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Optimal control of variational inequalities

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This paper is devoted to the study of optimal control problems of parabolic variational inequalities. We emphasize, particularly, the existence of optimality conditions and the numerical aspects. Some numerical examples are presented.

Introduction

Variational inequalities are used for the mathematical formulation of many physical problems in fluid mechanics (filtration problems), heat transfer (solidification, freezing), structural mechanics (elastoplasticity), etc... Such problems have been studied by many authors (J. L. Lions — G. Stampacchia [16], C. Baiocchi [3], D. Kinderlehrer — G. Stampacchia [12], G. Duvaut — J. L. Lions [7], A. Fasano — M. Primicerio [9], ...).

This paper is devoted to optimal control problems for such systems, more precisely, for systems governed by parabolic variational inequalities. The main difficulties appearing in the study of these problems consist in constructing optimality conditions and implementing efficient numerical methods.

In the elliptic case some results have been obtained by V. Barbu [1] using a penalty method and by F. Mignot — J. P. Puel [19] using the conical derivative.

In the parabolic case we consider in this paper, V. Barbu [2] and C. Saguez [23] have obtained optimality conditions by using a penalty method.

More recently, F. Mignot — J. P. Puel [20] and A. Bermudez — C. Saguez [4] have introduced a direct method which consists in transforming the control problem into a linear optimal control problem with nonconvex constraints on the state.

Numerical approximation of such problems have been studied by many authors (J. P. Yvon [24], M. Niezgódka — I. Pawlow [22], P. Neittaanmäki — D. Tiba [21], C. Saguez [23]). M. Larrecq — C. Saguez [14] have considered the optimal control of the secondary cooling system in a continuous casting process.

In this paper, after presentation of some physical problems (mainly the one-phase Stefan problem), we consider, from a general view-point, some optimal control problems for a parabolic variational inequality. We successively study existence of solution, optimality conditions and numerical methods. For more details we refer to C. Saguez [23] and A. Bermudez — C. Saguez [4].

1. Physical problems

1.1. The one-phase Stefan problem

An important class of problems the models of which are given by variational inequalities is constituted by those involving changes of phase of materials (solidification, freezing, ...). As an example let us consider the ice-water problem.

Let Ω be an open bounded subset of $\mathbb{R}^N$ $(N \leqslant 3)$. We assume that Ω is given as in figure 1.1, i.e. $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$.

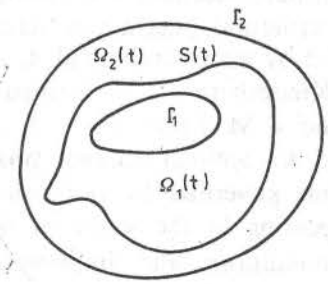

Fig. 1.1.

The domain Ω is divided into two time dependent subdomains Ω_1 and Ω_2 corresponding to water and ice, respectively.

The temperature, which is one of the unknowns of the system, is supposed to be constant and equal to zero in ice. In water it is a solution of the system:

$$\frac{\partial \theta_1}{\partial t} - \Delta\theta_1 = 0 \quad \text{in} \quad \bigcup_{s\in[0,T]} (\Omega_1(s)\times\{s\}) \tag{1.1}$$

$$\theta_{1|\Sigma_1} = v \text{ on } \Sigma_1 = \Gamma_1\times(0,T) \tag{1.2}$$

$$\theta_{1|\Sigma_2} = 0 \text{ on } \Sigma_2 = \Gamma_2\times(0,T) \tag{1.3}$$

$$\theta_{1|S(t)} = 0 \tag{1.4}$$

$$\left.\frac{\partial\theta_1}{\partial n}\right|_{S(t)} = -L\vec{v}\cdot\vec{n} \tag{1.5}$$

$$\theta_1(x,0) = \theta_0(x) \tag{1.6}$$

where L is the latent heat and $\vec{v}\cdot\vec{n}$ is the normal velocity of the free boundary $S(t)$.

This formulation corresponds to the so-called one-phase Stefan problem.

To transform this problem into a variational inequality we introduce a new unknown function $y(x,t)$ given by (G. Duvaut [5]):

$$y(x,t) = \int_0^t \theta(x,\tau)\,d\tau \tag{1.7}$$

with $\theta=\theta_1$ in Ω_1 and $\theta=\theta_2\equiv 0$ in Ω_2.

Then y is a solution of the following parabolic variational inequality

$$\begin{cases} \left(\dfrac{\partial y}{\partial t}, \varphi - y\right) + a(y, \varphi - y) \geqslant (f, \varphi - y) \text{ for all } \varphi\in K(t) \\ y(\cdot, t)\in K(t);\ y(x,0)=0 \end{cases} \tag{1.8}$$

$$y\in L^2(0,T;V),\ \frac{\partial y}{\partial t}\in L^2(0,T;L^2(\Omega)) \tag{1.9}$$

where

$$V = \{\varphi : \varphi\in H^1(\Omega),\ \varphi|_{\Gamma_2} = 0\} \tag{1.10}$$

$$K(t) = \{\varphi\in V : \varphi|_{\Gamma_1} = w,\ \varphi\geqslant 0 \text{ a.e. in } \Omega\} \tag{1.11}$$

$$a(y,\varphi) = \int_\Omega \nabla y\cdot\nabla\varphi\,dx \tag{1.12}$$

$$w(x,t) = \int_0^t v(x,\tau)\,d\tau \tag{1.13}$$

$$f(x,t) = \theta_0(x) - L(1-\chi_{\Omega_1(0)}). \tag{1.14}$$

REMARK 1.1. Similar methods can be used for the two-phase Stefan problems. We obtain a variational inequality of the following type (G. Duvaut [6]):

$$\begin{cases} \left(\frac{\partial y}{\partial t}, \varphi-\frac{\partial y}{\partial t}\right)+a\left(y, \varphi-\frac{\partial y}{\partial t}\right)+\psi(\varphi)-\psi\left(\frac{\partial y}{\partial t}\right) \geqslant\left(f, \varphi-\frac{\partial y}{\partial t}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } \varphi \in K \\ y(0)=0 \\ \qquad y(t) \in H^1(\Omega), \frac{\partial y}{\partial t}(t) \in K=\left\{\varphi \in H^1(\Omega): \varphi|_{\Gamma_1}=v\right\} \end{cases} \tag{1.15}$$

where ψ is a non-differentiable convex function.

1.2. An industrial example: the continuous casting

The expansion of continuous casting processes has been very important in the last ten years, principally for two reasons: an economic motivation, (the production cost is lower than for classical methods) and a metallurgical motivation connected to the quality of the steel. Due to these facts it is very interesting to develop methods to optimize the productivity of such a system.

The principle of the continuous casting process is to cast the steel in a mould whose bottom is constituted by the solidified ingot which is continuously extracted. A scheme is given in Figure 1.2.

The steel is continuously casted in a copper mould through a nozzle. At the end of this mould a very thin crust is solidified which is sufficient to avoid break — out.

In a second part the steel is cooled by a spray system divided into several independent zones. At the end of this part the steel is cut by a cutting torch.

We distinguish several types of products depending on their dimensions: slabs, blooms, billets.

To modelize this system we have the following equations:

Let T_1 (resp. T_2) be the temperature of the steel in the liquid (resp. solid) phase. Then T_1 satisfies the heat equation

$$\varrho V c_1 \frac{\partial T_1}{\partial t}-k_1 \Delta T_1=0 \tag{1.16}$$

where ϱ is the density, V the velocity of extraction, c_1 the heat capacity of the liquid and k_1 the thermal conductivity of the liquid.

Similarly, in the solid phase we have the equation

$$\varrho V c_2 \frac{\partial T_2}{\partial t}-k_2 \Delta T_2=0 \tag{1.17}$$

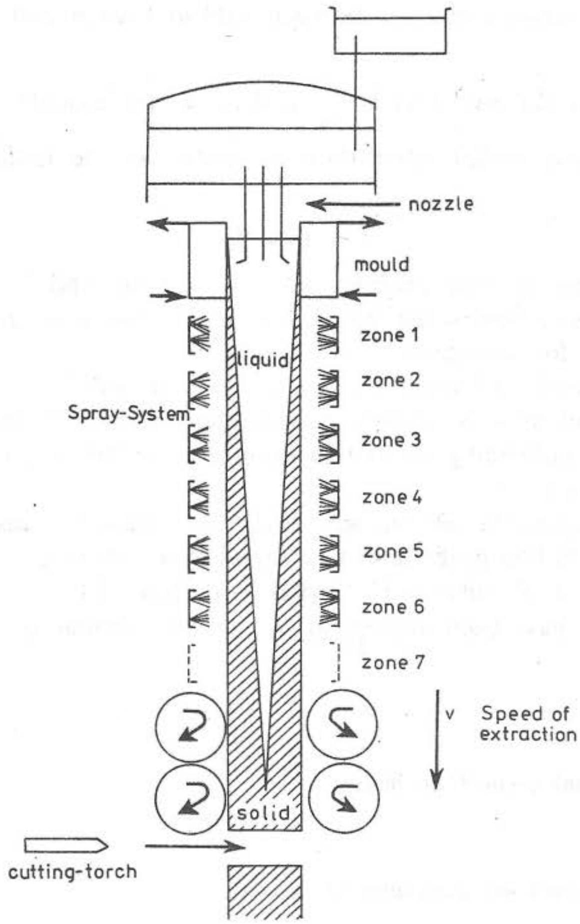

Fig. 1.2. Continuous casting process

where c_2 is the heat capacity of the solid steel and k_2 its thermal conductivity.

Along the front of solidification $S(t)$ (the free boundary) we have the following two conditions

$$T_{1|S(t)} = T_{2|S(t)} = T_S \text{ (temperature of solidification)}, \tag{1.18}$$

$$k_1 \vec{\nabla} T_1 \cdot \vec{n} - k_2 \vec{\nabla} T_2 \cdot \vec{n} = -L \vec{v} \cdot \vec{n}. \tag{1.19}$$

Finally we give boundary and initial conditions. To simplify, we consider the unique boundary condition

$$k \frac{\partial T}{\partial n} + h(T - T_w) = 0 \tag{1.20}$$

with h an exchange coefficient between steel and water and T_w the temperature of water.

In practice, the heat flux $k \frac{\partial T}{\partial n}$ is given in the mould and for the spray system we have similar conditions as above. As the initial condition we take

$$T(x, 0) = T_0(x). \tag{1.21}$$

The optimal control problem we study is to find h maximizing the velocity of extraction while respecting some structural and metallurgical constraints as for instance:

i) the quantity of water we can use is bounded

ii) the steel must be completely solidified before the cutting torch

iii) at the unbending point temperature must fulfill a condition of the type $T \notin (\theta_1, \theta_2)$

iv) the gradient of temperature along the boundary must be bounded.

More details on this problem can be found in M. Larrecq — C. Saguez [14], M. Larrecq — J. P. Birat — C. Saguez — J. Henry [13].

The results have been implement for several continuous casting processes in France.

2. Some optimal control problems

2.1. Distributed control with observation of the state

We suppose that the state of the system is a solution of the following parabolic variational inequality:

$$\begin{cases} \left(\frac{\partial y}{\partial t}, \varphi - y\right) + a(y, \varphi - y) \geqslant (f+u, \varphi - y) \text{ for all } \varphi \in K \\ y(\cdot, t) \in K;\ y(x, 0) = 0 \end{cases} \tag{2.1}$$

$$y \in L^2(0, T; H_0^1(\Omega)),\ \frac{\partial y}{\partial t} \in L^2(0, T; L^2(\Omega)) \tag{2.2}$$

where

$$K = \{\varphi \in H_0^1(\Omega) : \varphi \geqslant 0 \text{ a.e. in } \Omega\} \tag{2.3}$$

$$a(y, \varphi) = \int_\Omega (\nabla y\, \nabla \varphi + y\varphi)\, dx. \tag{2.4}$$

If f and u belong to $L^2(Q)$ $(Q = \Omega \times (0, T))$ then there exists a unique $y \in H^{2,1}(Q)$ satisfying (2.1).

REMARK 2.1. (C. Saguez [23])

If $f\in L^p(Q)$ and $u\in L^p(Q)$ then $y\in W_p^{2,1}(Q)$ and the estimate

$$\|y\|_{W_p^{2,1}(Q)} \leqslant C\left(\|f\|_{L^p(Q)}+\|u\|_{L^p(Q)}\right) \tag{2.5}$$

holds. ■

We are setting the following optimal control problem. Let u be the control, and y the observation. We introduce the cost function J as follows:

$$J(u)=\frac{1}{2}\|y-z_d\|^2_{L^2(Q)}+\frac{\nu}{2}\|u\|^2_{L^2(Q)} \tag{2.6}$$

(ν being a positive constant and z_d a given function in $L^2(Q)$).

Then the problem is to find $\bar{u}\in L^2(Q)$ such that:

$$J(\bar{u})\leqslant J(u) \text{ for all } u\in L^2(Q). \tag{2.7}$$

REMARK 2.2. In this paper we are considering the case without constraints on the control and the state. Problems with constraints can be studied by similar methods. ■

The existence of an optimal control follows from the result given hereafter.

PROPOSITION 2.1. The solution of the variational inequality (2.1) is weakly continuously dependent on the control variable u from $L^2(Q)$ into $H^{2,1}(Q)$.

Proof. Let $\{u_n\}$ be a sequence of controls such that $\{u_n\}\to u$ in $L^2(Q)$ weakly. We denote by y_n the solution of the variational inequality (2.1) corresponding to $u=u_n$. If we take $\varphi=0$ in (2.1), we get:

$$\left(\frac{\partial y_n}{\partial t}, y_n\right)+a(y_n, y_n)\leqslant (f+u_n, y_n) \tag{2.8}$$

from which it follows that $\|y_n\|_{L^2(0,T;H_0^1(\Omega))}$ is bounded.

On the other hand, there exists ξ_n such that:

$$\begin{cases} \dfrac{\partial y_n}{\partial t}+Ay_n+\xi_n=f+u_n \\ \xi_n\in\partial I_K(y_n), \end{cases} \tag{2.9}$$

and by using a penalty technique it can be shown that $\{\xi_n\}$ is bounded in $L^2(Q)$ and thus:

$$\|y_n\|_{H^{2,1}(Q)}\leqslant C. \tag{2.10}$$

We can now pass to the limit in (2.9) to obtain the result. ■

2.2. Boundary control with observation of the contact set

In this example we suppose that the state is a solution of the following problem:

$$\text{Find } y \in L^2(0, T; V), \frac{\partial y}{\partial t} \in L^2(Q),$$

such that:

$$\begin{cases} \left(\dfrac{\partial y}{\partial t}, \varphi - y\right) + a(y, \varphi - y) \geqslant (f, \varphi - y) \text{ for all } \varphi \in K(t) \\ y(\cdot, t) \in K(t);\ y(x, 0) = 0 \end{cases} \tag{2.11}$$

with V and $K(t)$ given by:

$$V = \{\varphi \in H^1(\Omega) : \varphi|_{\Gamma_2} = 0\} \tag{2.12}$$

$$K(t) = \{\varphi \in V : \varphi|_{\Gamma_1} = u,\ \varphi \geqslant 0 \text{ a.e. in } \Omega\}. \tag{2.13}$$

If f is given in $L^p(Q)$ $(p \geqslant 2)$ and u is given in $\mathscr{W}_p(\Sigma_1) = W_p^{2-\frac{1}{p}, 1-\frac{1}{2p}}(\Sigma_1)$ satisfying $u(x, 0) = 0$ and $u \geqslant 0$ a.e. on Σ, then the variational inequality (2.11) has a unique solution in $W_p^{2,1}(Q)$. Furthermore we have

$$\|y\|_{W_p^{2,1}(Q)} \leqslant C\left(\|f\|_{L^p(Q)} + \|u\|_{\mathscr{W}_p(\Sigma_1)}\right). \tag{2.14}$$

We take $p = 2$ and consider u as the control. Then we have the control space $U = H^{3/2, 3/4}(\Sigma_1)$.

Let $F(u)$ be the contact set (the region occupied by the ice in the example 1.1), defined by:

$$F(u) = \{(x, t) : y_u(x, t) = 0\} \tag{2.15}$$

and J the cost function:

$$J(u) = \frac{1}{2} \|\chi_{F(u)} - \chi_{F_d}\|^2_{L^2(Q)} + \frac{v}{2} \|u\|^2_U \tag{2.16}$$

where v is a positive constant, $\chi_{F(u)}$ denotes the characteristic function of the set $F(u)$ and χ_{F_d} is the characteristic function of a given subset F_d of Q.

We set the following optimal control problem:

$$\begin{cases} \text{Find } \bar{u} \in U \text{ such that:} \\ J(\bar{u}) \leqslant J(u) \text{ for all } u \in U. \end{cases} \tag{2.17}$$

REMARK 2.3. It is also possible to consider cost functions of the type

$$J(u) = \delta\left(F(u), F_d\right) + \frac{v}{2} \|u\|^2_U \tag{2.18}$$

where δ denotes the Hausdorff metric (see C. Saguez [23]). ■

REMARK 2.4. With the same approach one can take into account constraints of the type:

$$U_{ad} = \{u : F_d \subset F(u)\} \tag{2.19}$$

with F_d a given subset of Q. ■

In order to study the problem (2.17) we need the following assumption:

$$M(\{(x,t) : f(x,t) = 0\}) = 0 \tag{2.20}$$

(M denotes the Lebesgue measure).

The next proposition provides a continuity result.

PROPOSITION 2.2. The mapping $u \to \chi_{F(u)}$ is weakly-strongly continuous from U into $L^2(Q)$.

Proof. With the assumption (2.20) and the regularity of y, $\chi_{F(u)}$ is defined by:

$$f\chi_{F(u)} = f - \frac{\partial y}{\partial t} - \Delta y. \tag{2.21}$$

A similar method to that used in the proof of Proposition 1.1 shows that if $\{u_n\} \to u$ in U weakly then $\{y_n\}$ $(y_n = y(u_n))$ converges to $y = y(u)$ in $H^{2,1}(Q)$ weakly.

Let F_n denote the domain $F(u_n)$. We have

$$f\chi_{F_n} = f - \frac{\partial y_n}{\partial t} - \Delta y_n. \tag{2.22}$$

Since $\{\chi_{F_n}\}$ is a bounded sequence in $L^2(Q)$, there exists a subsequence, still denoted $\{\chi_{F_n}\}$, such that:

$$\{\chi_{F_n}\} \to l \text{ in } L^2(Q) \text{ weakly.} \tag{2.23}$$

By passing to the limit in (2.22) we obtain

$$fl = f - \frac{\partial y}{\partial t} - \Delta y \tag{2.24}$$

and then $l = \chi_F$.

The strong convergence can also be proved because χ_F is a characteristic function. ■

REMARK 2.5. In the case of the Hausdorff metric it is possible to prove that

$$\{F_n\} \to \tilde{F} \text{ with } M(\tilde{F} \Delta F) = 0,$$

provided $p > \sup\left(\frac{N+2}{2}, 2\right)$, $p \neq N+2$. ■

From the Proposition 2.2 we can deduce

PROPOSITION 2.3. The optimal control problem (2.17) has at least one solution.

Proof (see C. Saguez [23]). ■

3. Optimality conditions

In this chapter we consider the case of distributed control variables but the results can be extended to many other situations including also boundary controls. We assume $f \equiv 0$ for simplicity.

The problem of obtaining optimality conditions is difficult because the mapping $v \to y(v)$ is not differentiable but only Lipschitz continuous. Several authors studied this problem. In the elliptic case two principal methods have been used. The first one consists on introducing a penalty problem and then passing to the limit (V. Barbu [2]). The second one introduces the conical derivative of the mapping $v \to y(v)$ (F. Mignot — J. P. Puel [19]).

We present now two approaches for the parabolic case.

3.1. Penalty method

For this part, we do the following assumptions:

(H1) $U = L^p(Q) \cap V$, $V \subset L^2(Q)$ with compact embedding and

$$p > \sup\left(\frac{N+2}{2}, 2\right), \quad p \neq N+2.$$

(H2) The optimal control $\bar{u}$ verifies

$$M\left(\{(x,t): \bar{u}(x,t) = 0\}\right) = 0$$

and the associated state $\bar{y}$ is such that, if $\varphi \in H_0^{1,0}(Q)$ with $\varphi = 0$ a.e. in $\bar{F} = \{(x,t): \bar{y}(x,t) = 0\}$, then $\varphi \in H_0^{1,0}(Q - \bar{F})$.

Let us consider the following penalty problem:

— the state is solution of the nonlinear parabolic equation:

$$\begin{cases} \dfrac{\partial y_\varepsilon}{\partial t} - \Delta y_\varepsilon - \dfrac{1}{\varepsilon}\, y_\varepsilon^- = u \\ y_{\varepsilon|\Sigma} = 0 \\ y_\varepsilon(x,0) = 0 \end{cases} \tag{3.1}$$

— the cost function is defined by:

$$J_\varepsilon(u) = \frac{1}{2}\, \|y_\varepsilon - z_d\|^2_{L^2(Q)} + \frac{\nu}{2}\, \|u\|^2_U. \tag{3.2}$$

Then the relevant optimal control problem is:

$$\begin{cases} \text{Find } u_\varepsilon \in U \text{ such that} \\ J_\varepsilon(u_\varepsilon) \leqslant J_\varepsilon(u) \text{ for all } u \in U. \end{cases} \tag{3.3}$$

We prove that this problem has a solution u_ε and that every limit point of the sequence $\{u_\varepsilon\}$, as $\varepsilon \to 0$, is a solution of the optimal control problem (2.7). Then to obtain optimality conditions, we consider those corresponding to (3.3) and next pass to the limit as ε goes to zero.

We give the main steps of the proof.

STEP 1. A regularized penalty problem.

To get differentiability we regularize the problem (3.1)–(3.3). Let y_ε^η be defined by

$$\begin{cases} \dfrac{\partial y_\varepsilon^\eta}{\partial t} - \Delta y_\varepsilon^\eta + \dfrac{1}{\varepsilon}\varphi_\eta(y_\varepsilon^\eta) = u \\ y_{\varepsilon|\Sigma}^\eta = 0 \\ y_\varepsilon^\eta(x,0) = 0 \end{cases} \tag{3.4}$$

where φ_η is a regularization of $\varphi(x) = x^-$ such that

$$\varphi_\eta \in C^1(\mathbf{R}),\ \varphi_\eta(x)\,x \geqslant 0 \tag{3.5}$$

$$\varphi_\eta'(x) = \begin{cases} 1 \text{ if } x \leqslant \eta \\ \in [0,1] \text{ if } \eta \leqslant x \leqslant 0 \\ 0 \text{ if } x \geqslant 0. \end{cases} \tag{3.6}$$

Let J_ε^η denote the functional

$$J_\varepsilon^\eta(u) = \frac{1}{2}\|y_\varepsilon^\eta - z_d\|^2_{L^2(Q)} + \frac{\nu}{2}\|u\|^2_U. \tag{3.7}$$

Then the associated optimal control problem has a solution u_ε^η which satisfies the following optimality conditions:

$$\begin{cases} \dfrac{\partial y_\varepsilon^\eta}{\partial t} - \Delta y_\varepsilon^\eta + \dfrac{1}{\varepsilon}\varphi_n(y_\varepsilon^\eta) = u_\varepsilon^\eta \\ y_{\varepsilon|\Sigma}^\eta = 0 \\ y_\varepsilon^\eta(x,0) = 0, \end{cases} \tag{3.8}$$

$$\begin{cases} -\dfrac{\partial p_\varepsilon^\eta}{\partial t} - \Delta p_\varepsilon^\eta + \dfrac{1}{\varepsilon}\varphi_\eta'(y_\varepsilon^\eta)\,p_\varepsilon^\eta = 2(y_\varepsilon^\eta - z_d) \\ p_{\varepsilon|\Sigma}^\eta = 0 \\ p_\varepsilon^\eta(x,T) = 0, \end{cases} \tag{3.9}$$

$$p_\varepsilon^\eta + \nu u_\varepsilon^\eta = 0. \tag{3.10}$$

STEP 2: Optimality conditions for the penalty problem.

It can be shown that every limit point of the sequence $\{u_\varepsilon^n\}$ is a solution of the problem (3.1)–(3.3).

If u_ε is such a limit point then by passing to the limit in (3.8) the following optimality conditions are obtained:

$$\begin{cases} \dfrac{\partial y_\varepsilon}{\partial t} - \Delta y_\varepsilon - \dfrac{1}{\varepsilon} y_\varepsilon^- = u_\varepsilon \\ y_{\varepsilon|\Sigma} = 0 \\ y_\varepsilon(x, 0) = 0, \end{cases} \tag{3.11}$$

$$\begin{cases} -\dfrac{\partial p_\varepsilon}{\partial t} - \Delta p_\varepsilon + \dfrac{1}{\varepsilon} \chi_\varepsilon p_\varepsilon = 2(y_\varepsilon - z_d) \\ p_{\varepsilon|\Sigma} = 0 \\ p_\varepsilon(x, T) = 0, \end{cases} \tag{3.12}$$

$$p_\varepsilon + \nu u_\varepsilon = 0 \tag{3.13}$$

where $\chi_\varepsilon = \chi_{F_\varepsilon}$ and F_ε is defined by

$$F_\varepsilon = \{(x, t) : y_\varepsilon(x, t) \leqslant 0\}. \tag{3.14}$$

STEP 3: Passing to the limit as $\varepsilon \to 0$.

By using the assumptions (H1) and (H2) we first prove that

$$\chi_{F_\varepsilon} \to \chi_{\bar{F}} \text{ in } L^2(Q) \text{ strongly} \tag{3.15}$$

and

$$F_\varepsilon \to \tilde{F} \text{ in the Hausdorff metric sense,} \tag{3.16}$$

where

$$\bar{F} = \{(x, t) : \bar{y}(x, t) = 0\} \tag{3.17}$$

and $\tilde{F}$ satisfies

$$M(\tilde{F} \Delta \bar{F}) = 0. \tag{3.18}$$

Then it is possible to pass to the limit in (3.11)–(3.13) and to obtain the following result:

PROPOSITION 3.1. If $\bar{u}$ is a limit point of u_ε as $\varepsilon \to 0$ then $\bar{u}$ is a solution of the optimal control problem (2.7) and satisfies the optimality conditions:

$$\begin{cases} \left(\dfrac{\partial \bar{y}}{\partial t}, \varphi - \bar{y}\right) + a(\bar{y}, \varphi - \bar{y}) \geqslant (\bar{u}, \varphi - \bar{y}) \text{ for all } \varphi \in K(t) \\ \bar{y}(x, 0) = 0, \end{cases} \tag{3.19}$$

$$\left\{\begin{array}{l}\displaystyle\int_0^T \left\{\left(\bar{p}, \frac{\partial\varphi}{\partial t}\right) + a(\bar{p}, \varphi)\right\} dt = 2\int_0^T (\bar{y} - z_d, \varphi)\, dt \text{ for all } \varphi \in \mathscr{B}_0(\theta) \\ \bar{p} \in H_0^{1,0}(\theta)\end{array}\right. \tag{3.20}$$

where $\theta = Q - \bar{F}$ and $\mathscr{B}_0(\theta)$ is defined by:

$$\mathscr{B}_0(\theta) = \left\{\varphi \in \mathscr{F}(\theta) \colon \frac{\partial\varphi}{\partial t} \in \mathscr{F}(\theta),\ \varphi(x, 0) = 0\right\},\ \mathscr{F}(\theta) = \overline{\mathscr{D}(\theta)}^{H^{1,0}(\theta)},$$

$$\bar{p} + \nu\bar{u} = 0. \tag{3.21}$$

REMARK 3.1. For any solution $\tilde{u} \in U$ of the optimal control problem (2.17) we introduce the augmented cost function

$$\tilde{J}(u) = J(u) + \|u - \tilde{u}\|_U^2. \tag{3.22}$$

Then is clear that the unique solution of the optimal control problem associated with $\tilde{J}$ is $\tilde{u}$ and therefore we can deduce that every solution of the problem (2.7) satisfies the optimality conditions (3.19)–(3.21), by replacing J by $\tilde{J}$ in the previous proof. ■

3.2. A direct method

The idea is to transform the problem (2.7) into an optimal control problem for a linear parabolic state equation with state constraints.

For that we write the variational inequality (2.1) as follows:

$$\left\{\begin{array}{l}\dfrac{\partial y}{\partial t} + Ay = u - \xi \\ \xi \in \partial I_K(y) \\ y(x, 0) = 0.\end{array}\right. \tag{3.23}$$

We recall that $\xi \in \partial I_K(y)$ is equivalent to:

$$y \geqslant 0,\ \xi \leqslant 0,\ \langle y, \xi \rangle = 0 \tag{3.24}$$

(see for instance D. Kinderlehrer, G. Stampacchia [12]).

Therefore the optimal control problem (2.7) is equivalent to the following one:

— state equation:

$$\left\{\begin{array}{l}\dfrac{\partial y}{\partial t} + Ay = u - \xi \\ y(x, 0) = 0\end{array}\right. \tag{3.25}$$

— cost function:

$$J(u, \xi) = \frac{1}{2} \|y - z_d\|^2_{L^2(Q)} + \frac{\nu}{2} \|u\|^2_{L^2(Q)} \tag{3.26}$$

— feasible set of controls:

$$\{(u, \xi) : u \in L^2(Q), \xi \in L^2(0, T; H^{-1}(\Omega))\} \tag{3.27}$$

— constraints:

$$\xi \leqslant 0, \ y \geqslant 0, \ \langle y, \xi \rangle = 0. \tag{3.28}$$

By using the regularity of the optimal states (in particular, they belong to $C^0(Q)$) and classical results on the existence of Lagrange multipliers (D. Luenberger [12]) one can prove the following result.

PROPOSITION 3.2. If $\bar{u}$ is an optimal control then there exist $\bar{\xi} \in L^2(Q)$, a Borel measure μ and two nonnegative real numbers $\bar{r}, \bar{\lambda}$, such that:

$$\begin{cases} \dfrac{\partial \bar{y}}{\partial t} - \Delta \bar{y} + \bar{\xi} = \bar{u} \\ \bar{y}|_\Sigma = 0 \\ \bar{y}(x, 0) = 0, \end{cases} \tag{3.29}$$

$$\bar{y} \geqslant 0, \ \bar{\xi} \leqslant 0, \ \langle \bar{y}, \bar{\xi} \rangle = 0, \tag{3.30}$$

$$\int_Q \bar{p} \frac{\partial z}{\partial t} - \int_Q \bar{p} \Delta z = \bar{r} \int_Q (\bar{y} - z_d) z - \bar{\lambda} \int_Q \bar{\xi} z + \langle \bar{\mu}, z \rangle \tag{3.31}$$

for all $z \in H^{2,1}(Q)$, with $z|_\Sigma = 0$ and $z(x, 0) = 0$,

$$\bar{\mu} \leqslant 0, \ \langle \bar{\mu}, \bar{y} \rangle = 0, \tag{3.32}$$

$$\bar{r} \nu \bar{u} + \bar{p} = 0, \tag{3.33}$$

$$\int_Q \bar{p} (\xi - \bar{\xi}) + \bar{\lambda} \int_Q \bar{y} (\xi - \bar{\xi}) \leqslant 0 \text{ for all } \xi \in L^2(Q), \ \xi \leqslant 0. \tag{3.34}$$

■

REMARK 3.2. To prove that $\bar{r}$ is different from zero is an open problem. Recently F. Mignot — J. P. Puel [20], using the same transformation have obtained the following optimality conditions:

$$\begin{cases} \dfrac{\partial \bar{y}}{\partial t} - \Delta \bar{y} + \bar{\xi} = \bar{u} \\ \bar{y}|_\Sigma = 0 \\ \bar{y}(x, 0) = 0, \end{cases} \tag{3.35}$$

$$\bar{y} \geqslant 0, \ \bar{\xi} \leqslant 0, \ \langle \bar{y}, \bar{\xi} \rangle = 0, \tag{3.36}$$

$$\text{if } \delta \in \tilde{C}_{-\bar{\xi}} \text{ with } \langle \delta, y \rangle = 0 \text{ then } \langle \delta, \bar{p} \rangle \geqslant 0, \tag{3.37}$$

$$\begin{cases} \text{if } \varphi \in C_{\bar{y}} \cap W_0(0,T) \text{ with } \langle \bar{\xi}, \varphi \rangle = 0 \text{ then} \\ \left\langle \dfrac{\partial \varphi}{\partial t} - \Delta \varphi, \bar{p} \right\rangle - \displaystyle\int_Q (\bar{y} - z_d)\, \varphi \leqslant 0, \end{cases} \tag{3.38}$$

$$\bar{p} + \nu \bar{u} = 0 \tag{3.39}$$

where

$$C_y = \{\varphi \in L^2(0, T; H_0^1(\Omega)) : \exists t > 0, y + t\varphi \in K\} \tag{3.40}$$

$$\tilde{C}_\zeta = \{\delta \in L^2(0, T; H^{-1}(\Omega)) : \exists t > 0, \zeta + t\delta \geqslant 0\} \tag{3.41}$$

$$W_0(0, T) = \left\{ \varphi \in L^2(0, T; H_0^1(\Omega)) : \frac{\partial \varphi}{\partial t} \in L^2(0, T; H^{-1}(\Omega)), \varphi(0) = 0 \right\} \tag{3.42}$$

If $\bar{r}$ is different from zero it can be proved that (3.29)–(3.34) imply (3.35)–(3.39). ■

4. Numerical methods

In this chapter we give some numerical algorithms to solve optimal control problems for variational inequalities. More precisely we consider two methods, a penalty method and a Lagrangian method, corresponding to the two approaches used in the previous paragraph.

4.1. Penalty method

We transform the problem into a nonlinear optimal control problem by using a penalty technique. The variational inequality (2.1) is approximated by the non linear parabolic partial differential equation

$$\begin{cases} \dfrac{\partial y_\varepsilon}{\partial t} - \Delta y_\varepsilon + \dfrac{1}{\varepsilon} \varphi_\varepsilon(y_\varepsilon) = u + f \\ y_{\varepsilon|\Sigma} = 0 \\ y_\varepsilon(x, 0) = 0 \end{cases} \tag{4.1}$$

with φ_ε a C^1 regularization of the function $x \to -x^-$ satisfying:

$$\varphi_\varepsilon(0) = 0, \varphi_\varepsilon \text{ monotone}. \tag{4.2}$$

Under these assumptions the problem (4.1) has a unique solution and the mapping assigning the state to the control is continuously differentiable.

Therefore, replacing (2.1) by (4.1) we get an approximate optimal control problem which is differentiable. Optimality conditions for it are the following:

$$\begin{cases} \dfrac{\partial y_\varepsilon}{\partial t} - \Delta y_\varepsilon + \dfrac{1}{\varepsilon}\, \varphi_\varepsilon(y_\varepsilon) = f + u_\varepsilon \\ y_{\varepsilon|\Sigma} = 0 \\ y_\varepsilon(x, 0) = 0, \end{cases} \tag{4.3}$$

$$\begin{cases} \dfrac{\partial p_\varepsilon}{\partial t} - \Delta p_\varepsilon + \dfrac{1}{\varepsilon}\, \varphi'_\varepsilon(y_\varepsilon)\, p_\varepsilon = y_\varepsilon - z_d \\ p_{\varepsilon|\Sigma} = 0 \\ p_\varepsilon(x, T) = 0, \end{cases} \tag{4.4}$$

$$p_\varepsilon + \nu u_\varepsilon = 0. \tag{4.5}$$

From a numerical viewpoint it is very important to notice that the gradient of the cost function is given by

$$J'(u_\varepsilon) = p_\varepsilon + \nu u_\varepsilon. \tag{4.6}$$

This fact allows us to use descent methods to compute an optimal control. As an example we give the scheme implementing classical gradient algorithm but many other variants could be considered.

i) Initialization u_0, $n = 0$
ii) Compute the state $y_n = y(u_n)$ from (4.3)
iii) Compute the corresponding adjoint state p_n from (4.4)
iv) Compute the gradient of J by (4.6)
v) $u_{n+1} = u_n - \varrho_n J'(u_n)$
(ϱ_n can be chosen by using, for example, a Wolfe-like method)
vi) Test of convergence Yes $\to$ stop

No $\to n = n+1$, go to ii)

REMARK 4.1. We do not study techniques of discretization (for example by finite element methods) for which we refer to R. Glowinski, J. L. Lions, R. Trémolières [11], C. Elliott [8], P. Neittaanmäki, D. Tiba [21], R. Glowinski [10].

On the other hand, for numerical implementation we consider the adjoint state for the discretized optimal control problem rather than the one obtained from a discrete version of (4.4). ■

4.2. Lagrangian methods

We first notice that if $\bar{r}$ is different from zero then (3.29)–(3.34) are necessary conditions for $(\bar{u}, \bar{\xi}, \bar{\mu}, \bar{\lambda})$ to be a saddle point of the Lagrangian function:

$$L(u, \xi, \mu, \lambda) = \frac{1}{2} \|y(u, \xi) - z_d\|^2_{L^2(Q)} + \frac{\nu}{2} \|u\|^2_{L^2(Q)} + \langle \mu - \lambda\xi, y(u, \xi)\rangle \tag{4.7}$$

in the set

$$(L^2(Q) \times C) \times (D \times \mathbf{R}^+) \tag{4.8}$$

where C and D are given by:

$$C = \{\xi \in L^2(Q) : \xi \leqslant 0\} \tag{4.9}$$

$$D = \{\mu \in C^0(\bar{Q})' : \mu \leqslant 0\}. \tag{4.10}$$

This fact suggests to use Lagrangian algorithms to approximate an optimal control. Hereafter we recall the Uzawa's algorithm:

i) $\mu_0 \in C^0(\bar{Q})'$, $\lambda_0 \geqslant 0$, $n = 0$

ii) For given μ^n and λ^n, compute (u_n, ξ_n) solution of the optimal control problem defined by:

— state equation:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + \xi = f + u \\ y|_\Sigma = 0 \\ y(x, 0) = 0 \end{cases} \tag{4.11}$$

— cost function:

$$G(u, \xi) = \frac{1}{2} \|y(u, \xi) - z_d\|^2_{L^2(Q)} + \frac{\nu}{2} \|u\|^2_{L^2(Q)} + \langle \mu_n - \lambda_n \xi, y(u, \xi)\rangle \tag{4.12}$$

— feasible control set:

$$u \in L^2(Q), \xi \in C \tag{4.13}$$

(to solve this problem we can use, for instance, a classical gradient method, with projection on the constraint set C)

iii)

$$\mu_{n+1} = (\mu_n + \alpha_n y_n)^-, \alpha_n \in \mathbf{R}^+ \tag{4.14}$$

iv)

$$\lambda_{n+1} = (\lambda_n - \beta_n \langle \xi_n, y_n\rangle)^+ \tag{4.15}$$

v) test of convergence

Yes → stop
No → $n = n+1$, go to ii)

REMARK 4.2. Many other methods may be used to solve the constrained optimization problem (3.25)–(3.28) as for example augmented Lagrangian methods. ■

4.3. Examples

i) Example 1

Let us consider the following two-dimensional problem:

— The domain Ω is given as in figure 4.1, with $R_1 = 0.37$ and $R_2 = 0.77$

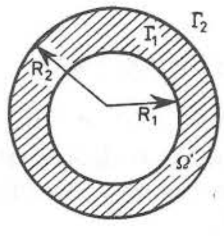

Fig. 4.1.

— The state y is the solution of the variational inequality (2.1) with $f \equiv -40$

— The cost function is defined by:

$$J(u) = \int_Q (y - z_d)^2 \, dx \, dt$$

with z_d being the solution of the variational inequality (2.1) corresponding to $f \equiv -40$ and $u(x, t) = 8\sqrt{t}\,\dfrac{\sqrt{x^4 + (R_1 - x)^4}}{R_1^2}$.

This problem is approximated by using triangular finite elements of degree 1. The discretized domain is given in figure 4.2.

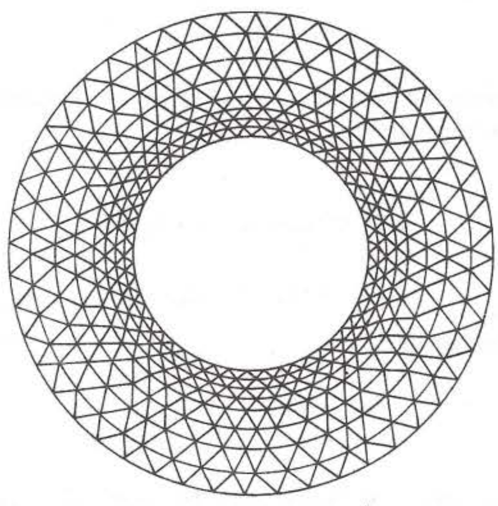

Fig. 4.2. 728 triangles, 482 nodes

The optimization problem is solved by a gradient method. Convergence is obtained after 10 iterations as is shown in table 4.1. The computed control after 10 iterations is given in figure 4.3.

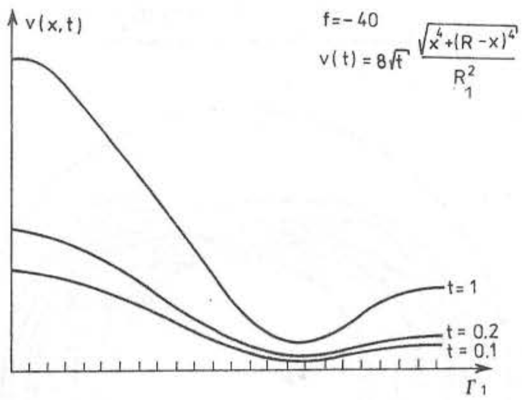

Fig. 4.3. Computed control along I_1

ii) Example 2

The state equation is similar to that of example 1 with $f \equiv -80$. The cost function is given by:

$$J(u) = \int_Q (\chi_{F(u)} - \chi_{F_d})^2 \, dx \, dt$$

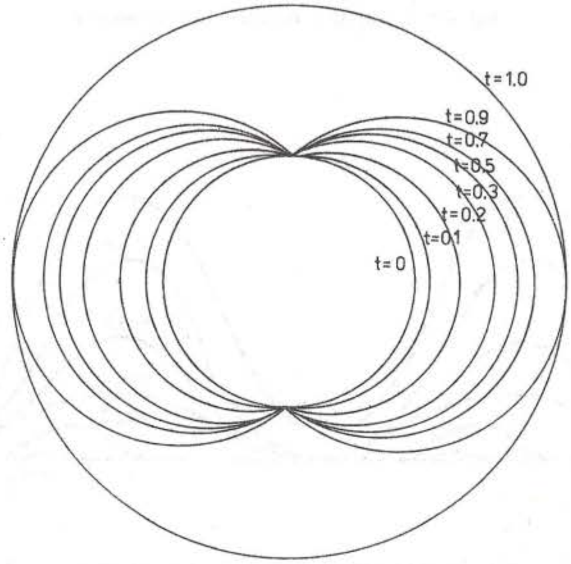

Fig. 4.4. Desired free boundary $\varrho_d = R_1 + t(R_2 - R_1)|\cos\theta|$

Table 4.1

Iterations	Cost function
1	$0.245\ 10^{2}$
2	0.935
4	$0.1\ 10^{-1}$
7	$0.16\ 10^{-2}$
10	$0.73\ 10^{-3}$

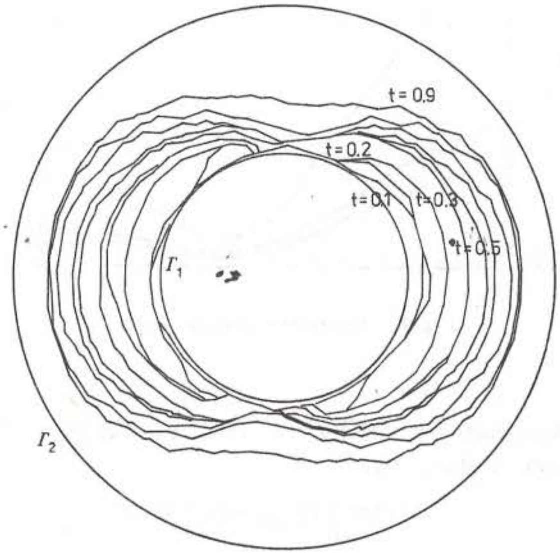

Fig. 4.5. Computed optimal free boundary

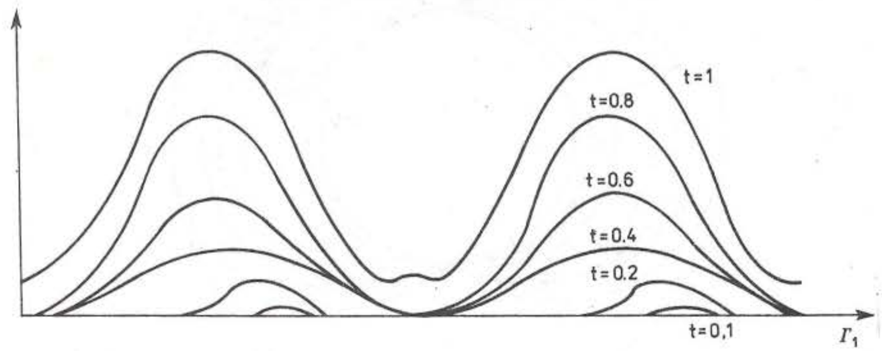

Fig. 4.6. Computed optimal control along Γ_1

with

$$F_d = \{(\varrho, \theta) : \varrho \leqslant R_1 + t\,(R_2 - R_1)\,|\cos\theta|\}.$$

In fact J is regularized as follows:

$$J_\eta(u) = \int_Q \left(\frac{\eta}{\eta + y} - \chi_{F_d}\right)^2 dx\,dt, \text{ with } \eta > 0.$$

Convergence of a gradient method is obtained after 8 iterations. Results are presented in figures 4.4, 4.5 and 4.6.

5. Conclusions

In this paper we have offered a general insight into optimal control problems for systems governed by parabolic variational inequalities. The first experience we have got shows that it is possible to adapt the classical approach to solve this kind of problems, in particular, from a numerical viewpoint.

Nevertheless, many theoretical and numerical problems are still open. This is the case, in particular, when the convex K given in (2.3) is replaced by

$$K = \{\varphi \in H_0^1(\Omega) : |\nabla\varphi|_2 \leqslant 1 \text{ a.e. in } \Omega\},$$

for instance. Such problems are of a great industrial interest in the design of elastic-plastic structures.

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Sterowanie optymalne nierówności wariacyjnych

Artykuł jest poświęcony analizie zadań sterowania optymalnego stawianych dla parabolicznych nierówności wariacyjnych. Szczególny nacisk kładzie się na możliwość sformułowania warunków optymalności oraz aspekty numeryczne zagadnień. Przedstawione są pewne przykłady numeryczne.

Оптимальное управление вариационными неравенствами

В работе рассматривается проблемы оптимального управления параболическими вариационными неравенствами. Особенное внимание уделяется возможности формулировки условий оптимальности, а также численным аспектам проблем. Представлены некоторые численные примеры.