Control
and Cybernetics
VOL. 14 (1985) No. 1—3

Numerical analysis of degenerate Stefan problems

by

MAREK NIEZGÓDKA*)

IRENA PAWŁOW

Systems Research Institute
Polish Academy of Sciences
Newelska 6, 01-447 Warsaw

Convergence properties of discrete approximations to degenerate problems of the Stefan type are discussed. The multidimensional problems are considered in their variational inequality formulations. Some results of the performed numerical experiments are presented.

1. Introduction

Discrete approximations to multi-phase Stefan problems in a multidimensional case were studied by several authors. A comprehensive numerical analysis of the approximations corresponding to the enthalpy formulation of the problems was, in particular, offered in [2, 3, 8]. In turn, the approximations of the problems in their freezing index formulation (as variational inequality) were studied in [5]. In the above-mentioned publications, results on the convergence of the discrete solutions, including estimates of the convergence rate, were given for the strongly parabolic Stefan problems.

Since the Stefan's structure is preserved also in some other multi-phase problems with free boundary, although without the strong parabolicity guaranteed, a natural question concerns the possibility of extending the convergence results from the parabolic onto a degenerate situation. This is motivated, for example, by fixed-domain models of electrochemical processes and partially saturated flows in porous media [1].

In this paper, discrete approximations of an evolution variational inequality, referring in particular, to a class of degenerate multi-phase Stefan

*) On leave at the Mathematical Institute of the University of Augsburg, supported by the Alexander von Humboldt Foundation.

problems, are studied. Results on the convergence, extending those valid in the parabolic case, are formulated. Some illustrating computational results are discussed.

The techniques developed in this paper are oriented onto the problems with non-constant boundary data on Γ. This is motivated by our interest in construction of discrete approximations for the related problems of boundary control.

2. Problem formulation

We shall consider the following evolution problem:

$$(S)\qquad \begin{cases} \gamma_0(y') + Ay \ni f_0 & \text{in } Q, \quad (2.1)\\ \partial_\nu y + g_0 y = g & \text{on } \Sigma, \quad (2.2)\\ y(0) = 0 & \text{in } \Omega, \quad (2.3)\end{cases}$$

where $A \in \mathscr{L}(V, V')$ is a linear operator, $V \triangleq H^1(\Omega)$, $\gamma_0 \subset R \times R$ is a maximal monotone graph (in general, multi-valued),
$\gamma_0(r) \equiv \tilde{\gamma}_0(r) + L\chi_0(r)$, $L \geqq 0$, $\tilde{\gamma}$ — Lipschitz continuous, monotone;
χ_0 — the Heaviside's graph (multi-valued).

Provided a specific form of the graph γ_0 and $A = -\Delta$, the problem (S) can refer to a multi-phase Stefan problem [4]. In particular, if the graph γ_0 is not strictly monotone, the Stefan problem appears no more parabolic.

In the Stefan problem,

$$y(x, t) = \int_0^t \theta(x, \tau)\, d\tau, \quad t \in [0, T],$$

where θ refers to temperature (in the thermal framework). In that problem, one prescribes $\theta(0) = \theta_0$ as an initial datum.

The problem (S) can be given the following weak formulation as a variational inequality [4]:

$$(P)\qquad \begin{cases} y \in W^{1,\infty}(0, T; V),\\ \big(\tilde{\gamma}_0(y'(t)) - f_0(t), z - y'(t)\big) + a\big(y(t), z - y'(t)\big) -\\ -\big(g(t), z - y'(t)\big)_\Gamma + \Psi(z) - \Psi(y'(t)) \geqq 0,\\ \forall z \in V, \text{ for a.a. } t \in [0, T], & (2.4)\\ y(0) = 0 \text{ in } \Omega & (2.5)\end{cases}$$

where $(.,.)$, $(.,.)_\Gamma$ are inner products in H and $L^2(\Gamma)$, respectively, $a(.,.)$ is the bilinear form corresponding to A,

$$\Psi(z) = L \int_\Omega \psi_0(z(x))\, dx, \quad \psi_0(r) = r^+.$$

The existence and uniqueness of the weak solution $y \in W^{1,\infty}(0, T; V)$ (if $\tilde{\gamma}_0$ is strictly monotone, then also $y \in H^2(0, T; H)$, $H \triangleq L^2(\Omega)$) of the problem (P) is guaranteed, provided the following hypotheses hold:
$(P1)$ $\tilde{\gamma}_0 \in W^{1,\infty}_{\text{loc}}(R)$, $\tilde{\gamma}_0(0) = 0$, $\tilde{\gamma}_0$ is non-decreasing;
$(P2)$ $A \in \mathscr{L}(V, V')$ is V-coercive:

$$\exists \alpha > 0 \text{ such that } \forall v \in V \ (Av, v) \geqq \alpha \|v\|_V^2;$$

$(P3)$ $f_0 \in H^2(0, T; H)$ (in the case of $\tilde{\gamma}_0$ strictly monotone, $f_0 \in H^1(0, T; H)$);
$(P4)$ $g \in H^2(0, T; L^2(\Gamma))$;
$(P5)$ $g_0 \in L^\infty(\Gamma)$, $g_0 \geqq 0$.

REMARK 1. In a physically motivated formulation of the problem, as already mentioned, one imposes an initial condition on y',

$$y'(0) = \theta_0. \tag{2.6}$$

For the correctness, one needs then to assume the compatibility condition of the form $f(0) = (\gamma_0)^0(\theta_0)$, where $(\gamma_0)^0$ denotes the least section of the graph γ_0.

The variational inequality (P) admits an alternative equivalent form.

LEMMA 1. (2.4) *can be equivalently written as*:

$$a\big(y(t), z - y'(t)\big) - \big(f_0(t), z - y'(t)\big) - \big(g(t), z - y'(t)\big)_\Gamma + \\ + \Phi(z) - \Phi\big(y'(t)\big) \geqq 0, \ \forall z \in V, \text{ for a.a. } t \in [0, T], \tag{2.7}$$

where

$$\Phi(z) \equiv B(z) + \Psi(z), \tag{2.8}$$

$$B: H \to R,\ B(z) \triangleq \int_\Omega \beta\big(z(x)\big)\, dx,\ \beta(r) \triangleq \int_0^r \tilde{\gamma}_0(s)\, ds,\ r \in R. \tag{2.9}$$

Proof. (2.7) follows from (2.4) in view of the convexity of B and due to the existence of the Gateaux differential DB(.), since

$$(DB(v), z) = \big(\tilde{\gamma}_0(v), z\big), \forall v, z \in H,$$
$$\big(\tilde{\gamma}_0(v), z - v\big) \leqq B(z) - B(v), \forall v, z \in H.$$

To show the reverse implication, take in (2.7)

$$z = y'(t) + \varkappa\big(w - y'(t)\big),$$

with an arbitrary $w \in V$, $\varkappa \in (0, 1)$, make use of the convexity of Ψ and, finally, pass to the limit with $\varkappa \to 0$:

$$0 \leqq a\big(y(t), w-y'(t)\big)-\big(f_0(t), w-y'(t)\big)-\big(g(t), w-y'(t)\big)_\Gamma+$$
$$+\lim_{\varkappa\to 0}\left\{\frac{1}{\varkappa}\left[B\big(y'(t)+\varkappa(w-y'(t))\big)-B\big(y'(t)\big)+\Psi\big(y'(t)+\varkappa(w-y'(t))\big)-\right.\right.$$
$$\left.\left.-\Psi\big(y'(t)\big)\right]\right\} \leqq a\big(y(t), w-y'(t)\big)-\big(f_0(t), w-y'(t)\big)-$$
$$-\big(g(t), w-y'(t)\big)_\Gamma+\big(\tilde{\gamma}_0(y'(t)), w-y'(t)\big)+\Psi(w)-\Psi\big(y'(t)\big). \qquad ■$$

In the further considerations, we shall confine ourselves to the most interesting degenerate case, assuming a special form of $\tilde{\gamma}_0$ (as in the Stefan problems, see [4]):

$$(P6) \qquad \tilde{\gamma}_0(r) \equiv \int_0^r \varrho(s)\,ds, \text{ where } 0 \leqq \varrho(s) \leqq \bar{\varrho} < \infty,\ s\in R.$$

3. Construction of approximations to Problem (P)

3.1. Continuous approximations

For any $\mu \in [0, 1)$ define the strictly monotone function

$$\tilde{\gamma}_\mu(r) \triangleq \tilde{\gamma}_0(r)+\mu r,\ r\in R \tag{3.1}$$

and introduce the corresponding

$$\gamma_\mu(r) = \tilde{\gamma}_\mu(r)+L\chi_0(r).$$

The resulting strongly parabolic problem $(P)^\mu$ (parabolic regularization of (P)) differs from (P) only in the form of the equation (2.1), now

$$\gamma_\mu(y'_\mu)+Ay_\mu \ni f_\mu \text{ in } Q, \tag{3.2}$$

where $f_\mu \triangleq f_0+\mu\theta_0$.

In turn, let us approximate the Heaviside's graph χ_0 by a family of smooth functions χ_ε ($\chi_\varepsilon \in C^3(R)$, see [4]), $\varepsilon \in (0, 1)$, so that for the relevant

$$\gamma_{\mu\varepsilon} \triangleq \tilde{\gamma}_{\mu\varepsilon}+L\chi_\varepsilon,$$

$\tilde{\gamma}_{\mu\varepsilon}$ being a twice differentiable approximation of $\tilde{\gamma}_\mu$ (if necessary), the inequality

$$0 < \mu \leqq D\gamma_{\mu\varepsilon}(r) \leqq \frac{C}{\varepsilon},\ r\in R \tag{3.3}$$

is satisfied with a finite constant C independent of μ, ε; besides, the mapping

$\gamma_{\mu\varepsilon}: H \to H$ is uniformly bounded,

$$\|\gamma_{\mu\varepsilon}(v)\|_H \leqq (\bar{\bar{\varrho}}+1)\,\|v\|_H + L\,(\text{meas}\,\Omega)^{1/2}.$$

The corresponding problem $(P)^{\mu}_{\varepsilon}$ differs from $(P)^{\mu}$ again only in the form of the equation, this time

$$\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon}) + A y_{\mu\varepsilon} = f_{\mu\varepsilon} \text{ in } Q, \tag{3.4}$$

instead of (3.2), with $f_{\mu\varepsilon}$ being a smooth approximation of f_{μ} (see [6]).

We shall also admit $\mu, \varepsilon = 0$ in the sequel, with $\mu = 0$ referring to Problem $(P)_{\varepsilon}$, $\varepsilon = 0$ — to Problem $(P)^{\mu}$.

The existence and uniqueness of the solutions y_{μ} and $y_{\mu\varepsilon}$ of Problems $(P)^{\mu}$ and $(P)^{\mu}_{\varepsilon}$, respectively, follows as for Problem (P) (see [4]).

For a detailed construction of the above regularizations we refer to [6].

3.2. Discrete approximations

We shall confine our further exposition, assuming in the discrete case
$(P7)$ $\theta_0 \in H^2(\Omega)$;
$(P8)$ $\Omega \subset R^2$ is a convex domain; $\mathcal{T}_h$, $h \in (0, 1]$, denotes a regular triangulation of Ω, with h referring to the mesh size; $V_h \subset V$ is the finite-dimensional subspace of functions $v_h \in V \cap C(\bar{\Omega})$, such that v_h is a polynomial of order $\leqq 1$ over each element of $\mathcal{T}_h$.

In view of $(P8)$, the problem is discretized in space by finite elements of the first order.

To discretize the problem in t, we divide the interval $[0, T]$ into N equal subintervals $[t_i, t_{i+1}]$, $i = 0, \dots, N-1$, with $t_i = ik$, $k = T/N$. In the sequel, we shall use the standard notations:

$$w^i(x) = w(x, t_i),\ \delta w^i = (w^{i+1} - w^i)/k,$$
$$w^{i+k} = w^i + \varkappa\,(w^{i+1} - w^i) = w^i + \varkappa\,k\,\delta w^i,$$
$$\delta w^{i+\varkappa} = (w^{i+1+\varkappa} - w^{i+\varkappa})/k = \delta w^i + \varkappa\,(\delta w^{i+1} - \delta w^i),\ \varkappa \in [0, 1],$$
$$\delta^2 w^i = (\delta w^{i+1} - \delta w^i)/k = (w^{i+2} - 2w^{i+1} + w^i)/k^2.$$

As a discrete counterpart of Problem $(P)^{\mu}_{\varepsilon}$ $(\mu, \varepsilon \geqq 0)$ we shall take the following:
Problem $(P)^{\mu}_{\varepsilon,h,k}$: $(\mu, \varepsilon \in [0, 1)$ — arbitrary$)$

$$\left\{\begin{array}{l} w^{i+1} = w^i + k\,\delta w^i, \\ a\,(w^{i+\varkappa}, z_h - \delta w^i) - (f^i_{\mu\varepsilon h}, z_h - \delta w^i) - (g^i, z_h - \delta w^i)_{\Gamma} + \Phi_{\mu\varepsilon}(z_h) - \\ -\Phi_{\mu\varepsilon}(\delta w^i) \geqslant 0,\ \forall z_h \in V_h,\ i = 0, \dots, N-1,\ w^0 = 0 \text{ in } \Omega, \end{array}\right. \tag{3.5}$$

where $\varkappa \in [0, 1]$ is an arbitrary parameter characterizing the type of the scheme,

$$\Phi_{\mu\varepsilon}(z) = B_{\mu\varepsilon}(z) + \Psi_\varepsilon(z),$$
$$B_{\mu\varepsilon}(z) = \int_\Omega \beta_{\mu\varepsilon}(z(x))\, dx,$$
$$\beta_{\mu\varepsilon}(r) = \int_0^r \tilde{\gamma}_{\mu\varepsilon}(s)\, ds, \; r \in R,$$
$$\Psi_\varepsilon(z) = L \int_\Omega \psi_\varepsilon(z(x))\, dx,$$
$$\psi_\varepsilon(r) = \int_0^r \chi_\varepsilon(s)\, ds, \; r \in R.$$

REMARK 2. System $(P)^\mu_{\varepsilon,h,k}$ can be equivalently formulated as the following nonlinear programming problem:

$$\left\{\begin{array}{l} \quad \text{For } i = 0, \ldots, N-1 \text{ determine } \delta w^i \text{ realizing} \\ \inf\{J^i_{\mu\varepsilon}(z);\; z \in V_h\}, \\ \quad \text{where} \\ J^i_{\mu\varepsilon}(z) = \frac{1}{2}\varkappa k a(z, z) + \Phi_{\mu\varepsilon}(z) + a(w^i, z) - (f^i_{\mu\varepsilon h}, z) - (g^i, z)_\Gamma. \end{array}\right.$$

This implies that in view of the construction of $\Phi_{\mu\varepsilon}$, by the Weierstrass theorem there exists a solution of Problem $(P)^\mu_{\varepsilon,h,k}$. This solution is unique for all $\varkappa \in [0, 1]$ in the parabolic case, and for $\varkappa \in (0, 1]$ in the degenerate case.

Problem $(P)^\mu_{\varepsilon,h,k}$ can be implemented in the form of a time-stepping algorithm which offers a numerical method of solving Problem (P). Some results of the relevant computational experiments are shown in Section 5 (see also [7]). The exposed way of approximating Problem (P) can be justified theoretically both in the parabolic and degenerate case.

As an auxiliary intermediate stage, let us introduce the following semi-discrete

Problem $(P)^\mu_{\varepsilon,h}$: $\mu, \varepsilon > 0$ — arbitrarily fixed

Determine $y_h \triangleq y_{\mu\varepsilon h} \in W^{1,\infty}(0, T; V_h)$, such that

$$\left\{\begin{array}{l} a(y_h(t), z_h - y_h'(t)) - (f_{\mu\varepsilon h}(t), z_h - y_h'(t)) - (g(t), z_h - y_h'(t))_\Gamma + \\ + \Phi_{\mu\varepsilon}(z_h) - \Phi_{\mu\varepsilon}(y_h'(t)) \geqq 0, \; \forall z_h \in V_h, \text{ for a.a. } t \in [0, T], \\ y_h(0) = 0 \text{ in } \Omega. \end{array}\right.$$

The existence and uniqueness of the solution $y_h \in W^{1,\infty}(0, T; V) \cap H^2(0, T; H)$ follows as in the case of Problem $(P)^\mu_\varepsilon$ (see [4]).

4. Convergence of approximations

The analysis of a convergence (and its rate) of the discrete solutions $y_{\mu\varepsilon hk}$ (of $(P)^{\mu}_{\varepsilon,h,k}$) to the solution y (of (P)), can be performed stepwise. In this paper we present only the principal steps of this analysis, referring to [6] for a detailed exposition.

Throughout we shall assume that the hypotheses $(P1)$–$(P6)$ are satisfied.

Proposition 1. $(P)^{\mu} \to (P)$ as $\mu \to 0+$. Let y_{μ}, y be the solutions of $(P)^{\mu}$ and (P), respectively. Then

$$\|y_{\mu}\|_{W^{1,\infty}(0,T;V)} + \mu^{1/2} \|y''_{\mu}\|_{L^2(Q)} \leqq C_0 \tag{4.1}$$

with a finite constant C_0 independent of μ. Let $\mu \to 0+$, then

$$y_{\mu} \to y \text{ weakly} - * \text{ in } W^{1,\infty}(0, T; V) \tag{4.2a}$$

with

$$\mu^{1/2} y'_{\mu} \to 0, \ \mu y''_{\mu} \to 0 \text{ strongly in } L^2(Q); \tag{4.2b}$$

besides,

$$\|y_{\mu} - y\|_{L^{\infty}(0,T;V)} + \mu^{1/2} \|y' - y'_{\mu}\|_{L^2(Q)} \leqq C_0 \mu^{1/2} \tag{4.2c}$$

with the same C_0 as in (4.1).

Proposition 2. $(P)^{\mu}_{\varepsilon} \to (P)^{\mu}$ as $\varepsilon \to 0+$, μ-fixed. Let $y_{\mu\varepsilon}, y_{\mu}$ be the solutions of $(P)^{\mu}_{\varepsilon}$ and $(P)^{\mu}$, respectively. Then the a priori bound (4.1) holds also for $y_{\mu\varepsilon}$; besides,

$$\|\Delta y_{\mu\varepsilon}\|_{C([0,T];H)} + \varepsilon^{1/2} \|\Delta y'_{\mu\varepsilon}\|_{L^2(Q)} \leqq C_0. \tag{4.3}$$

Assume, in addition to $(P1) \div (P6)$,

$(P9)$ meas $\{x \in \Omega; 0 < \theta_0(x) < \varepsilon\} \leqq C\varepsilon$, $C \neq C(\varepsilon)$.

Then

$$\|y_{\mu} - y_{\mu\varepsilon}\|_{L^{\infty}(0,T;V)} \leqq C_1 \varepsilon^{1/2} \tag{4.4}$$

with a finite constant C_1 independent of μ, ε.

Proposition 3. $(P)^{\mu}_{\varepsilon,h,k} \to (P)^{\mu}_{\varepsilon}$ as $h, k \to 0+$, μ, ε-fixed.

Assume, in addition to $(P1) \div (P6)$, also $(P7) \div (P9)$ and

$(P10)$ $\varkappa \in [1/2, 1]$.

Then the solutions $w = y_{\mu\varepsilon hk}$ of $(P)^{\mu}_{\varepsilon,h,k}$ are a priori bounded,

$$\max_{i=0,\dots,N} \|w^i\|_V + \max_{i=0,\dots,N-1} \|\delta w^i\|_V + \mu^{1/2} \Big(\sum_{i=0}^{N-2} k \|\delta^2 w^i\|_H^2\Big)^{1/2} \leqq$$
$$\leqq C_2 \{\|f_0\|_{H^2(0,T;H)} + \|g\|_{H^2(0,T;L^2(\Gamma))} + \|\theta_0\|_{H^2(\Omega)}\} \tag{4.5}$$

with a finite constant C_2 independent of μ, ε, h, k.

Besides,

(i) for any fixed $\varepsilon \in [0, 1)$, h, $k > 0$,

$$\max_{i=0,\dots,N} \|y^i_{0\varepsilon hk} - y^i_{\mu\varepsilon hk}\|_V \leqq C_2\, \mu^{1/2}, \tag{4.6a}$$

with the same C_2 as in (4.5);

(ii) for any fixed $\mu \in [0, 1)$, h, $k > 0$,

$$\max_{i=0,\dots,N} \|y^i_{\mu 0 hk} - y^i_{\mu\varepsilon hk}\|_V \leqq C\varepsilon^{1/2}, \tag{4.6b}$$

with a finite constant C independent of μ, ε, h, k.

PROPOSITION 4. $(P)^{\mu}_{\varepsilon,h} \to (P)^{\mu}_{\varepsilon}$ as $h \to 0+$, μ, ε-fixed. Let $y_{\mu\varepsilon k}$, $y_{\mu\varepsilon}$ be the solutions of $(P)^{\mu}_{\varepsilon,h}$ and $(P)^{\mu}_{\varepsilon}$, respectively. Then

$$\|y_{\mu\varepsilon h}\|_{W^{1,\infty}(0,T;V)} + \mu^{1/2}\, \|y''_{\mu\varepsilon h}\|_{L^2(Q)} \leqq C_0 \tag{4.7}$$

with the same C_0 as in (4.1). Besides, there exists a finite constant C_3 independent of μ, ε, h, such that

$$\|y_{\mu\varepsilon} - y_{\mu\varepsilon h}\|_{L^\infty(0,T;V)} + \mu^{1/2}\, \|y'_{\mu\varepsilon} - y'_{\mu\varepsilon h}\|_{L^2(Q)} \leqq C_3\, \frac{h}{\varepsilon^{1/2}}, \tag{4.8a}$$

or, expressed discretely in t,

$$\max_{i=0,\dots,N} \|y^i_{\mu\varepsilon} - y^i_{\mu\varepsilon h}\|_V + \mu^{1/2} \Big(\sum_{i=0}^{N-1} k\, \|(y'_{\mu\varepsilon})^i - (y'_{\mu\varepsilon h})^i\|^2_H\Big)^{1/2} \leqq$$
$$\leqq C_3 \left(\frac{h}{\varepsilon^{1/2}} + k\right). \tag{4.8b}$$

PROPOSITION 5. $(P)^{\mu}_{\varepsilon,h,k} \to (P)^{\mu}_{\varepsilon,h}$ as $k \to 0+$, μ, ε, h-fixed. Let $y^i_{\mu\varepsilon hk}$, $y^i_{\mu\varepsilon h}$ be the solutions of $(P)^{\mu}_{\varepsilon,h,k}$ and $(P)^{\mu}_{\varepsilon,h}$, respectively. Assume that all the hypotheses $(P1) \div (P10)$ are satisfied. Then there exists a finite constant C_4 independent of μ, ε, h, k, such that

$$\max_{i=0,\dots,N} \|y^i_{\mu\varepsilon hk} - y^i_{\mu\varepsilon h}\|_V + \mu^{1/2} \Big(\sum_{i=0}^{N-1} k\, \|\delta y^i_{\mu\varepsilon hk} - (y'_{\mu\varepsilon h})^i\|^2_H\Big)^{1/2} \leqq$$
$$\leqq C_4 \left[\left(\frac{k}{\mu^{1/2}\, h}\right)^{1/2} + \varkappa^{1/2}\, k\right]. \tag{4.9}$$

The assertions of Propositions $1 \div 5$ taken together yield the following estimate of the convergence rate.

THEOREM 1. $(P)^{\mu}_{\varepsilon,h,k} \to (P)$. *Let the hypotheses* $(P1) \div (P10)$ *be satisfied. Then for the solutions* y, $y_{\mu\varepsilon hk}$ *of Problems* (P) *and* $(P)^{\mu}_{\varepsilon,h,k}$, *respectively.*

$$\max_{i=0,\dots,N} \|y^i - y^i_{\mu\varepsilon hk}\|_V \leqq C\left[\mu^{1/2} + \varepsilon^{1/2} + \frac{h}{\varepsilon^{1/2}} + \left(\frac{k}{\mu^{1/2} h}\right)^{1/2} + k\right], \quad (4.10)$$

with a finite constant C *independent of* μ, ε, h, k.

In particular, let us assume $\mu = \mu_0 h$, $\varepsilon = \varepsilon_0 h$, $k \leqq k_0 h^{5/2}$, where $\mu_0, \varepsilon_0 \geqq 0$, $k_0 > 0$ are arbitrary constants. Then the assertion (4.10) takes the form

$$\max_{i=0,\dots,N} \|y^i - y^i_{\mu\varepsilon hk}\|_V \leqq Ch^{1/2}.$$

REMARK 3. The above results apply also to the problem (P) in an extended form, with Dirichlet conditions prescribed on a part Γ' of the boundary Γ (the Neumann or respectively mixed type conditions are imposed on the complement $\Gamma'' = \Gamma \setminus \Gamma'$). To this end, one only needs to introduce an appropriate shifted variable $\bar{y} = y - \omega$, $\omega \in W^{1,\infty}(0, T; V)$ — given, with the trace of ω' on Γ' specifing the Dirichlet data (see [6] for details).

5. Numerical solution of Problem (P)

The results on the convergence rate for the discrete approximations to Problem (P), given in section 4, provide a basis for setting up relevant numerical algorithms. In the sequel, we shall present some results of the performed computational experiments.

According to the discrete formulation $(P)^{\mu}_{\varepsilon,h,k}$ of the problem, we have employed the following time-stepping algorithm:

(0) Given parameters μ, ε, h, k and data: θ_0, g_0, g, f_0

(i) set the initial time t_i, $i = 0$

(ii) solve (3.5) with respect to δw^i

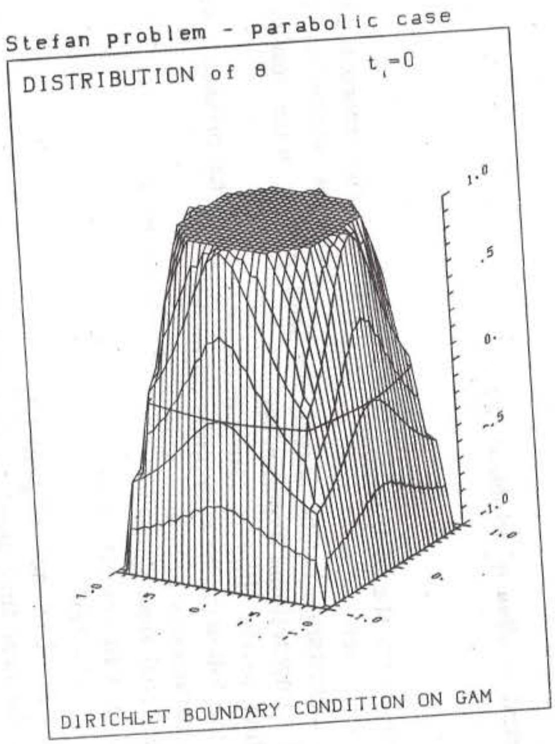
(iii) set $w^{i+1} = w^i + k\,\delta w^i$

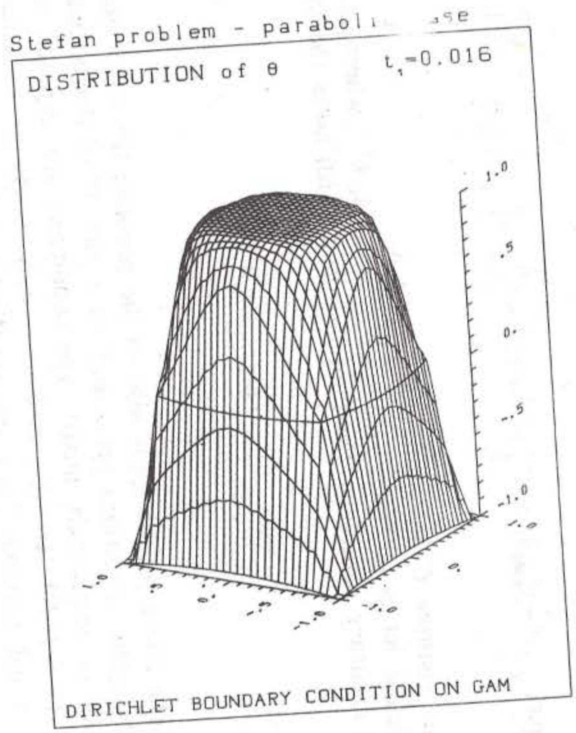
(iv) test if $i < N-1$; if not, skip to (vi)

(v) pass to the next time step, $i \to i+1$; return to (ii)

(vi) stop.

To perform the step (ii) of the algorithm, we solved the minimization problem:


Stefan problem - parabolic case
DISTRIBUTION of θ
t₁=0
1.0
.5
0.
-.5
-1.0
DIRICHLET BOUNDARY CONDITION ON GAM


Stefan problem - parabol
DISTRIBUTION of θ
t₁=0.016
1.0
.5
0.
-.5
-1.0
DIRICHLET BOUNDARY CONDITION ON GAM

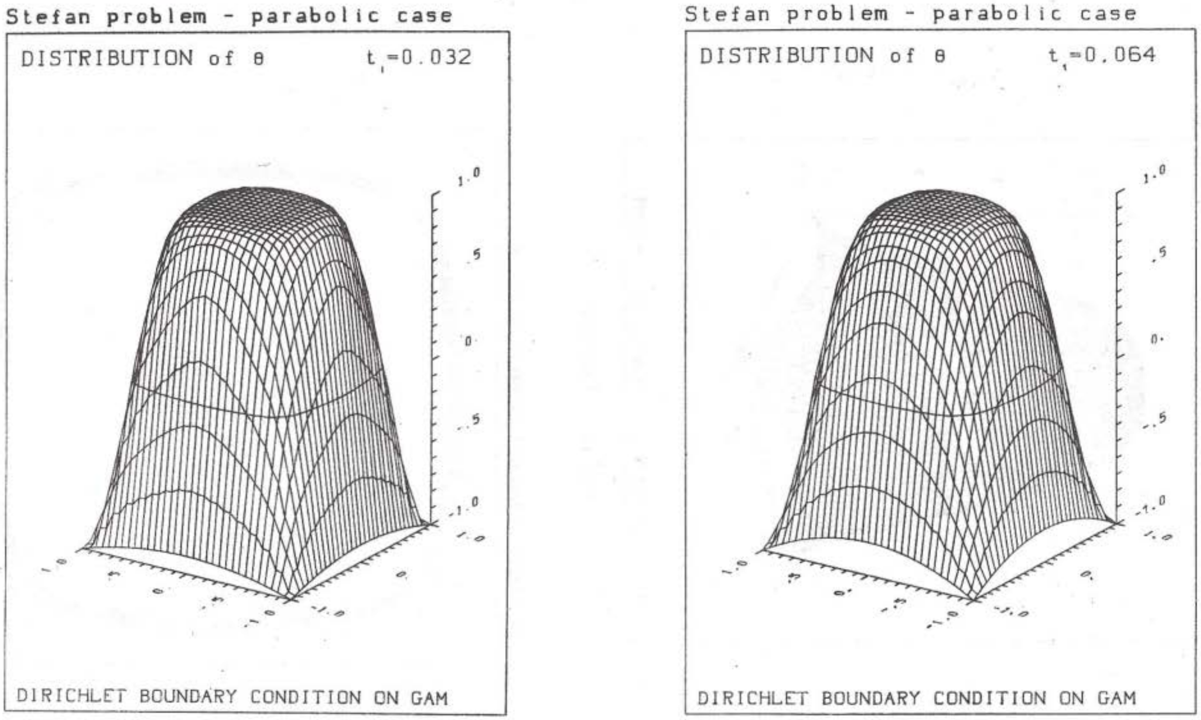


Fig. 1a. 1–4 Example 1

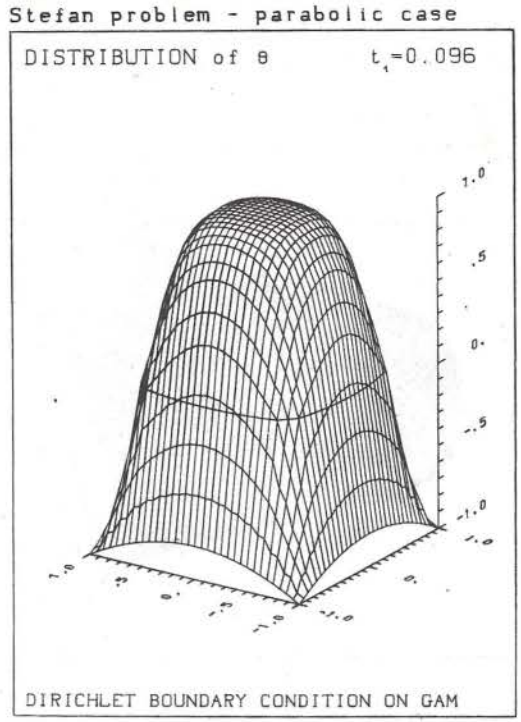


Fig. 1a. 5 Example 1

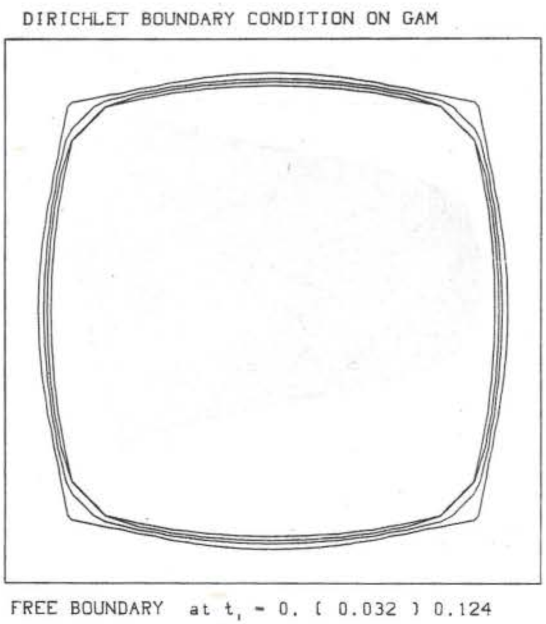


Fig. 1b. Example 1

Determine $\delta w^i \in V_h$, $i = 0, \ldots, N-1$, such that

$$J^i_{\mu\varepsilon}(\delta w^i + \delta\omega^i_h) = \inf\{J^i_{\mu\varepsilon}(z + \delta\omega^i_h);\ z \in V_h\},$$

where ω_h denotes projection of ω onto V_h.

The last formulation covers also the case of the Dirichlet data imposed on $\Gamma' \subset \Gamma$.

The computations were performed both in the parabolic and in the degenerate cases. Our interest was focused on the efficiency of the proposed scheme in the degenerate case, therefore:

(i) a practical accuracy of the parabolic regularization was test ($\mu \to 0+$),

(ii) influence of the degeneracy onto the number of iterations necessary for achieving the accuracy comparable with that, in the parabolic case was discussed.

The problem (P) was studied in the extended form, with the Dirichlet data prescribed on the whole Γ so that the free boundary (level set $\theta = 0$) did not touch Γ. As the domain we took $\Omega \equiv [-1, 1] \times [-1, 1] \subset R^2$, $T = 0.256$. We have admitted alternatively $\tilde{\gamma}(r) \equiv 0$ (degenerate case) or $\tilde{\gamma}(r) = r$, $r < 0$ and $\tilde{\gamma}(r) = 5r$, $r > 0$ (parabolic case). As the parabolic regularization parameters we chose $\mu = 10^{-n}$, $n = 1, 2, 3$. For solving the minimization problem in the step (ii) of the algorithm the SOR method was used in the case of the implicit scheme. A regular triangulation $\mathcal{T}_h$ of Ω was constructed including 289 nodes and 512 elements. The discretization parameters: $h = 0.125$, $k = 0.008$.

The behaviour of the approximate solutions is shown in Fig. 1 (parabolic case) and Fig. 2 (degenerate case), in both cases (a) refer to the evolution of the solution in time, (*b*) show the corresponding movement of the free boundary. As a representation of the solution y its time derivative $y' = \theta$ is used. In the parabolic case, θ can be interpreted as temperature (with freezing value 0), in the degenerate case — as potential or saturation, in particular (see also [1]).

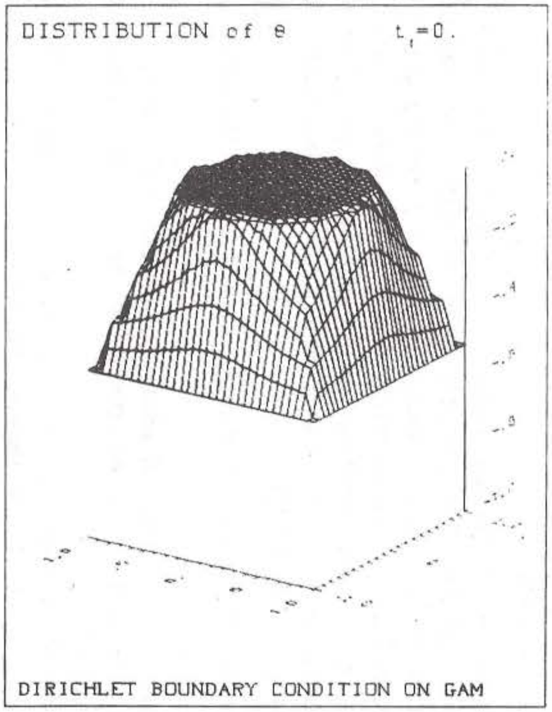
The results of the performed experiments led us to the following observations:

(i) accuracy of the numerical solutions was not influenced by the regularizations, one could see it by comparing the relevant evolution of the free boundaries;

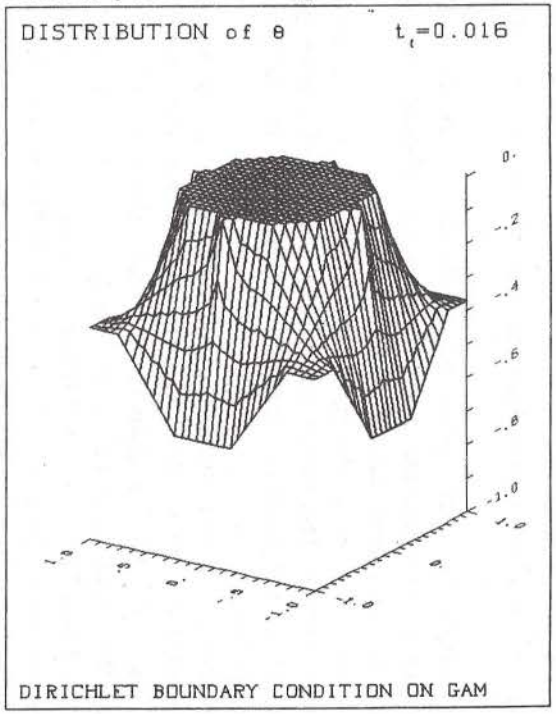
(ii) number of iterations in the SOR process was heavily dependent upon the shape of $\tilde{\gamma}$; in the degenerate case, to provide the same accuracy as in the parabolic case one needed in average two up to three times more iterations;

(iii) in the degenerate problem under consideration we were able to reproduce the shape of Γ at the free boundary, as it could be physically expected in the electrochemical machining process, in particular.

Stefan problem - degenerate case



Stefan problem - degenerate case



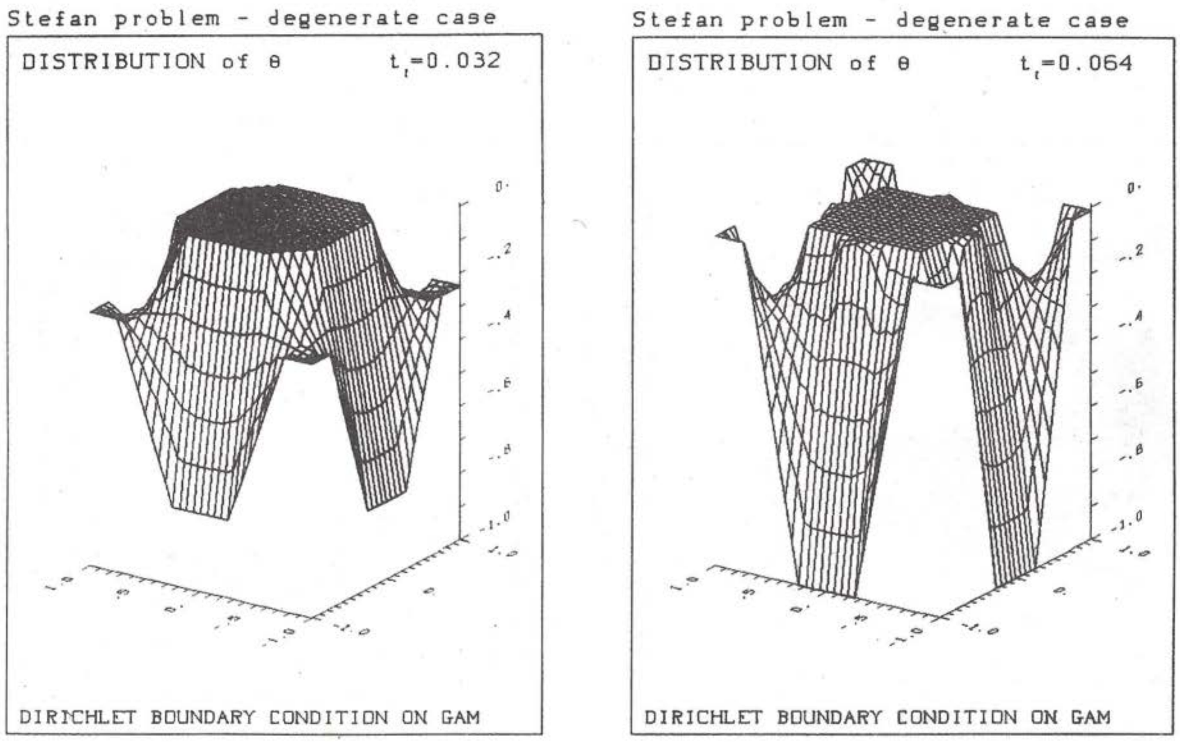


Fig. 2a. 1–4 Example 2

Stefan problem - degenerate case

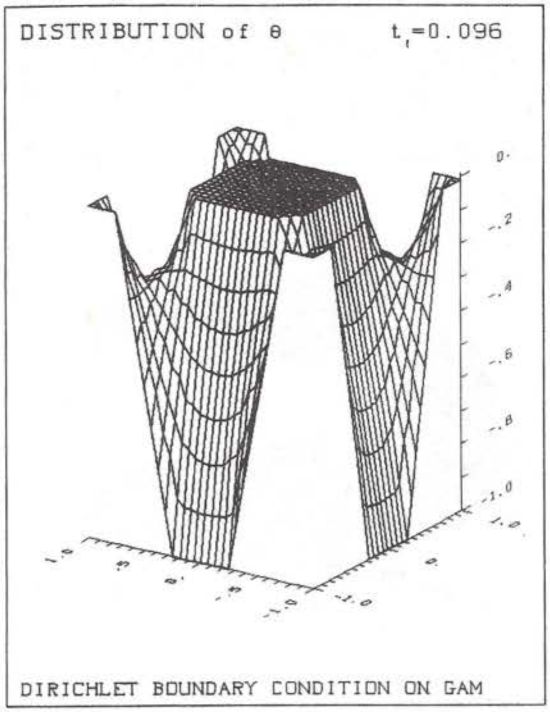
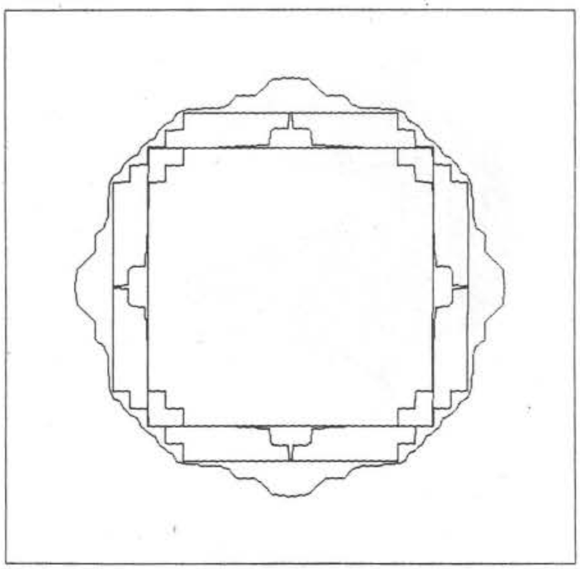


Fig. 2a. 5 Example 2

DIRICHLET BOUNDARY CONDITION ON GAM



FREE BOUNDARY at t_i = 0. (0.032) 0.124

Fig. 2b. Example 2

Acknowledgement

The computations were performed on NORSK DATA 540 at the Mathematical Institute of the University of Augsburg. The authors would like to thank Mr. K. Bernt for his assistance in carrying out the experiments.

References

[1] ELLIOTT C. M., OCKENDON J. R. Weak and Variational Methods for Moving Boundary Problems. Boston, Pitman, 1982.
[2] JEROME J. W. Approximation of Nonlinear Evolution Systems. New York, Academic Press, 1983.
[3] JEROME J. W., ROSE M. E. Error estimates for the multidimensional two-phase Stefan problem. *Math. of Comput.*, **39** (1982), 377–414.
[4] PAWLOW I. A variational inequality approach to generalized two-phase Stefan problem in several space variables. *Ann. Matem. Pura ed Applicata*, **131** (1982), 333–373.
[5] PAWLOW I. Approximation of an evolution variational inequality arising from free boundary problems. In: Optimal Control of Partial Differential Equations, K.-H. Hoffmann, W. Krabs, Eds. Basel, Birkhäuser-Verlag, 1984, 188–209.
[6] PAWLOW I. Approximation of variational inequality arising from a class of degenerate multi-phase Stefan problems, submitted to Numerische Mathematik.
[7] PAWLOW I., SHINDO Y., SAKAWA Y. Numerical solution of a multidimensional two-phase Stefan problem. *Numer. Func. Analysis & Optimization*, **8** (1985), 55–82.
[8] STREIT U. Zur Konvergenzordnung von Differenzenmethoden für die Approximation schwacher Lösungen instationärer Differentialgleichungen. *Beiträge zur Numer. Mathematik*, **12** (1984), 181–189.

Analiza numeryczna zdegenerowanych zagadnień Stefana

W pracy analizowana jest zbieżność dyskretnych aproksymacji zdegenerowanych zagadnień typu Stefana. Zagadnienia te są sformułowane w postaci nierówności wariacyjnych. Przedstawiona zostaje dyskusja wyników przeprowadzonych eksperymentów numerycznych.

Численный анализ проблем Стефана с особенностями

В работе рассматривается сходимость дискретных аппроксимаций для проблем типа Стефана с особенностями. Эти проблемы формулируется в виде вариационных неравенств. Представлена дискуссия результатов численных экспериментов.