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# A formula for extremum involving $\Phi$-convexity

by

PHAN QUOC KHANH

B 15, P. 11 Khu Kimlien
Hanoi, Vietnam

In the note the following formula

$$\inf \{F(x): Cx=\bar{y}\}= \sup_{\varphi \in \Phi} \inf \{F(x): \varphi(Cx)=\varphi(\bar{y})\}$$

is proved for a general $\Phi$-convex case. Known results for convex cases are obtained as its consequences.

A great number of extremal problems with constraints can be formulated as follows. Given a functional $F$ on a set $X$, a mapping $C$ from $X$ into a set $Y$ and a point $\bar{y} \in CX$; a problem is looking for

$$\inf \{F(x): Cx=\bar{y}\} .$$

Let $\Phi$ be a family of functionals on $Y$. A good situation is when the above problem can be reduced to another problem with a scalar constraint by the formula

$$\inf \{F(x): Cx=\bar{y}\}= \sup_{\varphi \in \Phi} \inf \{F(x): \varphi(Cx)=\varphi(\bar{y})\} . \tag{1}$$

Such reduction has been studied in [8] for the case when $X$ and $Y$ are Banach spaces, $C$ is a linear continuous operator and $\Phi=Y^*$. It is known also that formulae of type (1) have a close connection with PONTRYAGIN Maximum Principle in an abstract (nonHAMILTONian) form and with LAGRANGE multiplier methods. This gives us a trustful motivation for investigating formulae of type (1). The proofs of known such formulae base on the HAHN-BANACH theorem in a separation form. The separation technique has had a very big progress (for a survey of results on separation in linear spaces see [3]), but examining the proofs of previous considerations of formulae of type (1) one can see that only separations of points from sets are needed. These separations, of course, demand weaker conditions than for separations of sets and then can be extended to more general cases.

In the present note we prove (1) for a $\Phi$-convex case and derive, as consequences, known results about convex cases, including, in particular, some theorems of S. ROLEWICZ [7], [8].

$\Phi$-convexity was introduced by KY FAN [5]. After that numerous investigations were devoted to this subject. Let $X$ be an arbitrary set and $\Phi$ be a family of functionals from $X$ into extended real line $\bar{R}=R\cup\{-\infty\}\cup\{+\infty\}$. A subset $K$ of $X$ is said to be $\Phi$-convex if either $K=X$ or $K$ is an intersection of subsets each of the form $\{x\in X: f(x)\geqslant\gamma\}$, where $f\in\Phi$ and $\gamma\in\bar{R}$. These level sets of functionals in $\Phi$ are named, as customary, semispaces. Setting, for $K\subset X$ and $f\in\Phi$,

$$h(K,f)=\inf_{x\in K} f(x)\,,$$

we have

LEMMA. *A subset $K$ of $X$ is $\Phi$-convex if and only if it has the form*

$$K=\bigcap_{f\in\Phi}\{x\in X: f(x)\geqslant h(K,f)\}\,. \tag{2}$$

Proof. The sufficiency is evident by the definition of $\Phi$-convex sets. We show the necessity. Since $K\subset\{x: f(x)\geqslant h(K,f)\}$ the right-hand side of (2) includes $K$. On the other hand, if $K$ is $\Phi$-convex, then $K$ is of the form $K=\bigcap_\tau\{x: f_\tau(x)\geqslant\gamma_\tau\}$, where $f_\tau\in\Phi$, $\gamma_\tau\in\bar{R}$. We have $h(K,f_\tau)\geqslant\gamma_\tau$, then $\{x: f_\tau(x)\geqslant h(K,f_\tau)\}\subset\{x: f_\tau(x)\geqslant\gamma_\tau\}$. Therefore $K$ includes the right-hand side of (2). ∎

The couple $\{X,\Phi\}$ is called a $\Phi$-convexity space. A mapping $C$ from $\{X,\Phi\}$ into $\{Y,\Psi\}$ is called a cyrtomorphism if images and preimages of convex sets are convex.

PROPOSITION. For a mapping $C$ from $\{X,\Phi\}$ into $\{Y,\Psi\}$ be a cyrtomorphism it is necessery and sufficient that:

(i) for every subfamily $\{g_\tau)\subset\Phi$, $\{\gamma_\tau\}\subset\bar{R}$ and $y_0\in Y$ such that $C^{-1}y_0\subset\bigcup_\tau\{x: g_\tau(x)<<\gamma_\tau\}$ there be $\varphi\in\Psi$, $\beta\in\bar{R}$, satisfying

$$\varphi(y_0)<\beta\,,$$

$$\varphi(Cx)\geqslant\beta \text{ if } g_\tau(x)\geqslant\gamma_\tau \text{ for all } \tau;\ \text{that}$$

(ii) $CX$ be $\Psi$-convex; and that

(iii) for every $\{\varphi_\theta\}\subset\Psi$, $\{\beta_\theta\}\subset\bar{R}$ and $x_0\in X$ such that $\varphi_\theta(Cx_0)<\beta_\theta$ for some $\varphi_\theta$, $\beta_\theta$ from mentioned families, there exist $f\in\Phi$, $\delta\in\bar{R}$, satisfying

$$f(x_0)<\delta\,,$$

$$f(x)\geqslant\delta \text{ if } \varphi_\theta(Cx)\geqslant\beta_\theta \text{ for all } \theta\,.$$

Proof. Necessity. (i) The set $A:=C\bigcap_\tau\{x\in X: g_\tau(x)\geqslant\gamma_\tau\}$ is $\Psi$-convex and does not contain $y_0$. Then there is $\varphi\in\Psi$, $\beta\in\bar{R}$, such that the semispace $\{y\in Y: \varphi(y)\geqslant\beta\}$ contains $A$ and not $y_0$. Thus $\varphi$ is a required functional;

(ii) clear by definition;

(iii) the set $B:=\{x\in X: \varphi_\theta(Cx)\geqslant\beta_\theta$ for all $\theta\}$ is $\Phi$-convex and does not contain $x_0$. Thus a functional $f\in\Phi$ such that the semispace $\{x\in X: f(x)\geqslant\delta\}$ contains $B$ and not $x_0$ is a required one.

SUFFICIENCY. If $K$ is a $\Phi$-convex set, then $K$ is of the form $K=\bigcap_\tau\{x: g_\tau(x)\geqslant\gamma_\tau\}$. If $y_0\notin CK$, then $C^{-1}y_0\subset\bigcap_\tau\{x: g_\tau(x)<\gamma_\tau\}$. The semispace $\{y\in Y: \varphi(y)\geqslant\beta\}$, where $\varphi$ is the mentioned in condition (i) functional, contains $CK$ and not $y_0$. Thus $CK$ is $\Psi$-convex. Using (iii) we show in the same way that preimages of $\Psi$-convex sets are $\Phi$-convex. ∎

We define a family $\Phi_{epi}$ associated with the family $\Phi$ of functionals on $X$ by setting

$$\Phi_{epi}=\{\tilde f\in\bar R^{X\times R}: \tilde f(x,r)=r+f(x),\ f\in\Phi\}.$$

A functional $F: X\to\bar R$ is said to be $\Phi$-convex if its epigraph is $\Phi_{epi}$-convex. A functional $F: X\to\bar R$ is $\Phi$-convex if and only if

$$F(x)=\sup_\tau[-g_\tau(x)+a_\tau],$$

for some subfamily $\{g_\tau\}\subset\Phi$ and for all $x\in X$, where $a_\tau$ are from $R$ (see [4]).

THEOREM. *Let $C: X\to\{Y,\Phi\}$ be a mapping and $F\in\bar R^X$ (a functional from $X$ into $\bar R$) such that for each $r$ the set $CK_r:=C\{x: F(x)\leqslant r\}$ is $\Phi$-convex. Then, for $\bar y\in CX$,*

$$\inf\{F(x): Cx=\bar y\}=\sup_{\varphi\in\Phi}\inf\{F(x): \varphi(Cx)=\varphi(\bar y)\}. \tag{3}$$

Proof. Let $a$ stand for the left-hand side of (3) and $b$ for the right-hand one. Obviously $a\geqslant b$. Arguing by contradiction, suppose $a>b$. Taking $s\in(b,a)$ we see that $y\notin G_s:=CK_s$. The hypothesis on $\Phi$-convexity of $G_s$ and the above lemma entails the existence of $\varphi_0\in\Phi$ such that $\varphi_0(\bar y)<h(G_s,\varphi_0)$. Henceforth

$$CK_s\subset\{y\in Y: \varphi_0(y)\geqslant h(G_s,\varphi_0)\}\subset\{y: \varphi_0(y)>\varphi_0(\bar y)\}.$$

Consequently

$$\inf\{F(x): \varphi_0(Cx)=\varphi_0(\bar y)\}\geqslant s>b,$$

which contradicts the definition of number $b$. ∎

REMARK. (1) The hypothesis of the theorem is clearly fulfilled if $C$ is a cyrtomorphism for some $\Psi$-convexity on $X$ and $F$ is a $\Psi$-convex functional.

(2) The idea of the above proof is the same as in the known ones but a noteworthy advantage here is that the separation is performed without linear and topological structures and basing only on a general $\Phi$-convexity.

COROLLARY 1 [8]. *Let $X$ be a linear space and $Y$ be a linear topological space and let $C: X \to Y$ be a linear operator, and let $F: X \to \bar{R}$ be a convex functional such that, for each $r$, $CK_r = C\{x: F(x) \leqslant r\}$ is closed. Then, for $\bar{y} \in CX$,*

$$\inf \{F(x): Cx = \bar{y}\} = \sup_{\varphi \in Y^*} \inf \{F(x): <\langle \varphi, Cx \rangle = \langle \varphi, \bar{y} \rangle\}.$$

Proof. Setting $\Phi = Y^*$ one sees, by the assumption on closeness of $CK_r$, the $Y^*$--convexity of these sets, which, because of the theorc m, completes the proof. ■

COROLLARY 2 [7]. *Let $X$ be a linear normed space and $Y$ be a linear topological space and let $C: X \to Y$ be a linear operator possessing the property that $C\{x \in X: \|x\| \leqslant \leqslant 1\}$ is closed. Then, for $\bar{y} \in CX$,*

$$\inf \{\|x\|: Cx = \bar{y}\} = \sup_{\varphi \in Y^*} \inf \{\|x\|: <\varphi, Cx\rangle = \langle \varphi, \bar{y} \rangle\}.$$

This statement about a probl m of moments is a particular case of Corollary 1 and was proved by M.G. KREIN [1] for the case when $Y$ is finite-dimensional and by A.G. BUTKOVSKI [2] for the case $X = L_p$ and $Y = 1_p$ and by S. ROLEWICZ [7] for the case when $X$ and $Y$ are BANACH spaces.

COROLLARY 3 [8]. *Let $X = X_0^*$, $Y = Y_0^*$, $C = C_0^*$, where $X_0$ is a linear normed space, $Y_0$ is a linear topological space and $C_0: Y_0 \to X_0$ is a linear continuous operator. Then, for $\bar{y} \in CX$,*

$$\inf \{\|x\|: Cx = \bar{y}\} = \sup_{y_0 \in Y_0} \inf \{\|x\|: \langle x, C_0 y_0 \rangle = \langle \bar{y}, y_0 \rangle\}.$$

Proof. Set $\Phi = Y_0$. By compactness of the unit ball and the continuity of $C$ in the weak* topology, the sets $G_r := C\{x: \|x\| \leqslant r\}$ are weak* compact. Since $Y$ with weak* topology is a HAUSDORFF space $G_r$ are closed and then are $Y_0$-convex. ■

The three corollaries above are other versions of S. ROLEWICZ'S results. They can be proved also by HAHN-BANACH theorem.

To state other corollaries we consider the set $X^\alpha$, for fixed $\alpha \in (0, 1]$, of all functionals on a linear normed space $X$, satisfying the HÖLDER condition

$$f(x_1) - f(x_2) \leqslant K \|x_1 - x_2\|^\alpha,$$

and $f(0) = 0$. $X^\alpha$ is clearly a linear space with ordinary operations of functionals. It turns into a normed space by defining the following norm (see [6]):

$$\|f\|_\alpha = \sup_{\|z\| \leqslant 1} (f 0^\alpha)(z),$$

where $f 0^\alpha$ is the recession functional of degree $\alpha$ of functional $f$, that is a functional on $X$ with the epigraph

$$\text{epi}(f 0^\alpha) = \{(x, \nu) \in X \times R: \text{epi} f + (\lambda x, \lambda^\alpha \nu) \subset \text{epi} f, \ \forall \lambda > 0\}.$$

Let $X$ and $Y$ be linear normed spaces and $C: X \to Y$ be a linear continuous operator. For $x \in X$, $\varphi \in Y^\alpha$ we set $(C^\alpha \varphi)(x) = \varphi(Cx)$. Then $C^\alpha$ is a linear operator of $Y^\alpha$ into

$X^\alpha$ (since $\|C^\alpha\varphi\|_\alpha \leqslant \|C\|^\alpha \|\varphi\|_\alpha$). For $x\in X$, $f\in X^\alpha$, we set $x(f)=f(x)$, then $x$ is a linear functional on $X^\alpha$. We use $\sigma_\alpha(X)$ to designate the topology on $X^\alpha$ defined by the class $X$ of linear functionals on $X^\alpha$, that is the coarsest topology in which all $x\in X$ are continuous. The basis of neighborhoods of the origin in $X^\alpha$ consists of all sets of the form

$$U=\{f\in X^\alpha: |x_i(f)|<\varepsilon_i,\ i=1,\dots,n,\ x_i\in X,\ \varepsilon_i\in R\}. \tag{4}$$

$C^\alpha$ is a linear continuous operator from $Y^\alpha$ with topology $\sigma_\alpha(Y)$ into $X^\alpha$ with topology $\sigma_\alpha(X)$. Indeed,

$$(C^\alpha)^{-1}U=\{\varphi\in Y^\alpha: |x_i(C^\alpha\varphi)|<\varepsilon_i,\ x_i\in X\}=\{\varphi\in Y^\alpha: |Cx_i(\varphi)|<\varepsilon_i,\ Cx_i\in Y\}$$

is a neighborhood of the origin of $Y^\alpha$ in topology $\sigma_\alpha(Y)$ for each $U$ of the form (4). We have an analogy of ALAOGLU theorem: the unit ball $S:=\{f\in X^\alpha: \|f\|_\alpha\leqslant 1\}$ is compact in topology $\sigma_\alpha(X)$ (for the proof see [6]).

Let us define on the encountered set $X$ of linear functionals on $X^\alpha$ the following addition and multiplication with scalars (different from the linear operations in given linear normed space $X$):

$$(x_1\oplus x_2)(f)=x_1(f)+x_2(f),\ \text{for}\ x_1,x_2\in X,\ f\in X^\alpha,$$
$$(\gamma\circ x)(f)=\gamma x(f),\ \text{for}\ x\in X,\ \gamma\in R.$$

We denote the linear hull with respect to $\oplus$ and $\circ$ by Lin $X$. Next we extend $C$ to the whole Lin $X$ by the formula

$$(Cx)(\varphi)=x(C^\alpha\varphi),\ \text{for}\ x\in \operatorname{Lin} X,\ \varphi\in Y^\alpha.$$

Then we still have $x(f)=f(x)$ for $x\in \operatorname{Lin} X$, $f\in X^\alpha$. In fact, $x(f)=(\underset{\oplus}{\Sigma}\gamma_i\circ x_i)(f)=$ $=\Sigma\gamma_i x_i(f)=\Sigma\gamma_i f(x_i)=f(\underset{\oplus}{\Sigma}\gamma_i\circ x_i)=f(x)$. (Because Lin $X$ is a linear dual space of $X^\alpha$). Note that the topology on $X^\alpha$ defined by Lin $X$ also admits the basis (4), i.e. it coincides with $\sigma_\alpha(X)$. Indeed, for a neighborhood in the basis of the topology $\sigma_\alpha(\operatorname{Lin} X)$

$$V=\Big\{f\in X^\alpha: \Big|\sum_{\substack{\oplus\\ j=1}}^{k_i}\gamma_j^i\circ x_j^i(f)\Big|<\varepsilon_i,\ x_j^i\in X,\ \varepsilon_i\in R\Big\},$$

one finds a neighborhood in the basis (4)

$$U=\left\{f\in X^\alpha: |x_j^i(f)|<\frac{\varepsilon_i}{k_i|\gamma_j^i|}\right\}$$

contained in $V$. Thus $\sigma_\alpha(\operatorname{Lin} X)=\sigma_\alpha(X)$.

COROLLARY 4 [6]. *Let $X$ and $Y$ be linear normed spaces and let $C: X\to Y$ be a linear continuous operator. Then, for $f\in C^\alpha Y^\alpha$, $0<\alpha\leqslant 1$,*

$$\inf\{\|\varphi\|_\alpha: \varphi\in Y^\alpha,\ C^\alpha\varphi=f\}=\sup_{x\in \operatorname{Lin} X}\inf\{\|\varphi\|_\alpha: \varphi\in Y^\alpha,\ \varphi(Cx)=f(x)\}.$$

Proof. Set $\Phi=\text{Lin } X$. By the compactness of the unit ball of $Y^\alpha$ in $\sigma_\alpha(Y)$ and the continuity of $C^\alpha$, it is easy to see the Lin $X$-convexity of the sets $G_r := C^\alpha \{\varphi\in Y^\alpha : \|\varphi\|_\alpha \leqslant \varepsilon\}$. ■

Let us consider now convex functionals on a linear normed space $X$. Inasmuch as convex functionals different from constants may be HÖLDER continuous with exponent $\alpha$ only if $\alpha=1$, expecting a similar assertion as in Corollary 4 we need to investigate the set $X^0 \subset X^1$ of all LIPSCHITZ convex functionals on $X$. We see that the operator $C^1: Y^1 \to X^1$ defined by $(C^1\varphi)(x)=\varphi(Cx)$ is also an operator of $Y^0$ into $X^0$. In fact, for $\varphi\in Y^0$, $x_1, x_2 \in X$ and $\gamma\in[0, 1]$ we have

$$C^1\varphi(\gamma x_1+(1-\gamma)x_2)=\varphi(\gamma Cx_1+(1-\gamma)Cx_2)\leqslant$$
$$\leqslant \gamma\varphi(Cx_1)+(1-\gamma)\varphi(Cx_2)=\gamma(C^1\varphi)(x_1)+(1-\gamma)(C^1\varphi)(x_2),$$

that is $C^1\varphi\in X^0$.

Corollary [6]. *Let $X$, $Y$ and $C$ be as in Corollary 4. If $f\in C^1 Y^0$, then*

$$\inf\{\|\varphi\|_1 : \varphi\in Y^0,\ C^1\varphi=f\}=\sup_{x\in \text{Lin } X}\inf\{\|\varphi\|_1 : \varphi\in Y^0,\ \varphi(Cx)=f(x)\}.$$

Proof. Consider the mapping $C^1: \{Y^0, \text{Lin } Y\}\to\{X^0, \text{Lin } X\}$. We show at first the closedness in $\sigma_1(Y)$ of $S\cap Y^0$ ($S$ is the unit ball of $Y^1$). Assume $\varphi_n\in S\cap Y^0$, $\varphi_n\to\varphi$ in $\sigma_1(Y)$. By compactness of $S$ we have $\varphi\in S$. $\varphi$ is also convex since, for $y_1, y_2\in Y$, and $\gamma\in[0, 1]$,

$$\varphi(\gamma y_1+(1-\gamma)y_2)=\lim_{n\to\infty}\varphi_n(\gamma y_1+(1-\gamma)y_2)\leqslant$$
$$\leqslant\lim_{n\to\infty}[\gamma\varphi_n(y_1)+(1-\gamma)\varphi_n(y_2)]=\gamma\varphi(y_1)+(1-\gamma)\varphi(y_2).$$

Hence $S\cap Y^0$ is compact in $\sigma_1(Y)$ and then, because of the continuity of $C^1$ the set $C^1\{\varphi\in Y^0 : \|\varphi\|_1\leqslant r\}$ is Lin $X$-convex. ■

In [6] there were applications of Corollaries 4 and 5 in the observation theory. But it should be pointed out that some notations in the corresponding assertions in [6] are incorrect.

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## Wzór na ekstremum w przypadku $\Phi$-wypukłości

W ogólnym przypadku $\Phi$-wypukłym dowodzi się następującego wzoru

$$\inf\{F(x)\colon Cx=\bar{y}\}=\sup_{\varphi\in\Phi}\inf\{F(x)\colon \varphi(Cx)=\varphi(\bar{y})\}$$

z którego wynikają rezultaty znane dla zadań wypukłych.

## Формула экстремума для ф-выпуклого случая

В этой работе доказывается следующая формула

$$\inf\{F(x)\colon Cx=\bar{y}\}=\sup_{\varphi\in\Phi}\inf\{F(x)\colon \varphi(Cx)=\varphi(\bar{y})\}$$

для общего ф-выпуклого случая. Некоторые известные результаты для выпуклых случаев считаются её следствиями.