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Robust stability of decentralized ϵ-coupled control systems*)

by

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This paper deals with the problem of robustness of decentralized ε-coupled control systems. Bounds on perturbations in the coupling parameter are established and it is shown that it is possible to calculate a sector such that system stability is assured. The suggested approach is not crucially dependent on any particular method for decentralized control system design.

KEYWORDS: Robustness, Decentralized control, Large-scale systems, ε-Coupling, Stability.

1. Introduction

Reduction of computation and simplification of the control system structure are of particular concern in decentralized control of large scale systems [1]. The methods for model simplification*) can be divided into two classes: agregation methods [3] and perturbation methods. It is common practice to divide the perturbation methods into two subclasses of nonsingular perturbation (ε-coupling) [6] — [9] and singular perturbation [4] methods.

When perturbation methods are used, following his intuition and experience a designer neglects small parameters, that is, sets them to zero to produce a simplified system. However, the question is whether an approximate design can be satisfactorily applied to high order systems. A shortcoming of these methods is that they do not give an estimation of a range of the parameters in which the reduced solution can be used to stabilize the original system. Therefore, a fundamental problem in large scale system theory is to give conditions for the success of design based on simplified models — this is essentially a robustness problem.

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In this paper we consider a problem of robustness of decentralized ε-coupled systems subject to variations in the coupling parameter. A brief description of the paper is as follows. The design procedure and problem formulation are given in Section 2. The main results and robustness characterization are given in Section 3.

2. Design procedure and problem formulation

The ε-coupling approach is based on the notation of a nonsingular perturbation, i.e., a perturbation term is in the right-hand side of a differential equation. We define the ε-coupled decentralized control system as

$$(S_0)\begin{cases} \dot{x}(t)=\hat{A}x(t)+\varepsilon\bar{A}x(t)+\sum_i^k \tilde{B}_i u_i(t)+\varepsilon\sum_i^k \bar{B}_i u_i(t) & (1)\\ u_i(t)=F_i C_i x(t), \quad i=1,2,\dots,k & (2)\end{cases}$$

where $x\in R^n$ and $u_i\in R^{m_i}$ are the state and inputs of (S_0), and

$$A=\tilde{A}+\varepsilon(t)\bar{A}, \quad B_i=\tilde{B}_i+\varepsilon(t)\bar{B}_i \tag{3}$$

$$\tilde{A}=\operatorname{diag}(A_{ii}), \qquad \bar{A}=\begin{bmatrix} 0 & A_{12} & \cdots & A_{1k}\\ A_{21} & 0 & \cdots & A_{2k}\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ A_{k1} & A_{k2} & \cdots & 0\end{bmatrix} \tag{4}$$

$$\tilde{B}_i=\begin{bmatrix} 0\\ \cdot\\ \cdot\\ \cdot\\ B_{ii}\\ \cdot\\ \cdot\\ \cdot\\ 0\end{bmatrix} \qquad \bar{B}_i=\begin{bmatrix} B_{1i}\\ \cdot\\ \cdot\\ \cdot\\ B_{i-1,i}\\ 0\\ B_{i+1,i}\\ \cdot\\ \cdot\\ \cdot\\ B_{k,i}\end{bmatrix} \tag{5}$$

In the design procedure the model (1) and (2) can be employed to express the fact that a large scale system is composed of several similar subsystems which are uncoupled if a coupling parameter ε is neglected. The decentralized control methods

*) A great variety of reduced-order modeling techniques exist for general systems (see e.g. the bibliography of [2]).

for large scale systems based on ε-coupling approach follow directly from the results developed in [7] — [9]. It can be easily shown that the ε-coupling method can be applied to both approaches for decentralized control designs suggested in [10]. The first approach is based on minimization of the decentralized quadratic performance index

$$J=\int_0^\infty \left(x^T Qx+\sum_i^k u_i^T R_i u_i\right) dt \tag{6}$$

where $Q=Q^T\geqslant 0$, $R_i=R_i^T>0$, and

$$Q=\begin{bmatrix} Q_{11} & \varepsilon Q_{12} & \cdots & \varepsilon Q_{1k} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \varepsilon Q_{k1} & \varepsilon Q_{k2} & \cdots & Q_{kk} \end{bmatrix} \tag{7}$$

The second approach is based on computation of a complete state feedback (by linear quadratic methodology) and reduction to a specified control with a decentralized structure.

In both cases the computation of the decentralized control can be done by decoupling subsystem calculation by using Maclaurin series expansion. Then the control law can be expressed as

$$u_i(t)=-R_i^{-1}\hat{B}_i^T\hat{K}\hat{A}_i x(t), \qquad i=1,2,\ldots,k \tag{8}$$

where the values of the matrices $\hat{K}$ and $\hat{A}_i$ depend on the specific approach chosen; $\hat{B}_i=B_i(\varepsilon^0)$, $\hat{K}=K(\varepsilon^0)$ and $\hat{A}_i=A_i(\varepsilon_0)$ are calculated for some $\varepsilon=\varepsilon_0$.

When the control (8) is used in the actual system (S_0) the closed loop stability is in question. Because of continuity, for sufficiently small $\Delta\varepsilon=\varepsilon-\varepsilon_0$ it can be expected that the actual system will remain stable. Therefore, in what follows, it will be assumed that the closed loop system

$$\dot{x}(t)=\left(\hat{A}-\sum_i^k \hat{B}_i R_i^{-1}\hat{B}_i^T\hat{K}\hat{A}_i\right)x(t) \tag{9}$$

where $\hat{A}=A(\varepsilon_0)$, $\hat{B}_i=B_i(\varepsilon_0)$, $\hat{K}=K(\varepsilon_0)$ and $\hat{A}_i=A_i(\varepsilon_0)$, is stable. However, if during the operation of the system the coupling parameter considerably changes its value, then the stability of the system can be destroyed. A shortcoming of a weak coupling approach is that, in general case, it does not give estimation of range of the parameter ε in which the approximate solution can be used to stabilize the perturbed system. Therefore, it should be pointed out that it is not only important that there exists ε so that the closed loop system is stable, but also it is important to know for which range of values of ε the closed loop system remains stable. In the following section we give such an estimation.

Before that, notice that the value of the corresponding performance index is $J=x^T(t_0)\,Px(t_0)$, where P satisfies the matrix equation

$$\left(\hat{A}-\sum_i^k \hat{B}_i R_i^{-1} \hat{B}_i^T \hat{K}\hat{\Lambda}_i\right)^T P+P\left(\hat{A}-\sum_i^k \hat{B}_i \hat{R}_i^{-1} \hat{B}_i^T \hat{K}\Lambda_i\right)+T=0 \tag{10}$$

where

$$T=Q+\sum_i^k \hat{\Lambda}_i^T \hat{K}^T \hat{B}_i R_i \hat{B}_i^T \hat{K}\hat{\Lambda}_i \tag{11}$$

For ease in subsequent calculation it is assumed that the matrix T is a nonsingular matrix, thus guaranteeing [11] that P is positive definite.

3. Robustness characterization

Let the actual value of the coupling parameter $\varepsilon(t)$ be

$$\varepsilon(t)=\varepsilon_0+\Delta\varepsilon(t) \tag{12}$$

Then the perturbed system can be presented in the form

$$(S_1):\dot{x}(t)=\left(\hat{A}+\Delta\varepsilon\bar{A}-\sum_i^k(\hat{B}_i+\Delta\varepsilon\bar{B}_i)\,R_i^{-1}\,\hat{B}_i^T\,\hat{K}\hat{\Lambda}_i\right)x(t) \tag{13}$$

To estimate the range of the allowable perturbation $\Delta\varepsilon$ which does not affect system stability we give the following theorem.

Theorem 1. Let the closed loop system (S_1), eqn. (13), be stable for $\varepsilon=\varepsilon_0$, i.e. $\Delta\varepsilon=0$. Then it will remain stable as long as

$$\varepsilon(t)=\varepsilon_0+\Delta\varepsilon(t)\,,\ \text{for all}\ t\varepsilon[0,\infty) \tag{14}$$

where

$$\Delta\varepsilon(t)<\Delta\varepsilon_{\max}=\lambda_{\max}^{-1}(S) \tag{15}$$

$$S=T^{-1}\left(\left(\bar{A}-\sum_i^k \bar{B}_i R_i^{-1} \hat{B}_i^T \hat{K}\hat{\Lambda}_i\right)^T P+P\left(\bar{A}-\sum_i^k \bar{B}_i R_i^{-1} \hat{B}_i^T \hat{K}\hat{\Lambda}_i\right)\right) \tag{16}$$

matrices P and T are defined with (10) and (11), respectively, and $\lambda_{\max}(\cdot)$ denotes maximum eigenvalue of $(\cdot)$.

Proof. The proof proceeds by utilizing argument of Lyapunov theory. Consider the positive definite function $V(x)$ for the perturbed system (S_1) as

$$V(x)=x^T(t)\,Px(t) \tag{17}$$

Since P is positive definite matrix it remains to examine $V(x)$. Taking the time derivation of $V(x)$ along the solution of (S_1), it follows

$$\dot{V}(x)=-x^T\left(T-\Delta\varepsilon(t)\left(\left(\bar{A}-\sum_i^k \bar{B}_i R_i^{-1}\hat{B}_i^T \hat{K}\hat{\Lambda}_i\right)^T P+\right.\right.$$
$$\left.\left.+P\left(\bar{A}-\sum_i^k \bar{B}_i R_i^{-1}\hat{B}_i^T \hat{K}\hat{\Lambda}_i\right)\right)\right)x \qquad (18)$$

making the simplification by using the Lyapunov matrix equation (10). Asymptotic stability follows if $\dot{V}(x)$ is negative definite, which follows if

$$T-\Delta\varepsilon(t)\left(\left(\bar{A}-\sum_i^k \bar{B}_i R_i^{-1}\hat{B}^T \hat{K}\hat{\Lambda}_i\right)^T P+P\left(\bar{A}-\sum_i^k \bar{B}_i R_i^{-1}\hat{B}_i^T \hat{K}\hat{\Lambda}_i\right)\right)>0 \qquad (19)$$

To prove condition (15) recall the following lemma.

LEMMA 1. [12]: *If E and F are symmetric matrices and E is positive definite, there exists a nonsingular matrix G such that*

$$G^T(E+F)G=I+H \qquad (20)$$

where matrix H is a diagonal matrix whose elements are eigenvalues of $E^{-1}F$.

Therefore, using the results of Lemma 1, it can be easily concluded that the perturbed system (S_1) will remain asymptotically stable if the following inequality is satisfied

$$1-\Delta\varepsilon(t)\lambda_j\left(T^{-1}\left(\left(\bar{A}-\sum_i^k \bar{B}_i R_i^{-1}\hat{B}^T \hat{K}\hat{\Lambda}_i\right)^T P+\right.\right.$$
$$\left.\left.+P\left(\bar{A}-\sum_i^k \bar{B}_i R_i^{-1}\hat{B}_i^T \hat{K}\hat{\Lambda}_i\right)\right)\right)>0 \qquad (21)$$
$$j=1,2,\ldots,n$$

i.e.,

$$1-\Delta\varepsilon(t)\lambda_j(S)>0, \quad j=1,2,\ldots,n, \quad \text{for all } t\in[0,\infty) \qquad (22)$$

where the matrix S is defined by (16).

Now, under the assumption that $\lambda_{\max}(S)>0$, which is the usual case, it follows that

$$\Delta\varepsilon_{\max}=\lambda_{\max}^{-1}(S) \qquad (23)$$

i.e., the perturbed system (S_1) remains stable if

$$\Delta\varepsilon(t)\in[0,\Delta\varepsilon_{\max}), \quad \text{for all } t\in[0,-\infty) \qquad (24)$$

■

The results of Theorem 1 can be expresed alternatively in term of specific bounds on $\Delta\varepsilon(t)$, so that the results become easier to appreciate. This is done in the following lemma.

LEMMA 2. *If the perturbation parameter $\Delta\varepsilon(t)$ satisfies the condition*

$$|\Delta\varepsilon| \left(\|\bar{A}\|_s + \sum_i^k \|\bar{B}_i R_i^{-1} \hat{B}_i^T \hat{K}\hat{\Lambda}_i\|_s\right) < \frac{\lambda_{\min}(T)}{2\,\lambda_{\max}(P)} \tag{25}$$

$$\textit{for all } t \in [, \infty)$$

where the matrices P and T are defined by (10) *and* (11), *respectively, and* $\|\cdot\|_s$ *denotes spectral norm of* $(\cdot)$; *then the perturbed system remains stable.*

Proof. The proof is similar to the proof of Theorem 1. Choosing $V(x) = x^T(t)\,Px(t)$, where the matrix P is a positive definite solution of (10), as a Lyapunov function or the system (S_1), $\dot{V}(x)$ can be calculated as before to give

$$\dot{V}(x) = -x^T \Big(T - \Delta\varepsilon(t) \Big(\Big(\bar{A} - \sum_i^k \bar{B}_i R_i^{-1} \hat{B}_i^T \hat{K}\hat{\Lambda}_i\Big)^T P + P\Big(\bar{A} - \sum_i^k \bar{B}_i R_i^{-1} \hat{B}^T \hat{K}\hat{\Lambda}_i\Big)\Big)\Big) x \tag{26}$$

Now, notice that

$$x^T \Delta\varepsilon(t)\, P\Big(\bar{A} - \sum_i^k \bar{B}_i R_i^{-1} \hat{B}^T \hat{K}\hat{\Lambda}_i\Big) x < \|x\|_s^2 \, \|P\|_s \, \|\bar{A} + \sum_i^k \bar{B}_i R_i^{-1} \hat{B}_i^T \hat{K}\hat{\Lambda}_i\|_s |\Delta\varepsilon| \tag{27}$$

Therefore, from (26), (27) and condition (25) it follows that

$$x^T \Delta\varepsilon(t)\, P\Big(\bar{A} - \sum_i^k \bar{B}_i R_i^{-1} \hat{B}^T \hat{K}\hat{\Lambda}_i\Big) x < 1/2\lambda_{\min}(T)\, \|x\|_s^2 \tag{28}$$

and $\dot{V}(x)$ becomes

$$\dot{V}(x) < -x^T (T - \lambda_{\min}(T)\, I)\, x \tag{29}$$

It is easy to see that $\dot{V}(x) < 0$, for all $x(t)$, and, hence, the perturbed system remains stable. ∎

The perturbations acting on the coupling parameter $\varepsilon(t)$ are frequently not known accurately, although some estimate of their maximum bounds may be available. The results of Lemma 2 provide analytically verifiable conditions which can be used to evaluate robustness against the perturbations in the coupling parameter $\varepsilon(t)$. Any variations within the sector (25) cannot destabilize the nominal closed loop stable system.

The following corollary is easy to prove but is interesting as it includes the particular but important case when only the zero terms of the corresponding Maclaurin series expansion are included.

COROLLARY 1. *The closed loop system* (S_1) *with decentralized control law*

$$u_i(t) = -R_i^{-1} \hat{B}_i^T \hat{K}\hat{\Lambda}_i\, x(t), \quad i = 1, 2, \ldots, k \tag{30}$$

where

$$\hat{B}_i = B_i(0), \; \hat{K} = K(0), \; \hat{\Lambda}_i = \Lambda_i(0) \tag{31}$$

will remain stable as long as $\varepsilon < \Delta\varepsilon_{\max}$, *where* $\Delta\varepsilon$ *is defined by* (15) *for* $T=T(0)$, *and the matrix* $P(0)$ *is a positive definite solution of the linear equation*

$$\left(\tilde{A}-\sum_{i}^{k}\tilde{B}_i R_i^{-1}\hat{B}^T\hat{K}\hat{\Lambda}_i\right)^T P+P\left(\tilde{A}-\sum_{i}^{k}\tilde{B}_i R_i^{-1}\hat{B}_i^T\hat{K}\hat{\Lambda}_i\right)+ \\ +\sum_{i}^{k}\hat{\Lambda}_i^T\hat{K}\hat{B}_i R_i\hat{B}_i^T\hat{K}\hat{\Lambda}_i+Q=0 \qquad (32)$$

where

$$\tilde{A}=\operatorname{diag}(A_{ii}),\ Q=\operatorname{diag}(Q_{ii}) \text{ and } \tilde{B}_i=\begin{bmatrix}0\\ \cdot\\ \cdot\\ \cdot\\ B_{ii}\\ \cdot\\ \cdot\\ \cdot\\ 0\end{bmatrix},\quad i=1,2,\ldots,k \qquad (33)$$

Notice that the result of Corollary 1 is a generalization of result presented in [13], [14] where an explicit weak coupling condition which insures that the approximate solution for $\varepsilon=0$ of full state feedback, stabilizes the actual system, has been obtained.

So far, the robustness analysis has been restricted to the case when the perturbations in all interconnections are the same, i.e., $\Delta\varepsilon(t)$ has the same value for the whole system. However, in the case of most engineering systems, during the operation of the system, the perturbation coupling parameter may have different values for different interconnections. In what follows we consider the case when the perturbation matrices $\Delta\varepsilon\bar{A}$ and $\Delta\varepsilon\bar{B}_i$, $i=1,2,\ldots,k$, during the operation of the system become

$$A^1=\begin{bmatrix}0 & \Delta\varepsilon_2 A_{12} & \cdots & \Delta_{\varepsilon k}A_{1k}\\ \Delta\varepsilon_1 A_{21} & 0 & \cdots & \Delta_{\varepsilon k}A_{2k}\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ \Delta\varepsilon_k\Delta\varepsilon_1 & \Delta\varepsilon_2 A_{k2} & \cdots & 0\end{bmatrix}\qquad B_1^1=\begin{bmatrix}\Delta\varepsilon_i B_{1i}\\ \cdot\\ \cdot\\ \cdot\\ \Delta\varepsilon_i B_{i-1,i}\\ 0\\ \Delta\varepsilon_i B_{i+1,i}\\ \cdot\\ \cdot\\ \cdot\\ \Delta\varepsilon_i B_{ki}\end{bmatrix} \qquad (34)$$

In this case A^1 and B^1 can be presented as

$$A^1=\bar{A}E,\ B^1=\bar{B}E \qquad (35)$$

where

$$E=\operatorname{diag}(\Delta\varepsilon_i)\,, \quad i=1,2,\ldots,k\,, \quad B^1=[B_i^1\, B_i^1\, \cdots\, B_i^1] \tag{36}$$

and

$$\bar{B}=\begin{bmatrix} 0 & B_{12} & \cdots & B_{1k} \\ B_{21} & 0 & \cdots & B_{2k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ B_{k1} & B_{k2} & \cdots & 0 \end{bmatrix} \tag{37}$$

Now, the following lemma holds.

LEMMA 3. *The closed loop system*

$$\dot{x}(t)=\Big(\tilde{A}+A^1-\sum_i^k(\tilde{B}_i+B_i^l)\,R_i^{-1}\,\hat{B}_i^T\,\hat{K}\hat{\Lambda}_i\Big)\,x(t) \tag{38}$$

where A^1 and B^1 are defined by (35), *will remain stable if the following condition holds*

$$|\Delta\varepsilon_i|_{\max}\Big(||\tilde{A}||_s+\sum_i^k||\bar{B}_i||_s\,||R_i^{-1}\,\hat{B}_i^T\,\hat{K}\hat{\Lambda}_i||_s\Big)<\frac{\lambda_{\min}(T)}{2\,\lambda_{\max}(P)} \tag{39}$$

for all $t\in[0,\infty)$, where matrices T and P are defined with (10) *and* (11), *respectively.*

Proof. This is a straight forward generalization of Lemma 2. ∎

Although the expressions for the bounds on the perturbations in the coupling parameter $\varepsilon(t)$ appear to be complicated, they are, in fact, not very difficult to calculate. Once the decentralized control problem is solved a few further computations are needed to carry out the robustness analysis. Remember that matrices P and T are calculated as a part of a design procedure and the calculation of the suboptimal index performance. In addition, there are straightforward methods for $\lambda_{\max}$ calculations.

4. Conclusions

A computationally efficient method for robustness evaluation in decentralized ε-coupled control systems has been proposed. Bounds on perturbations in the coupling parameter have been established such that stability of the system is assured. The case when the perturbations on the coupling parameter have different values for different interconnections has been also considered. The results presented in this paper are readily applicable to practical situations in which a designer has estimates on the bounds of perturbations.

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Odporna stabilność zdecentralizowanych epsilonowo powiązanych systemów sterowania

W artykule rozważono zagadnienie odporności w zdecentralizowanych, epsilonowo powiązanych systemach sterowania. Podano ograniczenia na zmiany parametru powiązania i wykazano, że można znaleźć obszar, w którym stabilność jest zapewniona. Proponowane podejście nie zależy specjalnie od żadnej szczególnej metody projektowania zdecentralizowanych systemów.

Стойкая устойчивость децентрализованных ε-связанных систем управления

В статье рассматривается вопрос устойчивости в децентрализованных ε-связанных системах управления. Приведены ограничения на изменения параметра связи и показано, что можно найти область, в которой устойчивость обеспечена. Предлагаемый подход особо не зависит от какого-либо частного метода проектирования децентрализованных систем.