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# The models of thermocapillary motion

by

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Thermocapillary effect is thought to be one of the major causes of fluid motion in weak forced fields. This paper represents the review on mathematical models of thermocapillary motion. The results obtained in the Lavrentyev Institute of Hydrodynamics, Krasnoyarsk Computing Center of the Siberian Division of the USSR Academy of Sciences and Novosibirsk State University form the basis of this review.

## 1. Formulation of the problem

We consider nonisothermal motion of a viscous incompressible fluid with a free surface which for simplicity is assumed to be connected. Main attention is paid to partial or developed weightlessness. In this case, in the absence of a surfacely active matter, the surface tension coefficient as being function of temperature is the major cause of fluid motion.

A mathematical formulation of the above-mentioned problem is as follows [1, 2]. It is required to find a region $\Omega(t) \subset R^3$, $t \in (0, T)$ bounded by both a solid impermeable unmovable surface and a free surface $\Gamma(t)$ and a solution $\vec{v}(x, t), p(x, t), \theta(x, t)$ of the Oberbeck-Boussinesq system of equations

$$\begin{gathered} \vec{v}_t + \vec{v} \cdot \nabla \vec{v} = -\varrho^{-1} \nabla p + \nu \Delta \vec{v} - \beta \theta \vec{g}, \\ \nabla \cdot \vec{v} = 0, \theta_t + \vec{v} \cdot \nabla \theta = \chi \Delta \theta \end{gathered} \tag{1.1}$$

in the region satisfying the initial conditions $\Omega(0) = \Omega^0$

$$\vec{v}(x, 0) = \vec{v}^0(x), \ \theta(x, 0) = \theta^0(x), \ x \in \Omega^0, \tag{1.2}$$

the boundary conditions on the unmovable boundary

$$\vec{v} = 0, \ \theta = a(x, t), \ x \in \Sigma, \ t \in (0, T) \tag{1.3}$$

and the boundary conditions on the free surface $\Gamma(t)$ (for simplicity it is assumed that $\bar{\Sigma} \cap \bar{\Gamma} = \emptyset$)

$$p - 2\varrho\nu\vec{n}\cdot D\cdot\vec{n} = 2\sigma H - \varrho\vec{g}\cdot\vec{x} + C, \tag{1.4}$$

$$2\varrho\nu D\cdot\vec{n} - 2\varrho\nu\,(\vec{n}\cdot D\cdot\vec{n})\,\vec{n} = \nabla_\Gamma\,\sigma, \tag{1.5}$$

$$\nabla\theta\cdot\vec{n} + \alpha\,(\theta - b\,(x, t)) = 0, \tag{1.6}$$

$$\vec{v}\cdot\vec{n} = V_n,\ x \in \Gamma\,(t),\ t \in (0, T). \tag{1.7}$$

Here $\vec{v}$ is the velocity, $p$ is the pressure fluid deviation from a hydrostatic value, $\theta$ is the fluid temperature, $\vec{g} = \vec{g}\,(t)$ the gravitational acceleration. In (1.4)–(1.7) $\vec{n}$ denotes the unit vector of an external normal to $\Gamma$, $D = (\nabla\vec{v} + (\nabla\vec{v})^*)/2$ is the deformation velocity tensor, $H$ is the mean surface curvature, $\sigma = \sigma\,(\theta)$ is the surface tension coefficient, $C = C\,(t)$ is the parameter connected with a mean depth or volume of region $\Omega\,(t)$, $V_n$ is the velocity of $\Gamma$ displacement towards $\vec{n}$, $\nabla_\Gamma = \nabla - (\vec{n}\cdot\nabla)\,\vec{n}$ is the surface gradient. Positive values of $\nu$ (kinematic viscosity coefficient), $\varrho$ (density), $\chi$ (heat conductivity coefficient), $\beta$ (coefficient of thermal expansion), $\alpha$ (interphase heat exchange coefficient) are assumed constant. The prescribed vector function $\vec{v}^0$ and positive functions $\theta^0, a, b$ satisfy natural consistency conditions which are not specified here. Physically, the function $\sigma\,(\theta)$ assumes positive meanings also for pure liquids, $d\sigma/d\theta < 0$. (An anomalous thermocapillary effect observed in aqueous solutions of high-molecule alcohols [3] will not be considered). For simplicity, let us assume that

$$\sigma = \sigma_0 - \varkappa\,(\theta - \theta_0) \tag{1.8}$$

where $\sigma_0, \theta_0, \varkappa$ (temperature coefficient of surface tension) are positive constants. An import limiting case of (1.6) when $\alpha = 0$ corresponds to a heat insulated free boundary. Instead of (1.6) it is suitable to consider the condition

$$\theta = b\,(x, t),\ x \in \Gamma\,(t),\ t \in (0, T), \tag{1.9}$$

formally obtained from (1.6) by the limiting transition $\alpha \to \infty$.

In the last equation of (1.1) the term which describes the heat inflow due to kinetic energy dissipation is not taken into account. Its value is expressed by $2\nu c^{-1}\,D\!:\!D$, where $c$ is the heat capacity of a fluid. In the case of convective motion the above-mentioned term achieves significant values only in the boundary layers generated by a great temperature drop along the free surface.

## 2. Basic parameters

The relations (1.1)–(1.8) represent a basic mathematical model of gravitational capillary convection in a homogeneous fluid. A relative role of either of the convection mechanism is characterized by the dimensionless parameter $L = \varrho g \beta h^2 \varkappa^{-1}$, where $h$ is the extent of region $\Omega$ in the direction of a gravity force, $g = |\vec{g}|$. If $L \ll 1$, the thermocapillary effect contribution into convection appears dominating.

Technological experiments in weightlessness usually are made at $h < 5$ cm [4]. The value of $g$ was about $10^{-2}$ cm/s$^2$ for an unmanned flight of the space station "Salute-6" and about 1 cm/s$^2$ for a regular work of the spacecraft crew [5] Since for the majority of liquids $\varrho\beta \cdot \varkappa^{-1} < 10^{-2}\, s^2/\mathrm{cm}^3$, the parameter $L$ does not exceed $0.25 \times 10^{-2}$ and 0.25 for the first and second case, respectively.

The intensity of thermocapillary convection is characterized by the Marangoni number $M = \varkappa\theta_* h/\varrho\nu\chi$, and that of gravitational convection — by the Grasshof number $Gr = LM$ (here $\theta_*$ is the characteristic temperature drop). Sometimes instead of $M$ it is appropriate to introduce the modified Marangoni number $Ma = M/Pr$, where $Pr = \nu/\chi$ is Prandtl number.

The parameter $A = \varkappa\theta_*/\sigma_0$ responsible for the free surface deformation by surface thermocapillary forces is of great importance in our considerations. As usual, this parameter is small, e.g., in the experiments with germanium crystallization using the CRYSTAL setup it was $A < 0.09$ [6], and on detecting thermocapillary bubble drift using the PION setup — $A < 0.05$ [4].

In the case of an equilibrium form of an isothermal free surface the Bond number $Bo = \varrho g h^2/\sigma_0$ is the determining parameter. In the conditions of an orbital flight it is about $10^{-4}$–$10^{-3}$. The Biot number $Bi = \alpha h$ characterizes the intensity of heat exchange between liquid and gaseous phases. In contrast to the above-mentioned values, the parameter $\alpha$, together with the Biot number, is determined with a significantly less accuracy, due to a semi-empirical character of (1.6).

## 3. Exact solutions

The problem (1.1)–(1.8) only recently became an object of mathematical considerations. Up to now an unknown boundary, nonlinearity and high order of the system of equations have not allowed us to obtain general results guaranteeing unique solvability of an initial boundary-value problem. Its numerical solution appears very complicated due to the same reasons. Therefore a construction of any classes of special solutions of (1.1)–(1.8) is of great interest. Each special solution can be regarded as a model of some thermocapillary motion.

In [7] the group-theoretic analysis of differential equations [8] is applied to investigate the invariance properties of the boundary value problem (1.1)–(1.8) for $\vec{g} = 0$, besides the conditions on the function $a(x, t)$, $b(x, t)$ are formulated, such that the problem has invariant solutions. Among the solutions described in [7] those are to be emphasized which describe plane or axisymmetrical deformation of a liquid film due to the gaseous phase temperature inhomogeneity having the form $b = b_2(t)\, r^2 + b_0(t)$, where $r = |x_1|$ and $r = (x_1^2 + x_2^2)^{1/2}$ in the plane and axisymmetrical case, respectively.

Another example illustrates a plane stationary self-similar flow in the sector where one boundary is free and heat isolated and the other is a solid wall [9]; here $a(x)$ is the power function of polar radius. The velocity field at the origin has a singularity of source or sink type. Earlier an axisymmetrical stationary selfsimilar solution was built with a plane free boundary, the latter being the position of a heat source [10].

The class of exact solutions of (1.1)–(1.8) is not wide in the general case $\vec{g} \neq 0$. The situation considered by Birikh [11] and its generalization [12] are of particular importance. These solutions correspond to the case $\vec{g} =$ const. and describe plane stationary flows, where the free boundary is a plane, $x_2 = 0$, and the functions $a, b$ are linearly dependent on $x_1$.

## 4. A plane stationary problem

It is assumed that $\vec{v}, \theta, p$ and $a$, $b$, $c$ do not depend on $x_3$ and $t$, and $\Sigma$ and $\Gamma$ are the cylindrical surfaces with a generatrix parallel to the axis $x_3$, the surface $\Gamma$ being unmovable, so that $V_n = 0$. In this case the solution of (1.1)–(1.8) describes a plane stationary motion. The above problem, under additional assumptions $\vec{g} = 0$ and $\bar{\Sigma} \cap \bar{\Gamma} = \emptyset$ was first investigated in [13] using a complex representation of the solution of Navier-Stokes equation [14]. The existence and uniqueness of the solution at small Marangoni numbers, $Ma$, was proved, an iterative method of its numerical solution was justified and branching of rest and quasisolid rotation was studied.

Let us consider the latter result in more detail. It is assumed that $\Sigma$ is the circumference $\tau = (x_1^2 + x_2^2)^{1/2} = R_0$ rotating round its centre with an angular velocity $\omega$, and that $\vec{g} = 0$, $a =$ const., $b =$ const. Then the problem (1.1), (1.3)–(1.8) has an exact solution, which, in a rotating system of coordinates, corresponds to rest with temperature distribution $\theta = \theta_* \ln(R/r)$, where $\theta_* =$ const. $> 0$, and circular free boundary $\Gamma: r = R > R_0$ (basic solution). In this case $a = \theta_* \ln 1/\varepsilon$, $b = -\theta_*\, Bi$, $\varepsilon = R_0/R$, $Bi = \alpha R$ (Biot number). Let us introduce the dimensionless parameter $We = \varrho\nu\chi/\sigma_0 R$ (Weber number), $Bo = \varrho\omega^2 R^3/\sigma_0$ (analog of Bond number) and $M = \varkappa\theta_* R/\varrho\nu\chi \ln 1/\varepsilon$ (Marangoni number), the latter being the bifurcational parameter.

It has been proved in [15] that there exist secondary flows branching from the basic one when

$$M_k = M_k^0 \frac{1 + c_k\, Bi}{1 + a_k\, We\, (Bo + k^2 - 1)^{-1}}, \quad k = 1, 2, \ldots$$

where $M_k^0$, $c_k$, $\alpha_k$ are the positive functions of parameter $\varepsilon$. The numbers $M_k^0$ corresponding to the undeformable heat isolated free boundary were previously calculated in [16]. It is obvious that taking into account deformability $\Gamma$ gives rise to a decrease in spectral numbers $M_k$.

The secondary flow has a structure $2k$ of convective shafts and is symmetrical with respect to the fixed straight line passing through the origin. The spectral numbers $M_k$ are at a minimum at some $k = k_m(\varepsilon)$. The number of $k_m$ can be estimated as follows: the convective shaft size along an angular coordinate $\varphi$ must have an order of the layer thickness $R - R_0$.

The methods described in [13] admit a natural generalization for the cases when $g =$ const. and the lines $\Sigma$ and $\Gamma$ have contact points. It deserves a remark that in [17] the class of exact solutions of the linearized system (1.1) has been found, which described thermocapillary motion in a circular region or half-plane. In that case the free boundary is adjacent to the solid wall and the temperature distribution along it can be arbitrary.

## 5. A weakly deformable free surface

As it has been mentioned above, the question of solvability of (1.1)–(1.8) in the general case remains open. In addition to technical difficulties, the absence of a closed statement of the problems of such kind in the cases when surfaces $\Sigma$ and $\Gamma$ have a moving contact line is a serious handicap for solving this problem (see a discussion of this problem and references cited in [18]). In the meantime, in applications there can arise situations when the problem (1.1)–(1.8) may be splitted with a high accuracy into two successively solvable problems, of the determination of the form of the free boundary and determination of temperature and flow field in the known region. Validity of the condition $\vec{g} =$ const., $A = \varkappa\theta_*/\sigma_0 \ll 1$ provides a good justification for such decomposition (all other determining parameters are of the order of unity). It should be noted that just this scheme is used to calculate thermocapillary motion (see, e.g., [19–21]). The works [22, 23] represent an exception to this rule.

Thus, it is assumed that the parameter $A$ is small, $\vec{g}$ is idependent of $t$, and it is required to find the components $\vec{v}, p, \theta$ of the solution of (1.1)–(1.8) in the form of formal power series over the whole non-negative powers of parameter $A$. A similar expansion is used for both the function determining the free boundary (symbolically it can be represented as $\Gamma = \Gamma^{(0)} + A\Gamma^{(1)} + \dots$) and the functional $C$ entering the condition (1.4): $C = C^{(0)} + AC^{(1)} + \dots$ The corresponding characteristic scales of time, velocity, modified pressure and value of $C$ are respectively $\varrho\nu/\varkappa\theta_* h$, $\varkappa\theta_*/\varrho\nu$, $\varkappa\theta_*/h$ and $\sigma_0/h$. The characteristic linear size $h$ and temperature drop $\theta_*$ are assumed a priori known.

As far as $A$ is concerned, if $A \to 0$ then we obtain

$$2\sigma_0 H^{(0)} + \varrho \vec{g} \cdot \vec{x} + C^{(0)} = 0, \tag{5.1}$$

which in principle gives us the surface $\Gamma^{(0)}$. For definiteness the liquid is assumed to move in an open vessel, then the following boundary condition is added to (5.1):

$$\vec{n}^{(0)} \cdot \vec{n}_\Sigma = \cos \gamma, \tag{5.2}$$

which is to be fulfilled on the surface $\zeta$ between surfaces $\Gamma^{(0)}$ and $\Sigma$. Here $\vec{n}^{(0)}$ and $\vec{n}_\Sigma$ are the unit vectors of normals to $\Gamma^{(0)}$ and $\Sigma$, $\gamma$ is the boundary angle determined from the properties of contacting media.

An extensive bibliography in [1] is devoted to the problem (5.1), (5.2); also the works [24, 25] deserve an attention in this respect. Further we postulate that this problem has a classical solution $\Gamma^{(0)} \in C^2$ (Sufficient conditions for this are $\Sigma$ is the cylindrical surface whose directrix is a curve line of class $C^3$; $\vec{s} \cdot \vec{g} > 0$, where $\vec{s}$ is the vector parallel to the generatrix $\Sigma$ and directed from the gaseous phase to the liquid one; $0 < \gamma < \pi$ [26]).

Zero order terms $\vec{v}^{(0)}, p^{(0)}, \theta^{(0)}$ in the power expansion of the solution with respect to $A$ satisfy the equation (1.1) in the known region $\Omega^{(0)}$ bounded by surfaces $\Sigma$ and $\Gamma^{(0)}$, initial conditions (1.3) on the surface $\Sigma$ and conditions (1.5)–(1.7) on $\Gamma^{(0)}$. In the latter conditions $\vec{n}$ is to be substituted by $\vec{n}^{(0)}$, $\nabla_\Gamma \sigma$ by $-\varkappa \nabla_{\Gamma^{(0)}} \theta^{(0)}$ and it should be noted that $V_n = 0$, since $\Gamma^{(0)}$ is unmovable.

So far any mathematical facts have not been established for (1.1)–(1.3), (1.5)–(1.8), even though this problem is of natural origin and great significance in view of applications. The uniqueness of the classical solution alone is proved for it in a relatively simple manner. It is difficult to prove the solvability of this problem "in global" due to the absence at present of sufficiently strong a priori estimates.

Now let us give a stationary formulation of the problem under consideration. Index (0) of he required functions and symbols $\Gamma^{(0)}$ and $\vec{n}^{(0)}$ will be omitted. It is required to find the solution $\vec{v}(x), p(x), \theta(x)$ of the system of equations

$$\begin{gathered} \vec{v} \cdot \nabla \vec{v} = -\varrho^{-1} \nabla p + \nu \Delta \vec{v} - \beta \theta \vec{g}, \\ \nabla \cdot \vec{v} = 0, \ \vec{v} \cdot \nabla \theta = \chi \Delta \theta \end{gathered} \tag{5.3}$$

in the region $\Omega$ with the boundary $\Sigma \cup \Gamma \cup \zeta$ satisfying the boundary conditions

$$\vec{v} = 0, \theta = a(x), \ x \in \Sigma, \tag{5.4}$$

$$2\varrho\nu D \cdot \vec{n} - 2\varrho\nu (\vec{n} \cdot D \cdot \vec{n}) \vec{n} = -\varkappa \nabla_\Gamma \theta, \ x \in \Gamma, \tag{5.5}$$

$$\nabla\theta \cdot \vec{n} + \alpha (\theta - b(x)) = 0, \ \vec{v} \cdot \vec{n} = 0, \ x \in \Gamma. \tag{5.6}$$

If the functions $a$ and $b$ slightly differ from a constant, it is appropriate to use the method of successive approximations to solve the problem (5.3)–(5.6). The results obtained in [27] on the boundary value problem for a stationary system of Navier-Stokes equations with mixed boundary conditions (5.4)–(5.6) allow us to hope for the convergence of an iterative process at low Marangoni numbers.

## 6. A linear model

As a rule, in order to understand a mathematical nature of non-linar problems, it is appropriate first to investigate relevant linearized models. In the problem under consideration the simplest linear model arises from linearization (1.1)–(1.7) at rest ($\vec{v} = 0$) with a constant temperature distribution. Such linearization is based on the representation $a = \theta_* + \varepsilon f_1(x, t)$, $b = \theta_* + \varepsilon f_2(x, t)$, where $\theta_* = \text{const.}$, $\varepsilon$ is the small parameter and $f_1$ and $f_2$ are the smooth functions independent of $\varepsilon$.

It is readily seen that after series expanding of the solution in series in terms $A$ in zero approximation the free surface $\Gamma$ is determined as an equilibrium capillary surface in a gravity force. In the first approximation there arises a problem in the known region $\Omega$ which is splitted into two successively solvable problems. The first, trivial problem consists in finding the solution of a linear heat equation with the conditions (1.3), (1.6). The second is the problem of finding the solution of the system of Stokes equations with inhomogeneous conditions on the free boundary $\Gamma$. The right hand side of the equation (1.1), where the term $\vec{v} \cdot \nabla \vec{v}$ is omitted, and the right-hand sides of (1.4), (1.5) include the pre-determined function $\theta$ and its derivatives. Probably this problem has not been studied in a general statement, though a necessary mathematical apparatus in reality has been already developed (see Chapter VI of the book [1]). At the same time, there is a great number of works, where the above-mentioned problem has been solved for specific regions $\Omega$ of a rather simple form (see the paper [28], where the case of an infinite circular cylinder with a free boundary has been considered in detail, as well as references therein).

The proof of existence and uniqueness theorems in the linearized problem has mainly a methodological meaning. However, a linar model plays a principal role in the study of the problem of Lyapunov motion stability. The thermocapillary instability mechanism was first studied theoretically by Pearson in [29]. He investigated the instability of a weightless plane layer with an undeformable free boundary heated from below. If $a = \text{const.}$, $b = \text{const.}$, liquid is at rest, and its temperature is a linear function of the coordinate normal to the layer boundaries. This equilibrium is stable only if $M < M^*(Bi)$, where $M$ is Marangoni number calculated from the temperature drop on the boundaries and thickness of the layer. When $M > M^*$, the equilibrium becomes unstable. The function $M^* (Bi)$ monotonically increases with increasing Biot number $(Bi)$, $M^* \approx 79.6$, when $Bi = 0$, $M^* \to \infty$ when $Bi \to \infty$ [29].

In [30] the problem of stability is studied for the situation more typical for weightlessness, when both layer bounaries are free. It is worth to mention also the results of Chapter VIII in book [1], concerning the problem of thermocapillary equilibrium stability of the liquid filling in a spherical layer (one boundary is free and the other is solid) or a rectangular channel (here a free boundary is plane that corresponds to a wetting angle $\pi/2$); in both cases $\vec{g} = 0$.

As regards the thermocapillary stationary motion stability, the first publications on this matter appeared quite recently [31, 32]. They are devoted to the stability of plane and axisymmetrical flows [11, 33], where free boundaries are straight lines or cylindrical surfaces, and temperature is a linear function of a homogeneous coordinate. It should be noted that the resulting spectral problems do not admit a full separation of variables. Application of the longitudinal coordinate "freezing" in the condition (1.4), however, makes it possible to reduce the above problems to eigenvalue problems for a system of ordinary differential sixth-order equations (This approach has been used long ago in the analysis of flow stability in a boundary layer). By combination of asymptotical and numerical methods the authors of [31, 32] built numerical curves on the "wave number — Marangoni number" plane, studied different limiting cases of disturbance behavior, calculated asymptotics of eigenvalues and eigenfunctions. It has been shown, in particular, that taking into account the free boundary deformability can lead to the flow instability with respect to long-wave disturbances.

## 7. Motion in a thin layer

Since thermocapillary motion is induced by surface forces, the thermocapillary effect is expected to be most significant in thin liquid layers. Approximate equations describing thermocapillary convection in a thin layer were derived by a number of authors under different simplifying assumptions. This has been done successfully in the works [34–36], where a review of previous investigations is also available.

The liquid is assumed to be bounded by a solid plane $x_3 = 0$ and free surface $x_3 = \eta(x_1, x_2, t)$, which is projected uniquely onto the above plane. Let us also assume for simplicity that $\beta = 0, \vec{g} = (0, 0, -g) = \text{const}$. The basic hypothesis, which allows to simplify radically the problem (1.1)–(1.8) consists in the assuming that there exist two characteristic length scales in the motion, namely the longitudinal scale $l$ and the transversal scale $h$, thereby $\varepsilon = h/l \ll 1$. For $h$, it is natural to choose $\max \eta_0(x_1, x_2)$, where

$\eta_0 = \eta(x_1, x_2, 0)$ is given function taking non-negative values according to its physical sense. For $l$, it is appropriate to choose the diameter supp $\eta_0$ in the case of a finite function $\eta_0$ (this corresponds to a drop on a plane), and the least period $\eta_0$ in the case when this function is periodical, etc.

By an asymptotic integration (when $\varepsilon \to 0$) of the equations (1.1) across the layer and eliminating the functions $\vec{v}, p$ due to the conditions (1.3)–(1.5), (1.7), we obtain the equation describing the layer thickness evolution $\eta$:

$$\eta_t - \nabla \cdot \left[ \frac{\varkappa \eta^2}{2\varrho\nu} \nabla \theta_\Gamma + \frac{\eta^3}{3\nu} \nabla \left( g\eta - \frac{\sigma_0}{\varrho} \Delta \eta \right) \right] = 0 \tag{7.1}$$

(throughout Section 7 $\nabla$ and $\Delta$ denote two-dimensional gradient and Laplacian respectively. If free boundary temperature satisfies the first-kind condition (1.9), $\theta_\Gamma = b(x, t)$ and the equation (7.1) becomes closed. This situation was considered in [35] where a one-dimensional analog of (7.1) was obtained for $\sigma_0 = 0$. For a uniqueness of the function $\eta$ the following initial condition is to be added to (7.1):

$$\eta = \eta_0(x_1, x_2) \text{ at } t = 0. \tag{7.2}$$

If (1.6) is taken as a condition for the surface temperature, in the thin layer approximation we have

$$\theta_\Gamma = \frac{a + \alpha b \eta}{1 + \alpha \eta}, \tag{7.3}$$

where $a$, $b$ are the known functions of $x_1, x_2$ and $t$, $\alpha = \text{const.} > 0$ is the interface heat exchange coefficient.

In an important specific case when $a$ and $b$ were constant the problem (7.1)–(7.3) was studied in [34], $a \neq b$. It is expedient to emphasize such qualitative results as the existence of stationary solutions with a finite support (the wetting angle $\gamma$ on the drop boundary is to be of the order of $\varepsilon$) and the possibility to stabilize a liquid layer with constant thickness positioned on a "ceiling" ($g < 0$) by thermocapillary forces for sufficiently small values of dimensionless parameter $\varrho\,|g|\,h^2/\varkappa\,|a-b|$.

The solvabiliry of the Cauchy problem (7.1)–(7.3) for finite initial data $\eta_0$ is an open question. It seems that is succeeds in a large measure — it is so due to the difficulties typical for the problems of viscous fluid with a moving line of a three, phase contact [18].

## 8. Marangoni boundary layer

Our above considerations were based on the assumption that the Marangoni number is of order of unity. In some technological experiments under the conditions of weightlessness (see, for example, [6]) thermocapillary motion with high Marangoni numbers are realized. This fact stimulated the development of a Marangoni boundary layer theory [37–39]. We will restrict ourselves to formulation of the simplest boundary layer of this theory in the plane stationary case, ignoring buoyancy forces ($L \ll 1$) and free boundary deformation ($A \ll 1$).

It is required to find the solution $w(x, \psi)$ of the equation

$$w_x = \nu \sqrt{w}\, w_{\psi\psi} - 2dp/dx \tag{8.1}$$

in the region $\Pi_e = \{x, \psi : 0 < x < l, \psi > 0\}$, satisfying the conditions

$$\begin{gathered} w = w_0(\psi) \text{ if } x = 0, \psi \geqslant 0, \\ w_\psi = \tau(x) \text{ if } 0 \leqslant x \leqslant l, \psi = 0, \\ w \to V^2(x) \text{ if } 0 \leqslant x \leqslant l, \psi \to \infty . \end{gathered} \tag{8.2}$$

The relations (8.1), (8.2) have been expressed in the von Mises variables [40], so that $x$ is the free boundary arc length reckoned from some point on it, $\psi$ is the stream function, $w$ the longitudinal velocity square. Pressure $p$ is connected with the outer flow velocity $V$ by the equality $2p + V^2 = \text{const}$. The function $\tau(x)$ is proportional to a tangential deviation of temperature on the free boundary.

The problem (8.1)–(8.2) differs from the classical Prandtl problem studied in [41] by the kind of the boundary condition at $\psi = 0$. Generally speaking, (8.1) does not degenerate on the free boundary, however, its degeneration is possible when $\psi \to \infty$. This occurs when the Marangoni boundary layer flow is conjugated with a rest $V = 0$. The function $w_0$ is assumed to have a finite support, and the conditions which guarantee a global solvability of the problem (8.1), (8.2) are fulfilled: the function $w_0$ satisfies Lipschitz condition and is non-negative, the function $\tau$ is continuous and negative. Then at any fixed $x > 0$ the support of function $w$ will also be finite. This disturbance localization effect has no analog in the Prandtl boundary layer theory.

The problem (8.1), (8.2) was investigated in [42]. Let us formulate one of the results of that work. Let $w_0 > 0$, $dw_0/d\psi < 0$ for $0 \leqslant \psi < \infty$, $w_0 \in C^{2+\alpha}[0, \infty)$, $0 < \alpha < 1$, $V > 0$ for $0 \leqslant x \leqslant l$, $V \in C^{1+\alpha/2}[0, l]$, $w_0(\infty) = V^2(0)$, $\tau < 0$ for $0 \leqslant x \leqslant l$, $\tau \in C^{(1+\alpha)/2}[0, l]$. Then at any $l > 0$ there exists a unique solution $w \in C^{2+\alpha, 1+\alpha/2}(\bar{\Pi}_e)$ of (8.1), (8.2), thereby $w > 0$, $w_\psi < 0$ for $(x, \psi) \in \Pi_e$. It should be emphasized that in contrast to the Prandtl problem, for a global solvability of the problem under consideration "in a global" there is no need to impose the condition $p_x \leqslant 0$ for $x \in [0, l]$.

It should be noted that on the basis of the Marangoni boundary layer theory an algorithm for calculating thermocapillary flow in an ampule partially filled by the semiconductor melting has been developed [43].

We do not consider here the complex problems of a nonstationary Marangoni boundary layer which almost have not been studied up to now. A special publication of the author will be devoted to such problems.

## 9. A two-layer system

The thermocapillary convection model (1.1)–(1.8) under consideration and its simplified variants correspond to the idealized situation, when a dynamical effect of a gaseous phase on a liquid one can be ignored, and a thermal effect is modelled by the condition (1.6). In the meantime, already in [44] it has been shown that taking into account transfer processes in the gaseous phase effects qualitatively in the neutral curve behaviour over the range of small wave numbers of the Pearson problem [29].

In this connection a question arises: to what extent is the model proposed in Section 7 limiting from the point of view of the "two-layer" approach, in which one layer is liquid and the other is gaseous? To answer this question it is appropriate to derive equations describing thermocapillary motion in the "liquid-gas" system within the framework of long-wave approximation. The system is closed between two solid planes $x_3 = 0$ and $x_3 = h$. The interface has the equation $x_3 = \eta(x_1, x_2, t)$. The plane $x_3 = 0$ bounding liquid is sustained at a constant temperature $\theta_l$, and the plane $x_3 = h$ contacting with gas is heated up to a constant temperature $\theta_g$.

Let us denote by $\varrho_*, \mu_*, \lambda_*$ the ratios of densities, dynamic coefficients of viscosity and heat conductivity coefficients of gas and liquid. Typical values of these parameters are: $\varrho_* \sim 10^{-3} - 2 \times 10^{-3}$, $\mu^* \sim 10^{-2} - 2 \times 10^{-2}$, $\lambda_* \sim 5 \times 10^{-2} - 10^{-1}$. Therefore the limiting transition in the equations of long-wave approximation $\varrho_* \to 0$, $\mu_* \to 0$ at fixed value of $\lambda_* > 0$ seems to be justified. In the limit we obtain exactly the relations (7.1), (7.3), where $a = (1 - \lambda_*)^{-1} (\theta_g - \lambda_* \theta_l)$, $b = \theta_g$ (This result was obtained by the author together with L. G. Badratinova). Hence it can be concluded that the model of thermocapillary motion in a thin layer [34] attracting the condition of thermal contact with gaseous phase (1.6) is more realistic than the model suggested in [35].

The thermocapillary effect in the two-layer system lacks so far a satisfactory theoretic treatment. Let us note the works [45, 46] where finite-amplitude motion in a rectangular cavity filled by two viscous unmixing liquids has been studied. Among the most interesting results of these works are those on periodical and non-periodical oscillations caused by the thermocapillary mechanism.

## 10. Bubble motion

Let us consider the problem of steady-state motion in weightlessness of a gaseous bubble with radius $R$, caused by temperature field inhomogeneity in a liquid $\theta(x)$, such that $\nabla\theta \to \vec{g} = \text{const.} \neq 0$ when $x \to \infty$. This problem, by virtue of its significance, was the subject of a study (see [47] and references cited there).

To solve this problem, it is necessary to consider the given system as a two-component one and next to perform the limit passage $\varrho_* \to 0$; $\mu_* \to 0$, $\lambda_* \to 0$.

The problem of thermocapillary drift of a drop of one liquid into the other admits an effective analytical solution in the case of small modified Marangoni numbers $Ma = \varkappa R^2 |\vec{g}|/\varrho v^2$ (For the given pair of media the condition of smallness of $Ma$ can be realized by choosing a sufficiently small value of $R^2 |\vec{g}|$. For example, for an air bubble in pure water at $\theta = 20°\text{C}$ we have $Ma < 1$ if $R^2 |\vec{g}| < 10^{-3}$ cm · grad). It should be noted that the smallness condition of the Marangoni number in the case under consideration automatically gives rise to the small parameter $A$ responsible for the free boundary deformation. Thus, the expansion of the solution over small parameter $Ma$ bears a resemblence to the procedure described in Section 5. Moreover, the deviation of the bubble shape from a spherical one is of the order $Ma^2$ at $Ma \to 0$ [47]. The expression for velocity $\vec{V}$ of the thermocapillary bubble drift takes the form

$$\vec{V} = \frac{\varkappa R\vec{g}}{2\varrho v}[1 + 0(Ma)] \text{ if } Ma \to 0.$$

If the liquid contains may bubbles with a mean distance $l$ between them being $R \ll l \ll \text{diam.}\,\Omega$, the representations of a heterogeneous medium mechanism can be applied to the description of thermocapillary motion in such a system. A phenomenological model of thermocapillary motion in a gas-liquid system was suggested in [48] (see also [49]). The two results obtained by the above-mentioned model are of a particular importance. A uniform distribution of bubbles of equal radius at constant temperature gradient and constant gravity acceleration is unstable. A nonlinear evolution of one-dimensional initial distribution of smoothed ledge type causes a sharp steepening of concentration profile with increasing time [50].

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## Modele ruchów termokapilarnych

Efekt termokapilarności jest uważany za jeden z głównych przypadków ruchu płynów w polach słabych oddziaływań. Praca stanowi przegląd modeli matematycznych ruchów termokapilarnych. Podstawę przeglądu stanowią wyniki otrzymane w Instytucie Hydrodynamiki im. Ławrentiewa i Krasnojarskim Ośrodku Obliczeniowym Oddziału Syberyjskiego AN ZSRR, a także w Uniwersytecie w Nowosibirsku.

## Модели термо-капилярного движения

Эффект термокапилярности является одним из самых существенных случаев движения жидкости в полях слабых действий. В работе представлены результаты, которые получены в Институте Гидродинамики им. Лаврентьева, Красноярским Вычислительном Центре Сибирского Отделения АН СССР и в Новосибирском Государственном Университете.