów matematycznych.

W artykule rozpatruje się zagadnienia modelowe, jednowymiarowej wersji tej teorii. Określa się pola stężeń, prędkości ruchu wzdłuż komórek rozdzielonych przez błonę oraz kształt tej ostatniej jako rozwiązanie zagadnienia brzegowego, sformułowanego dla układu dwóch nieliniowych równań parabolicznych drugiego rzędu, równania hiperbolicznego pierwszego rzędu i dwóch równań różniczkowych zwyczajnych pierwszego rzędu z odpowiednimi warunkami brzegowymi i początkowymi. Poprzez sprowadzenie zagadnienia do układu równań całkowych dowodzi się lokalnego istnienia i jednoznaczności rozwiązania, wykorzystując twierdzenie o punkcie stałym.

### **Модельная задача теории переноса неэлектролитов через деформируемые полупроницаемые мембраны**

Теория конвективно-диффузионного массо-переноса через деформируемые полупроницаемые мембраны не сопротивляющиеся деформациям, представленная в статьях автора [9], [10], нуждается в изучении ее математических аспектов.

В статье рассматривается модельная задача одномерной версии этой теории. Поля концентраций, скоростей движения вдоль ячеек, разделенных мембраной и форма последней подлежат определению как решение краевой задачи, сформулированной для системы двух нелинейных параболических уравнений второго порядка, гиперболического уравнения первого порядка и двух обыкновенных дифференциальных уравнений первого порядка с соответствующими краевыми и начальными условиями. Локальная теорема существования и единственности доказана с помощью сведения задачи к системе интегральных уравнений с применением принципа сжатых отображений.

