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On the identification of heat conductivity and latent heat in a one-phase Stefan problem

by

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A method for the identification of the heat conductivity matrix and the latent heat in a multidimensional one-phase Stefan problem from measurements of the temperature distribution is proposed.

1. Introduction

This paper is concerned with the identification of parameters in variational inequalities which arise as variational formulation of one-phase Stefan problems. Following [3], we consider the following situation:

Let $G \subset \mathbf{R}^m$ denote a bounded domain whose boundary consists of two connected $C^{2+\alpha}$ — hypersurfaces Γ_0 and Γ_1, with Γ_0 lying inside Γ_1 and bounding a simply connected domain G_0. Let B_R be a sufficiently large ball with center 0 containing G; define $\Omega := B_R \setminus G_0$, $Q_T := \Omega \times (0, T)$. We consider the problem:

Find functions $t = s(x)$ and $\theta(x, t)$, $0 < t < T$, $x \in \Omega$, which satisfy

$$s(x) = 0, \text{ if } x \in \Omega, \tag{1.1}$$

$$\theta_t - \nabla \cdot (k^*(x) \nabla \theta) = 0, \text{ in } \{(x, t): x \in \Omega, t > s(x)\}, \tag{1.2}$$

$$\theta = 0 \text{ if } t = s(x), x \in \Omega \setminus G, \tag{1.3}$$

$$\nabla s(x) \cdot k^*(x) \nabla \theta(x, s(x)) = -L^*(x), x \in \Omega \setminus G, \tag{1.4}$$

$$\theta(x, 0) = h^*(x), \text{ on } G \times \{0\}, \tag{1.5}$$

$$\theta(x, t) = g^*(x, t), \text{ on } \Gamma_0 \times [0, T]. \tag{1.6}$$

Here $g^*(x,t)$ and $h^*(x)$ are positive functions in $C^{2+\alpha}(\Gamma_0\times[0,T])$ and $C^\alpha(\bar{G})$, respectively. For the latent heat we assume $L^*\in L^\infty(\Omega\setminus G)$ and $L^*(x)\geqq L_0>0$, a.e. on $\Omega\setminus G$, while the heat conductivity matrix $k^*=(k^*_{ij})$ has entries $k^*_{ij}\in L^\infty(\Omega)$ and satisfies $\zeta\cdot k^*(x)\,\zeta\geqq\alpha_1|\zeta|^2$, $\forall\zeta\in\mathbf{R}^m$, a.e. on Ω, with some $\alpha_1>0$.

Introduction of the freezing index u^* leads (see [3]) to the variational inequality:

Find $u^*\in K$ which solves

$$\int_\Omega u_t^*(v-u^*)\,dx+\int_\Omega\nabla(v-u^*)\cdot k^*(x)\nabla u^*\,dx-\int_\Omega f^*(v-u^*)\,dx+$$
$$+\int_{\Omega\setminus G}L^*(x)(v-u^*)\,dx\geqq 0,\ \text{a.e. on } (0,T), \forall v\in K. \tag{1.7}$$

Here we have set:

$$f^*(x):=\begin{cases}h^*(x), & x\in G\\ 0, & x\in\Omega\setminus G,\end{cases} \tag{1.8}$$

$$K:=\{v\in H^1(Q_T): v\geqq 0 \text{ a.e. in } Q_T,\ v=\Psi \text{ on } (\partial\Omega\times(0,T))\cup(\bar{\Omega}\times\{0\})\}, \tag{1.9}$$

where

$$\Psi(x,t):=\begin{cases}\int_0^t g^*(x,\tau)\,d\tau, & \text{if } x\in\Gamma_0,\ t>0,\\ 0, & \text{if } t=0 \text{ or } |x|=R.\end{cases} \tag{1.10}$$

The purpose of this paper is to give an approximate solution for the following inverse problem:

(IP) Given $u^*\in K$, $g^*>0$ in $C^{2+\alpha}(\Gamma_0\times[0,T])$ and $h^*>0$ in $C^\alpha(\bar{G})$, find a matrix function $k^*\in K_1^*$ and a function $L^*\in K_2^*$ such that (1.7) holds.

For the sets of physically admissible parameters we assume:

$$K_1^*:=\{k=(k_{ij})\in L^\infty_{\text{sym}}(\Omega): \|k\|_{L^\infty(\Omega)}\leqq\beta_1,\ \zeta\cdot k(x)\,\zeta\geqq\alpha_1|\zeta|^2, \forall\zeta\in\mathbf{R}^m, \text{ a.e. in } \Omega\}, \tag{1.11}$$

$$K_2^*:=\{L\in L^\infty(\Omega\setminus G): \alpha_2\leqq L(x)\leqq\beta_2, \text{ a.e. in } \Omega\}. \tag{1.12}$$

Here $\beta_i\geqq\alpha_i>0$, $i=1,2$, are chosen constants (prescribed by the experimental circumstances) and $L^\infty_{\text{sym}}(\Omega)$ denotes the set of symmetric matrix functions with entries in $L^\infty(\Omega)$. Moreover the norms in the spaces of $L^2(\Omega)$ — or $L^\infty(\Omega)$ — scalar, vector or matrix functions are all denoted by $\|\cdot\|_{L^2(\Omega)}$ or $\|\cdot\|_{L^\infty(\Omega)}$, respectively. For further convenience let us agree to make use of the Einstein summation convention, if appropriate.

We may point out at this place that the results obtained here can be applied to other problems having the same structure as well. For a rather general class of identification problems in variational inequalities we refer to [4]. Note, however, that (IP) cannot be subsumed under the problems considered there.

It is not possible to show the existence of a solution to (IP) by our method. Instead, our method relies strongly on the assumption that the following solvability condition (which is assumed to hold throughout) is satisfied:

(A1) (IP) has at least one solution.

From the practical point of view (A1) appears to be reasonable.

2. Asymptotic regularization

Instead of (IP) we solve its finite dimensional approximation. To this end, let $V \subset H^1(Q_T)$, $W_1 \subset L^\infty_{\text{sym}}(\Omega)$, $W_2 \subset L^\infty(\Omega \backslash G)$ denote finite dimensional subspaces which satisfy the following compatibility conditions:

(A2) $P_i(K_i^*) \subset K_i^*$, where P_i is the $L^2(\Omega)$ — orthogonal projection operator onto W_i, $i = 1, 2$.

(A3) $u^* \in V$, $u \in V \Rightarrow u(0) = 0$

(A4) $\int_0^T \nabla u(t) \otimes \nabla (u(t) - v(t))\, dt \in W_1, \ \forall u, v \in K \cap V.$

(A5) $\int_0^T (u(t) - v(t))\, dt_{|\Omega \backslash G} \in W_2, \forall u, v = K \cap V.$

Here we have used the following notation: For $u = (u_i)$, $v = (v_i) \in \mathbf{R}^m$ the matrix $u \otimes v \in \mathbf{R}^{m,m}$ is defined by: $(u \otimes v)_{ij} := \frac{1}{2}(u_i v_j + u_j v_i)$. Moreover, in (A5) the restrictions onto $\Omega \backslash G$ are meant.

It is clear that (A4) can be satisfied if $\int_0^T u(t)\, dt \in H^{1,\infty}(\Omega)$ for every $u \in V$ which then has to hold for $u = u^*$, in particular. In a numerical calculation one would approximate Ω by a suitably triangulated domain Ω_h with piecewise linear boundary and chose piecewise linear (for V) and piecewise constant (for W_1) finite elements.

Moreover one may chose $W_2 = \text{span}\,\{\int_0^T v_1(t)\, dt_{|\Omega \backslash G}, \ldots, \int_0^T v_N(t)\, dt_{|\Omega \backslash G}\}$ if $V = \text{span}\,\{v_1, \ldots, v_N\}$. Note that also (A2) holds with these choices for the subspaces.

From (A3) there follows $u^* \in K \cap V$, hence $K \cap V \neq \emptyset$. We finally assume

(A6) $K_1^* \cap W_1 \neq \emptyset$, $K_2^* \cap W_2 \neq \emptyset$.

The finite dimensional approximation to (IP) then reads:

(IP,V) Given $u^* \in K$, $g^* > 0$ in $C^{2+\alpha}(\Gamma_0 \times [0, T])$ and $h^* > 0$ in $C^\alpha(\bar{G})$, find $k^* \in K_1^*$ and $L^* \in K_2^*$ s.t.

$$\int_0^T \int_\Omega (u_t^* - f^*)(v - u^*)\, dx\, dt + \int_0^T \int_\Omega \nabla (v - u^*) \cdot k^*(x) \nabla u^*\, dx\, dt + \int_0^T \int_{\Omega \setminus G} L^*(x)(v - u^*)\, dx\, dt \geqq 0, \forall v \in K \cap V. \tag{2.1}$$

By (A1) the solution set $\mathbf{L}(V)$ of (IP, V) is nonempty. Moreover, $\mathbf{L}(V)$ is convex.

We now construct a sequence of systems of variational inequalities. We consider the problem

(P) Given $(u_0, k_0, L_0) \in (K \cap V) \times (K_1^* \cap W_1) \times (K_2^* \cap W_2)$. find $(u_{n+1}, k_{n+1}, L_{n+1}) \in (K \cap V) \times (K_1^* \cap W_1) \times (K_2^* \cap W_2)$, $n \geqq 0$, such that

$$\int_0^T \int_\Omega \left(\frac{u_{n+1} - u_n}{\sigma} + \frac{d}{dt} u_{n+1} - f^* \right)(v - u_{n+1})\, dx\, dt + \int_0^T \int_\Omega \nabla (v - u_{n+1}) \cdot k_{n+1}(x) \nabla u_{n+1}\, dx\, dt + \int_0^T \int_{\Omega \setminus G} L_{n+1}(x)(v - u_{n+1})\, dx\, dt \geqq 0, \forall v \in K \cap V, \tag{2.2}$$

$$\int_\Omega \left(\frac{k_{n+1} - k_n}{\varepsilon} - \int_0^T \nabla u_{n+1} \otimes \nabla (u_{n+1} - u^*)\, dt \right)_{ij} (\eta - k_{n+1})_{ij}\, dx \geqq 0, \quad \forall \eta \in K_1^* \cap W_1, \tag{2.3}$$

$$\int_{\Omega \setminus G} \left(\frac{L_{n+1} - L_n}{\lambda} - \int_0^T (u_{n+1} - u^*)\, dt \right)(\zeta - L_{n+1})\, dx \geqq 0, \quad \forall \zeta \in K_2^* \cap W_2. \tag{2.4}$$

Here $\sigma > 0$, $\varepsilon > 0$, $\lambda > 0$ are scaling factors. Note that $u_{n+1} \in K \cap V$ implies the initial condition $u_{n+1}(0) = 0$.

Let us for the moment assume that (P) has a solution $\{(u_n, k_n, L_n)\}$, and let $(k^*, L^*) \in \mathbf{L}(V)$ be arbitrary. We put $w_n := u_n - u^*$, $r_n := k_n - k^*$, $s_n := L_n - L^*$.

Substitution of $v=u_{n+1}$ into (2.1) and of $v=u^*$ into (2.2) yields

$$\int_0^T\int_\Omega (u_t^*-f^*)\,w_{n+1}\,dx\,dt+\int_0^T\int_\Omega \nabla w_{n+1}\cdot k^*(x)\,\nabla u^*\,dx\,dt+$$

$$+\int_0^T\int_{\Omega\setminus G} L^*(x)\,w_{n+1}\,dx\,dt\geqq 0\geqq \int_0^T\int_{\Omega\setminus G} L_{n+1}(x)\,w_{n+1}\,dx\,dt+$$

$$+\int_0^T\int_\Omega\left(\frac{u_{n+1}-u_n}{\sigma}+\frac{d}{dt}\,u_{n+1}-f^*\right)w_{n+1}\,dx\,dt+$$

$$+\int_0^T\int_\Omega \nabla w_{n+1}\cdot k_{n+1}(x)\,\nabla u_{n+1}\,dx\,dt,$$

whence

$$0\geqq \int_0^T\int_\Omega w_{n+1}\cdot\frac{d}{dt}\,w_{n+1}\,dx\,dt+\int_0^T\int_\Omega \frac{w_{n+1}-w_n}{\sigma}\,w_{n+1}\,dx\,dt+$$

$$+\int_0^T\int_\Omega \nabla w_{n+1}\cdot k^*(x)\,\nabla w_{n+1}\,dx\,dt+\int_0^T\int_\Omega \nabla w_{n+1}\cdot r_{n+1}\,\nabla u_{n+1}\,dx\,dt+$$

$$+\int_0^T\int_{\Omega\setminus G} s_{n+1}\,w_{n+1}\,dx\,dt\geqq\frac{1}{2}\,\|w_{n+1}(T)\|^2_{L^2(\Omega)}+$$

$$+\int_0^T\int_\Omega \frac{w_{n+1}-w_n}{\sigma}\,w_{n+1}\,dx\,dt+\alpha_1\int_0^T \|\nabla w_{n+1}\|^2_{L^2(\Omega)}\,dt+$$

$$+\int_0^T\int_\Omega \nabla w_{n+1}\cdot r_{n+1}\,\nabla u_{n+1}\,dx\,dt+\int_0^T\int_{\Omega\setminus G} s_{n+1}\,w_{n+1}\,dx\,dt. \qquad (2.5)$$

Next we substitute $\eta:=P_1\,k^*$ into (2.3) (note that $P_1\,k^*\in K_1^*\cap W_1$ by (A2)). Since $k^*-P_1\,k^*\in W_1^\perp$ we obtain from (A4):

$$0\geqq\int_\Omega\left(\frac{k_{n+1}-k_n}{\varepsilon}-\int_0^T \nabla u_{n+1}\otimes\nabla w_{n+1}\,dt\right)_{ij}(k_{n+1}-P_1\,k^*)_{ij}\,dx=$$

$$= \int_\Omega \left(\frac{r_{n+1}-r_n}{\varepsilon} - \int_0^T \nabla u_{n+1} \otimes \nabla w_{n+1}\, dt \right)_{ij} (r_{n+1})_{ij}\, dx =$$

$$= \int_\Omega \left(\frac{r_{n+1}-r_n}{\varepsilon} \right)_{ij} (r_{n+1})_{ij}\, dx - \int_0^T \int_\Omega \nabla w_{n+1} \cdot r_{n+1} \nabla u_{n+1}\, dx\, dt. \qquad (2.6)$$

Finally, substitute $\zeta := P_2 L^*$ into (2.4), use $L^* - P_2 L^* \in W_2^\perp$ and (A5) in order to obtain:

$$0 \geqq \int_{\Omega\setminus G} \left(\frac{L_{n+1}-L_n}{\lambda} - \int_0^T w_{n+1}\, dt \right) (L_{n+1} - P_2 L^*)\, dx =$$

$$= \int_{\Omega\setminus G} \frac{s_{n+1}-s_n}{\lambda} s_{n+1}\, dx - \int_0^T \int_{\Omega\setminus G} s_{n+1} w_{n+1}\, dx\, dt. \qquad (2.7)$$

Adding (2.5), (2.6), (2.7) gives

$$0 \geqq \frac{1}{2} \|w_{n+1}(T)\|^2_{L^2(\Omega)} + \alpha_1 \int_0^T \|\nabla w_{n+1}\|^2_{L^2(\Omega)}\, dt +$$

$$+ \int_0^T \int_\Omega \frac{w_{n+1}-w_n}{\sigma} \cdot w_{n+1}\, dx\, dt + \int_{\Omega\setminus G} \frac{s_{n+1}-s_n}{\lambda} s_{n+1}\, dx +$$

$$+ \int_\Omega \left(\frac{r_{n+1}-r_n}{\varepsilon} \right)_{ij} (r_{n+1})_{ij}\, dx. \qquad (2.8)$$

Standard arguments imply

$$\frac{1}{\sigma} \int_0^T \|w_{n+1}\|^2_{L^2(\Omega)}\, dt + \frac{1}{\varepsilon} \|r_{n+1}\|^2_{L^2(\Omega)} + \frac{1}{\lambda} \|s_{n+1}\|^2_{L^2(\Omega\setminus G)} \leqslant$$

$$\leqq \frac{1}{\sigma} \int_0^T \|w_n\|^2_{L^2(\Omega)}\, dt + \frac{1}{\varepsilon} \|r_n\|^2_{L^2(\Omega)} + \frac{1}{\lambda} \|s_n\|^2_{L^2(\Omega\setminus G)}, \forall\, n \geqq 0, \qquad (2.9)$$

and upon summation,

$$\frac{1}{2\sigma}\int_0^T \|w_{n+1}\|^2_{L^2(\Omega)}\,dt+\frac{1}{2\varepsilon}\|r_{n+1}\|^2_{L^2(\Omega)}+\frac{1}{2\lambda}\|s_{n+1}\|^2_{L^2(\Omega\setminus G)}+$$

$$+\frac{1}{2}\sum_{k=0}^{n}\|w_{k+1}(T)\|^2_{L^2(\Omega)}+\alpha_1\sum_{k=0}^{n}\int_0^T\|\nabla w_{k+1}\|^2_{L^2(\Omega)}\,dt\leqslant$$

$$\leqq\frac{1}{2\sigma}\int_0^T\|w_0\|^2_{L^2(\Omega)}\,dt+\frac{1}{2\varepsilon}\|r_0\|^2_{L^2(\Omega)}+\frac{1}{2\varepsilon}\|s_0\|^2_{L^2(\Omega\setminus G)}. \tag{2.10}$$

Thus we have proved the a-priori estimate

$$\sup_{n\in\mathbb{N}}\ [\int_0^T\|w_n\|^2_{L^2(\Omega)}\,dt+\|r_n\|^2_{L^2(\Omega)}+\|s_n\|^2_{L^2(\Omega\setminus G)}]+$$

$$+\sum_{k=1}^{\infty}\|w_k(T)\|^2_{L^2(\Omega)}+\sum_{k=1}^{\infty}\int_0^T\|\nabla w_k\|^2_{L^2(\Omega)}\,dt\leqq C<+\infty, \tag{2.11}$$

where C depends on σ, ε, λ, u_0, k_0, L_0, u^*, k^*, L^*, but not on V, W_1, W_2.

Next we show:

THEOREM 2.1. *Let (A1)–(A6) hold. Then (P) has a solution.*

Proof. Let $X:=V\times W_1\times W_2$ and $\mathbf{K}:=(K\cap V)\times(K_1^*\cap W_1)\times(K_2^*\cap W_2)$. Then $\mathbf{K}\neq\emptyset$ is closed and convex, and $\dim X<+\infty$. We endow X with the L^2 — scalar product

$$\langle(u,k,L),(\tilde u,\tilde k,\tilde L)\rangle:=\int_0^T\int_\Omega u\tilde u\,dx\,dt+\int_\Omega k_{ij}\tilde k_{ij}\,dx+$$

$$+\int_{\Omega\setminus G}L\tilde L\,dx,\ \text{for } (u,k,L),(\tilde u,\tilde k,\tilde L)\in X. \tag{2.12}$$

Let us define the operators $T_n, S_n: X\to X'$ (= dual of X) by

$$T_n(u,k,L):=(u_t,\Theta,\Theta), \tag{2.13}$$

and

$$[S_n(u,k,L),(v,\eta,\zeta)]:=\int_0^T\int_\Omega\left(\frac{u-u_n}{\sigma}-f^*\right)v\,dx\,dt+$$

$$+\int_0^T\int_\Omega \nabla v\cdot k\,\nabla u\,dx\,dt+\int_0^T\int_{\Omega\setminus G} Lv\,dx\,dt+\int_\Omega\left(\frac{k-k_n}{\varepsilon}\right)_{ij}\eta_{ij}\,dx-$$

$$-\int_\Omega\left(\int_0^T \nabla u\otimes\nabla(u-u^*)\,dt\right)_{ij}\eta_{ij}\,dx+$$

$$+\int_{\Omega\setminus G}\frac{L-L_n}{\lambda}\zeta\,dx-\int_{\Omega\setminus G}\int_0^T(u-u^*)\,dt\,\zeta\,dx, \quad (2.14)$$

where $[\cdot,\cdot]$ is the dual pairing between the elements of X and X'. Obviously the existence is proved for every $n\geqq 0$ if there exists $(u,k,L)\in\mathbf{K}$ such that

$$\langle T_n(u,k,L),(v,\eta,\zeta)-(u,k,L)\rangle+$$
$$+[S_n(u,k,L),(v,\eta,\zeta)-(u,k,L)]\geqq 0,\ \forall\,(v,\eta,\zeta)\in\mathbf{K}. \quad (2.15)$$

Since S_n is maximal monotone, the result follows from standard results on variational inequalities (see [6], p. 197, Satz 2.7) if S_n is bounded, continuous, pseudomonotone and coercive with respect to $\Theta\in X'$. We only show the coercivity as the other properties are obvious in our case.
For $(u,k,L)\in\mathbf{K}$ we have:

$$[S_n(u,k,L),(u,k,L)]=\int_0^T\int_\Omega\left(\frac{u-u_n}{\sigma}-f^*\right)u\,dx\,dt+$$

$$+\int_0^T\int_\Omega \nabla u\cdot k\,\nabla u\,dx\,dt+\int_0^T\int_{\Omega\setminus G} Lu\,dx\,dt+\int_\Omega\left(\frac{k-k_n}{\varepsilon}\right)_{ij}k_{ij}\,dx-$$

$$-\int_\Omega\left(\int_0^T \nabla u\otimes\nabla(u-u^*)\,dt\right)_{ij}k_{ij}\,dx+$$

$$+\int_{\Omega\setminus G}\frac{L-L_n}{\lambda}L\,dx-\int_{\Omega\setminus G}\int_0^T(u-u^*)\,L\,dt\,dx\geqq$$

$$\geqq\frac{1}{\sigma}\int_0^T\|u\|_{L^2(\Omega)}\left\{\|u\|_{L^2(\Omega)}-\|u_n\|_{L^2(\Omega)}-\sigma\,\|f^*\|_{L^2(\Omega)}\right\}dt+$$

$$+\int_0^T\int_\Omega \nabla u\cdot k\,\nabla u^*\,dx\,dt+\int_0^T\int_{\Omega\setminus G} Lu^*\,dx\,dt+$$

$$+\frac{1}{\varepsilon}\|k\|_{L^2(\Omega)}\{\|k\|_{L^2(\Omega)}-\|k_n\|_{L^2(\Omega)}\}+$$

$$+\frac{1}{\lambda}\|L\|_{L^2(\Omega\setminus G)}\{\|L\|_{L^2(\Omega\setminus G)}-\|L_n\|_{L^2(\Omega\setminus G)}\}.$$

Since $\dim V<+\infty$, we have with some $\gamma>0$:

$$\int_0^T\int_\Omega \nabla u\cdot k\nabla u^*\,dx\,dt+\int_{\Omega\setminus G} L\int_0^T u^*\,dt\,dx\geqq$$

$$\geqq -\beta_1\gamma\{\int_0^T\|u\|^2_{L^2(\Omega)}dt\}^{1/2}\{\int_0^T\|\nabla u^*\|^2_{L^2(\Omega)}dt\}^{1/2}-$$

$$-T^{1/2}\|L\|_{L^2(\Omega\setminus G)}\{\int_0^T\int_{\Omega\setminus G}u^{*2}\,dx\,dt\}^{1/2}.$$

Hence, there exists $R>0$ such that $[S_n(u,k,L),(u,k,L)]>0$, for all $(u,k,L)\in\mathbf{K}$ which satisfy

$$\|(u,k,L)\|^2:=\int_0^T\|u\|^2_{L^2(\Omega)}dt+\|k\|^2_{L^2(\Omega)}+\|L\|^2_{L^2(\Omega\setminus G)}\geqq R.$$

This concludes the proof. ■

THEOREM 2.2. *Let (A1)–(A6) hold, and let* $\{(u_n,k_n,L_n)\}$ *solve (P). Then*

$$\int_0^T\|\nabla(u_n-u^*)\|^2_{L^2(\Omega)}dt\to 0,\ \|w_n(T)\|^2_{L^2(\Omega)}\to 0,\ \text{as } n\to\infty; \tag{2.16}$$

$$(k_n,L_n)\to(k_\infty,L_\infty),\ \text{where } (k_\infty,L_\infty)\in\mathbf{L}(V). \tag{2.17}$$

Proof. (2.16) is a direct consequence of (2.11). Note that $u_n-u^*=0$ a.e. on $\partial\Omega\times(0,T)$, which means that $(u_n-u^*)(t)\in\mathring{H}^1(\Omega)$ a.e. and hence by Poincaré's inequality, $\|u_n-u^*\|_{L^2(0,T;H^1(\Omega))}\to 0$ as $n\to\infty$. Moreover, (2.11) implies that $\{k_n\}$ and $\{L_n\}$ are bounded in $L^2(\Omega)$ and $L^2(\Omega\setminus G)$, respectively. By the finite dimensionality of $W_1\subset L^\infty_{\text{sym}}(\Omega)$ and $W_2\subset L^\infty(\Omega\setminus G)$ it follows that for a subsequence, again denoted by $\{k_n\}$ and $\{L_n\}$, we have $k_n\to k_\infty\in W_1$, strongly in $L^\infty_{\text{sym}}(\Omega)$ and $L_n\to L_\infty\in W_2$, strongly in $L^\infty(\Omega\setminus G)$.
Thus $k_\infty\in K_1^*\cap W_1$ and $L_\infty\in K_2^*\cap W_2$.

Moreover, there holds for every $v\in K\cap V$ and $n\geqq 0$:

$$\lim_{n\to\infty}\int_0^T\int_\Omega\frac{d}{dt}u_{n+1}(v-u_{n+1})\,dx\,dt=$$

$$= \lim_{n\to\infty} \left\{ \int_0^T \int_\Omega \frac{d}{dt}\, u_{n+1}\, v\, dx\, dt - \frac{1}{2}\, \|u_{n+1}(T)\|^2_{L^2(\Omega)} \right\} =$$

$$= \lim_{n\to\infty} \left\{ \int_\Omega u_{n+1}(T)\, v(T)\, dx - \int_0^T \int_\Omega u_{n+1}\, v_t\, dx\, dt \right\} - \frac{1}{2}\, \|u^*(T)\|^2_{L^2(\Omega)} =$$

$$= \int_\Omega u^*(T)\, v(T)\, dx - \int_0^T \int_\Omega u^*\, v_t\, dx\, dt - \frac{1}{2}\, \|u^*(T)\|^2_{L^2(\Omega)} =$$

$$= \int_0^T \int_\Omega u_t^*\, v\, dx\, dt - \int_0^T \int_\Omega u_t^*\, u^*\, dx\, dt.$$

Also we obtain:

$$\int_0^T \int_\Omega \nabla u_{n+1} \cdot k_{n+1}\, \nabla (v - u_{n+1})\, dx\, dt =$$

$$= \int_0^T \int_\Omega \nabla w_{n+1} \cdot k_{n+1}\, \nabla (v - u_{n+1})\, dx\, dt +$$

$$+ \int_0^T \int_\Omega \nabla u^* \cdot (k_{n+1} - k_\infty)\, \nabla (v - u_{n+1})\, dx\, dt +$$

$$+ \int_0^T \int_\Omega \nabla u^* \cdot k_\infty\, \nabla (v - u_{n+1})\, dx\, dt \to$$

$$\to \int_0^T \int_\Omega \nabla u^* \cdot k_\infty\, (v - u^*)\, dx\, dt, \text{ as } n \to \infty.$$

Finally:

$$\int_0^T \int_{\Omega\setminus G} L_{n+1}\, (v - u_{n+1})\, dx\, dt \to \int_0^T \int_{\Omega\setminus G} L_\infty\, (v - u^*)\, dx\, dt, \text{ as } n \to \infty.$$

Hence passing to the limit in (2.2) yields that (k_∞, L_∞) solves (IP,V).

It remains to show that the limit does not depend on the choice of the subsequences. To this end, recall that (2.9) and (2.10) hold for any $(k^*, L^*) \in \mathbf{L}(V)$.

In particular,

$$\frac{1}{\sigma} \int_0^T \|u_{n+1} - u^*\|^2_{L^2(\Omega)}\, dt + \frac{1}{\varepsilon}\, \|k_n - k^*\|^2_{L^2(\Omega)} + \frac{1}{\lambda}\, \|L_n - L^*\|^2_{L^2(\Omega\setminus G)}$$

decreases and converges to some limit $\delta \geqq 0$.

But $\int_0^T \|u_{n+1} - u^*\|^2_{L^2(\Omega)} dt \to 0$, and thus $\frac{1}{\varepsilon} \|k_n - k^*\|^2_{L^2(\Omega)} + \frac{1}{\lambda} \|L_n - L^*\|^2_{L^2(\Omega\setminus G)} \to \delta$.

Hence $\frac{1}{\varepsilon}\|k_\infty - k^*\|^2_{L^2(\Omega)} + \frac{1}{\lambda}\|L_\infty - L^*\|^2_{L^2(\Omega\setminus G)} = \delta$, for every limit point (k_∞, L_∞) of $\{(k_n, L_n)\}$. Now let (k^1_∞, L^1_∞), (k^2_∞, L^2_∞) be any two limit points of $\{(k_n, L_n)\}$. Then

$$\frac{1}{\varepsilon}\|k^1_\infty - k^*\|^2_{L^2(\Omega)} + \frac{1}{\lambda}\|L^1_\infty - L^*\|^2_{L^2(\Omega\setminus G)} =$$

$$= \frac{1}{\varepsilon}\|k^2_\infty - k^*\|^2_{L^2(\Omega)} + \frac{1}{\lambda}\|L^2_\infty - L^*\|^2_{L^2(\Omega\setminus G)}$$

for any $(k^*, L^*) \in \mathbf{L}(V)$. But $(k^1_\infty, L^1_\infty) \in \mathbf{L}(V)$, and thus

$$\frac{1}{\varepsilon}\|k^2_\infty - k^1_\infty\|^2_{L^2(\Omega)} + \frac{1}{\lambda}\|L^2_\infty - L^1_\infty\|^2_{L^2(\Omega\setminus G)} = 0,$$

from which the assertion follows. ■

Next we show that the limit points (k_∞, L_∞) of (P) are stable with respect to perturbations in the initial data:

THEOREM 2.3. *Let (A1)–(A6) hold. Assume initial data* $(u^\nu_0, k^\nu_0, L^\nu_0) \in (K \cap V) \times (K^*_1 \cap W_1) \times (K^*_2 \cap W_2)$, $0 \leqq \nu \leqq \nu_0$, $\nu_0 > 0$, *are given such that*

$$u^\nu_0 \to u^0_0,\ k^\nu_0 \to k^0_0,\ L^\nu_0 \to L^0_0, \text{ as } \nu \to 0+. \tag{2.18}$$

Let $\{u^\nu_n, k^\nu_n, L^\nu_n)\}$ solve (IP,V) with respect to the initial data $(u^\nu_0, k^\nu_0, L^\nu_0)$, and let $(k^\nu_n, L^\nu_n) \to (k^\nu_\infty, L^\nu_\infty)$ as $n \to \infty$. Then

$$k^\nu_\infty \to k^0_\infty,\ L^\nu_\infty \to L^0_\infty, \text{ as } \nu \to 0+, \tag{2.19}$$

provided $\varepsilon > 0$ is sufficiently small.

Proof. Put for $n \geqq 0$ and for $0 \leqq \nu \leqq \nu_0$: $q^\nu_n := u^\nu_n - u^0_n$, $y^\nu_n := k^\nu_n - k^0_n$, $z^\nu_n := L^\nu_n - L^0_n$, $w^\nu_n := u^\nu_n - u^*$.

From (2.2)–(2.4) there follows

$$\int_0^T \int_\Omega \left(\frac{u^\nu_{n+1} - u^\nu_n}{\sigma} + \frac{d}{dt} u^\nu_{n+1} - f^* \right) q^\nu_{n+1}\, dx\, dt +$$

$$+ \int_0^T \int_\Omega \nabla q^\nu_{n+1} \cdot k^\nu_{n+1} \nabla u^\nu_{n+1}\, dx\, dt + \int_0^T \int_{\Omega\setminus G} L^\nu_{n+1} q^\nu_{n+1}\, dx\, dt +$$

$$+ \int_\Omega \left(\frac{k^\nu_{n+1} - k^\nu_n}{\varepsilon} \right)_{ij} (y^\nu_{n+1})_{ij}\, dx - \int_0^T \int_\Omega \nabla w^\nu_{n+1} \cdot y^\nu_{n+1} \nabla u^\nu_{n+1}\, dx\, dt +$$

$$+\int_{\Omega\setminus G}\left(\frac{L^{\nu}_{n+1}-L^{\nu}_{n}}{\lambda}\right) z^{\nu}_{n+1}\,dx-\int_0^T\int_{\Omega\setminus G} w^{\nu}_{n+1}\,z^{\nu}_{n+1}\,dx\,dt\leqslant$$

$$\leqslant 0\leqslant\int_0^T\int_{\Omega}\left(\frac{u^0_{n+1}-u^0_n}{\sigma}+\frac{d}{dt}\,u^0_{n+1}-f^*\right) q^{\nu}_{n+1}\,dx\,dt+$$

$$+\int_0^T\int_{\Omega}\nabla q^{\nu}_{n+1}\cdot k^0_{n+1}\,\nabla u^0_{n+1}\,dx\,dt+\int_0^T\int_{\Omega\setminus G} L^0_{n+1}\,q^{\nu}_{n+1}\,dx\,dt+$$

$$+\int_{\Omega}\left(\frac{k^0_{n+1}-k^0_n}{\varepsilon}\right)_{ij}(y^{\nu}_{n+1})_{ij}\,dx-\int_0^T\int_{\Omega}\nabla w^0_{n+1}\cdot y^{\nu}_{n+1}\,\nabla u^0_{n+1}\,dx\,dt+$$

$$+\int_{\Omega\setminus G}\left(\frac{L^0_{n+1}-L^0_n}{\lambda}\right) z^{\nu}_{n+1}\,dx-\int_0^T\int_{\Omega\setminus G} w^0_{n+1}\,z^{\nu}_{n+1}\,dx\,dt.$$

Subtracting the right-hand side from the left-hand side yields:

$$0\geqslant\int_0^T\int_{\Omega}\left(\frac{q^{\nu}_{n+1}-q^{\nu}_n}{\sigma}+\frac{d}{dt}\,q^{\nu}_{n+1}\right) q^{\nu}_{n+1}\,dx\,dt+$$

$$+\int_{\Omega}\left(\frac{y^{\nu}_{n+1}-y^{\nu}_n}{\varepsilon}\right)_{ij}(y^{\nu}_{n+1})_{ij}\,dx+\int_{\Omega\setminus G}\left(\frac{z^{\nu}_{n+1}-z^{\nu}_n}{\lambda}\right) z^{\nu}_{n+1}\,dx+$$

$$+\int_0^T\int_{\Omega}\{\nabla q^{\nu}_{n+1}\cdot k^{\nu}_{n+1}\,\nabla u^{\nu}_{n+1}-\nabla w^{\nu}_{n+1}\cdot y^{\nu}_{n+1}\,\nabla u^{\nu}_{n+1}-$$

$$-\nabla q^{\nu}_{n+1}\cdot k^0_{n+1}\,\nabla u^0_{n+1}+\nabla w^0_{n+1}\cdot y^{\nu}_{n+1}\,\nabla u^0_{n+1}\}\,dx\,dt.$$

Rearrangement of the terms in the last integral gives:

$$I=\int_0^T\int_{\Omega}\{\nabla q^{\nu}_{n+1}\cdot k^0_{n+1}\,\nabla q^{\nu}_{n+1}-\nabla q^{\nu}_{n+1}\cdot y^{\nu}_{n+1}\,\nabla w^0_{n+1}\}\,dx\,dt\geqslant$$

$$\geqslant\alpha_1\int_0^T\|\nabla q^{\nu}_{n+1}\|^2_{L^2(\Omega)}\,dt-$$

$$-\|y^{\nu}_{n+1}\|_{L^\infty(\Omega)}\Big(\int_0^T\|\nabla q^{\nu}_{n+1}\|^2_{L^2(\Omega)}\,dt\Big)^{1/2}\Big(\int_0^T\|\nabla w^0_{n+1}\|^2_{L^2(\Omega)}\,dt\Big)^{1/2}\geqslant$$

$$\geqslant \alpha_1 \int_0^T \|\nabla q_{n+1}^\nu\|_{L^2(\Omega)}^2\, dt - \left\{\alpha_1 \int_0^T \|\nabla q_{n+1}^\nu\|_{L^2(\Omega)}^2\, dt + \right.$$

$$\left. + \frac{1}{4\alpha_1} \|y_{n+1}^\nu\|_{L^\infty(\Omega)}^2 \int_0^T \|\nabla w_{n+1}^0\|_{L^2(\Omega)}^2\, dt \right\} \geqslant$$

$$\geqslant -\frac{C\varepsilon}{4\alpha_1} \int_0^T \|\nabla w_{n+1}^0\|_{L^2(\Omega)}^2\, dt \cdot \frac{1}{\varepsilon} \|y_{n+1}^\nu\|_{L^2(\Omega)}^2 \geqslant$$

$$\geqslant -\frac{C\varepsilon}{4\alpha_1} \int_0^T \|\nabla w_{n+1}^0\|_{L^2(\Omega)}^2\, dt \cdot (\varrho_{n+1}^\nu)^2,$$

where for $n \geqslant 0$

$$(\varrho_n^\nu)^2 := \frac{1}{\sigma} \int_0^T \|q_n^\nu\|_{L^2(\Omega)}^2\, dt + \frac{1}{\varepsilon} \|y_n^\nu\|_{L^2(\Omega)}^2 + \frac{1}{\lambda} \|z_n^\nu\|_{L^2(\Omega\setminus G)}^2. \tag{2.20}$$

Now

$$\frac{1}{\sigma} \int_0^T \int_\Omega q_n^\nu q_{n+1}^\nu\, dx\, dt + \frac{1}{\varepsilon} \int_\Omega (y_n^\nu)_{ij} (y_{n+1}^\nu)_{ij}\, dx +$$

$$+ \frac{1}{\lambda} \int_{\Omega\setminus G} z_n^\nu z_{n+1}^\nu\, dx \leqslant \varrho_n^\nu \varrho_{n+1}^\nu. \tag{2.21}$$

Hence

$$0 \geqslant (\varrho_{n+1}^\nu)^2 - \varrho_n^\nu \varrho_{n+1}^\nu - \gamma_{n+1} (\varrho_{n+1}^\nu)^2,$$

with $\gamma_{n+1} := \frac{C\varepsilon}{4\alpha_1} \int_0^T \|\nabla w_{n+1}^0\|_{L^2(\Omega)}^2\, dt.$

Now chose $\varepsilon > 0$ so small that for every $n \geqslant 0$, $\gamma_{n+1} \leqslant \varkappa < 1$ which is possible in view of (2.16).

Then for $n \geqslant 0$:

$$\varrho_{n+1}^\nu (1 - \gamma_{n+1}) \leqslant \varrho_n^\nu, \text{ or } \varrho_{n+1}^\nu \leqslant (1 + \delta_{n+1}) \varrho_n^\nu,$$

where

$$\delta_{n+1} := \frac{\gamma_{n+1}}{1 - \gamma_{n+1}} \leqslant \frac{\gamma_{n+1}}{1 - \varkappa}.$$

Induction yields with $\hat{\varkappa} := \frac{C\varepsilon}{4\alpha_1} \cdot \frac{1}{1-\varkappa}$:

$$\varrho_n^\nu \leqslant \varrho_0^\nu \prod_{k=1}^{n} (1+\delta_k) \leqslant \varrho_0^\nu \prod_{k=1}^{n} \exp(\delta_k) = \varrho_0^\nu \exp\Big(\sum_{k=1}^{n} \delta_k\Big) \leqslant$$

$$\leqslant \varrho_0^\nu \cdot \exp\Big(\hat{\varkappa} \sum_{n=1}^{\infty} \int_0^T \|\nabla w_{n+1}^0\|_{L^2(\Omega)}^2 \, dt\Big).$$

The latter factor is by (2.11) finite.

Letting $n \to \infty$ we obtain

$$\frac{1}{\varepsilon} \|k_\infty^\nu - k_\infty^0\|_{L^2(\Omega)}^2 + \frac{1}{\lambda} \|L_\infty^\nu - L_\infty^0\|_{L^2(\Omega)}^2 \leqslant C\varrho_0^\nu,$$

where C does not depend on ν.
The assertion now follows from (2.18). ■

Remark:

Stability results with respect to the data u^*, f^* in elliptic problems were derived in [1] and [2].

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O identyfikacji współczynnika przewodnictwa cieplnego i utajonego ciepła przemiany w jednofazowym zagadnieniu Stefana

Praca dotyczy identyfikacji, zależnych od zmiennych przestrzennych, współczynników ewolucyjnych nierówności wariacyjnych. Jako prototyp rozważań przyjęto nierówność wariacyjną odpowiadającą wielowymiarowemu jednofazowemu zagadnieniu Stefana. Wielkościami mierzonymi są w tym przypadku temperatura (albo indeks zastygania), ciepło właściwe, ponadto dane początkowe i brzegowe, natomiast parametrami identyfikowanymi są utajone ciepło przemiany oraz macierz współczynników przewodnictwa cieplnego. Przedstawiono metodę numeryczną rozwiązywania takich zagadnień oraz omówiono jej zbieżność i stabilność.

Об идентификации коэффициента теплопроводности и скрытой теплоты фазового перехода в однофазной задаче Стефана

Работа касается идентификации зависящих от пространственных координат коэффициентов эволюционных вариационных неравенств. Основой рассуждений принято вариационное неравенство соответствующее многомерной однофазной проблеме Стефана. Измерению подлежит температура (либо „freezing index"), собственная теплота, кроме того начальные и краевые данные, идентификации подлежит скрытая теплота фазового перехода и матрица коэффициентов теплопроводности. Представляется численный метод решения таких проблем и обсуждается его сходимость и устойчивость.