Control
and Cybernetics
VOL. 14 (1985) No. 1—3

On the length of the free boundary of a minimal surface

by

GERHARD DZIUK
Institut für Angewandte Mathematik
Universität Bonn
Wegelerstrasse 6
D-5300 Bonn, BRD

Let S be a smooth surface in Euclidean 3-space and let C be a smooth curve having its end points on S. Consider a surface x bounded by S and C which is a stationary point of Dirichlet's integral. We give an optimal estimate for the length of the trace of x on S. The proof of S. Hildebrandt and J. C. C. Nitsche is generalized to surfaces x with branch points of odd order on the free boundary.

1. Introduction

Let us have a look at a configuration in $\mathbf{R}^3$ consisting of a smooth Jordan arc C having its end points P_1 and P_2 on a smooth surface S and no other points in common with S. Consider a surface x which has minimal area among all surfaces bounded by C and S. This variational problem can be illustrated beautifully by experiments with soap films whire S is represented by a thin plate of plastic material and C is a thin wire fixed to S at two points (see fig. 1). Note that we shall work with supporting surfaces S without boundary and so S could be for example a torus or a sphere.

Let us formulate the basic variational problem. Denote by B the upper half of the unit disc in the (u, v)-plane $B = \{(u, v)|u^2+v^2 < 1, v > 0\}$ and by $\partial^+ B$ its boundary portion $\{(u, v)|u^2+v^2 = 1, v > 0\}$ and by I the interval $(-1, 1)\times\{0\}$ on the real axis. Let $Z(C, S)$ be the set of all surfaces $x = x(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v)) \in C^0(B) \cap H^{1,2}(B)$ which are bounded by C and S in the following sense: x maps $\partial^+ B$ continuously and in a weakly monotonic manner onto C such that $x(-1, 0) = P_1$, $x(1, 0) = P_2$

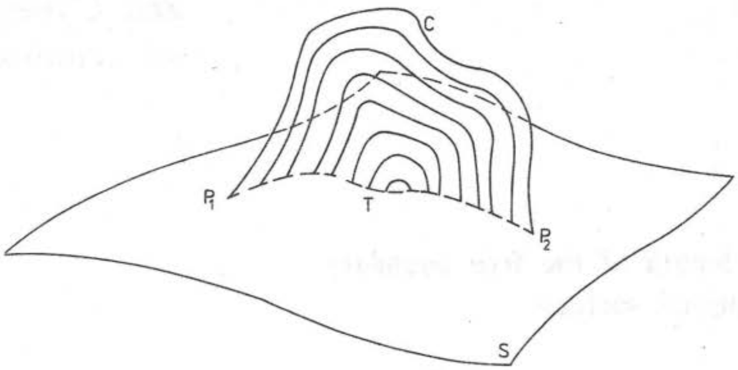

Fig. 1

and $x(0, 1) = P_3$ for some fixed third point P_3 on C, different from P_1 and P_2, while

$$\operatorname{dist}\big(x(u, v), S\big) = \inf_{y \in S} |x(u, v) - y| \to 0 \tag{1}$$

as $(u, v) \to (u_0, 0) \in I$. The boundary condition on I is a free boundary condition in the sense that x may vary on the supporting surface S. But note that the boundary condition on $\partial^+ B$ is free on the one-dimensional manifold C too. It just has to satisfy the three point condition.

We set

$$d(C, S) = \inf_{y \in Z(C, S)} D(y),$$

where

$$D(y) = \int_B |\nabla y|^2 \, du \, dv$$

is Dirichlet's integral, and consider the variational problem (V) to find some surface x in the class $Z(C, S)$ such that

$$D(x) = d(C, S).$$

It is well known that this variational problem has at least one solution, see ch. VI in [1]. In general there is no uniqueness for this problem. Each solution of (V) is a minimal surface i.e.

$$\Delta x = 0, \; x_u \cdot x_v = 0, \; |x_u| = |x_v| \tag{2}$$

on B. It also minimizes the area functional

$$A(x) = \int_B |x_u \wedge x_v| \, du \, dv$$

in the class $Z(C, S)$, see [12]. For a survey of the results up to 1975 we refer to [13], pp. 447–474.

Several authors have investigated the boundary behaviour of solutions of the variational problem (V). There is satisfactory information concerning the regularity. Roughly spoken the results are to the effect that up to the boundary x is as smooth as the bounding contour C resp. S itself is. For the behaviour at the "fixed" boundary $\partial^+ B$ see [13], pp. 281–325. The crucial point in proving regularity at the free boundary I is to show continuity of x on $B \cup I$. Condition (1) only implies continuity for the distance function but not for x itself. For solutions of (V), i.e. for area minimizing minimal surfaces, regularity of the free boundary has been proved by W. Jäger in [10], J. C. C. Nitsche in [12] and [14], and by K. H. Goldhorn and S. Hildebrandt in [5]. The most important observation concerning the free boundary is that a solution x meets the supporting surface S orthogonally. This has been proved by W. Jäger in [10] even for stationary solutions. M. Grüter, S. Hildebrandt, and J. C. C. Nitsche in [6] and the author in [2, 3] proved regularity of the free boundary for stationary points of Dirichlet's integral, i.e. for surfaces x which are not area minimizing. Regularity results for surfaces S with boundary can be found in [7] and [8].

Besides the question of boundary regularity one of the most interesting problems concerning partially free minimal surfaces is to estimate the length of the trace

$$T = \{x(u, 0) \,|\, |u| < 1\}$$

of x. In [9] Hildebrandt and Nitsche proved the estimate

$$L(T) \leqslant L(C) + \frac{a}{R} A(x) \tag{3}$$

for the length of the trace of a stationary surface x in $Z(C, S)$ with a constant $a < 7$. Here L represents the length of the curve indicated. A. Küster [11] succeeded in proving the optimal estimate with $a = 2$. $1/R$ represents a bound for the principal curvatures of S. The authors assumed that S satisfies a two-sided R-sphere condition and that the minimal surface x had no branch points of odd order on the free boundary. A branch point of a minimal surface is a point $w_0 = u_0 + iv_0$ where the gradient of x vanishes. In the neighbourhood of a branch point one has an asymptotic expansion of the form

$$x_w(w) = c\,(w - w_0)^m + o\,(|w - w_0|^m)$$

where $x_w = (x_u - ix_v)/2$, with some number $m \in \mathbf{N}$ and some vector $c \in \mathbf{C}$, $c \neq 0$, m is called the order of the branch point w_0. The problem with branch points of odd order is such in which $x_v(u, 0)$ changes direction as u passes through a branch point $(u_0, 0)$. For better interpretation of such singular points we refer to [9]. The authors exclude branch points on I under certain conditions, see Thms. 4 and 5 there. The most important fact is that for minimizing minimal surfaces branch points on I can be excluded.

The aim of this paper is to prove the estimate (3) without assuming the non existence of branch points of odd order on the free boundary. The idea of proof is due to S. Hildebrandt. J. C. C. Nitsche and A. Küster. Our task is just some sort of a regularization. First we say something about the basic situation, i.e. about admissible supporting surfaces and stationary solutions. We also list some regularity properties. Then we prove inequality (3).

2. Stationary Free Minimal Surfaces

First we suppose that $Z(C, S)$ is not empty. This is guaranteed for example if the end points P_1 and P_2 of C can be connected by a rectifiable Jordan arc on S. We assume that C is of class $C^{1,\alpha}$ and that C meets S at P_1 and P_2 under angles greater than zero. For weaker assumptions see [9]. We now specify what an admissible supporting surface is. For example every compact surface S in $\mathbf{R}^3$ given by $f(x)=0$, $f \in C^{m,\mu}(\mathbf{R}^3)$, $\nabla f \neq 0$ on S, represents such a surface.

1. Definition. Let S be a 2-dimensional $C^{m,\mu}$-manifold in $\mathbf{R}^3$ with the following properties: For every point $x_0 \in S$ there exist some neighbourhood U_0 of x_0 in $\mathbf{R}^3$ and some real-valued function $f_0 \in C^{m,\mu}(U_0)$ with $\nabla f_0 \neq 0$ in U_0 and $f_0(x)=0$ iff $x \in S \cap U_0$. There is some positive number d and there are functions $\xi, a, n \in C^{m-1,\mu}(U_d)$ in the strip $U_d = \{x \in \mathbf{R}^3 | \operatorname{dist}(x, S) < d\}$ such that every $x \in U_d$ can be expressed uniquely as

$$x = a(x) + \xi(x)\, n(x),$$

where $a(x) \in S$, $n(x)$ is normal to S at $a(x)$, $|n(x)| = 1$, and $|\xi(x)| = \operatorname{dist}(x, S)$. In addition to that we assume that

$$\sup_{U_d} |\nabla n| < \infty. \tag{4}$$

Then we call S an admissible supporting surface of class $C^{m,\mu}$ ($m \in \mathbf{N}$, $m \geqslant 2$, $0 \leqslant \mu \leqslant 1$).

It should be noticed that this definition contains two decisive assumptions. First there is a global strip about S, embedded in $\mathbf{R}^3$, and secondly (4) is a global condition on the curvature of S if S is unbounded.

From 1. Definition one easily concludes the following statements; see also [4], appendix.

2. Lemma. *$\xi \in C^{m,\mu}(U_d)$, $n(x) = \nabla \xi(x)$, $H(x)\, n(x) = 0$ on U_d, where we define $H_{ik}(x) = \xi_{x_i x_k}(x)$ $(i, k = 1, 2, 3)$. The nonzero eigenvalues of H can be esti-*

mated by

$$\frac{k}{1-k\,|\xi|},$$

where $k=\max\{|k_1|,|k_2|\}$ *and* k_1, k_2 *are the principal curvatures of* S *at the footpoint* a. *At last* $k \leqslant 1/d$.

3. Definition. A family $x_\varepsilon \in Z(C,S)$ $(|\varepsilon|<|\varepsilon_0|)$ is called an admissible variation of $x\in Z(C,S)$ if $x_\varepsilon = x+\varepsilon z(\varepsilon,\cdot)$, where $z(\varepsilon,\cdot)\in H^{1,2}(B)$ and

$$\frac{d}{d\varepsilon} D(x_\varepsilon)|_{\varepsilon=0}$$

exists. x is called stationary if this expression vanishes for all admissible variation x_ε of x. x is called minimizing if

$$D(x)\leqslant D(y)$$

for every $y\in Z(C,S)$.

For a detailed discussion of admissible variations the interested reader is referred to [11], § 3. Of course, every minimizing surface is stationary. But there are surfaces in $Z(C,S)$ which are stationary but not area minimizing.

W. Jäger has proved that a stationary surface meets the supporting surface orthogonally; see [10].

4. Lemma. *Let* S *be an admissible supporting surface of class* C^2 *and let* $x\in Z(C,S)$ *be stationary. Then*

$$x_v = x_v\cdot n\cdot x\; n\cdot x$$

on I.

Here we have used the regularity of x on I. Let us just collect the regularity properties of stationary solutions of (V).

5. Lemma. 1. *Regularity at the "fixed" boundary. Assume that* C *is a regular curve of class* $C^{m,\mu}$ $(m\in\mathbf{N}, 0\leqslant\mu<1)$. *Then* $x\in C^{m,\mu}(B\cup\partial^+B)$ *if* $\mu\neq 0$ *and* $x\in C^{m-1,\nu}(B\cup\partial^+B)$ *for every* $\nu\in(0,1)$ *if* $\mu=0$.

2. *Regularity at the free boundary. Let* S *be an admissible supporting surface of class* $C^{m,\mu}$ $(m\in\mathbf{N}, m\geqslant 2, 0\leqslant\mu<1)$. *Then* $x\in C^{m,\mu}(B\cup I)$ *if* $\mu\neq 0$ *and* $x\in C^{m-1,\nu}(B\cup I)$ *for every* $\nu\in(0,1)$ *if* $\mu=0$.

In both cases we have assumed that x *is stationary.*

References for the proof of this Lemma were given in the introduction to this paper.

3. The Length of the Trace

In this part of the paper we shall prove the estimate (3) for the length of the trace of a stationary minimal surface. Due to an idea of S. Hildebrandt an J.C.C. Nitsche we do not need continuity for x at the corner points $(-1{,}0)$ and $(0, 1)$.

6. THEOREM. *Let S be an admissible surface of class C^2 and let C be a regular arc of class $C^{1,v}$. If $x\in Z(C, S)$ is stationary then*

$$L(T)\leqslant L(C)+\frac{2}{d}A(x).$$

7. COROLLARY. *Under the assumptions of the theorem, $x\in C^{0,\mu}(\bar{B})$ for some $\mu\in(0, 1)$.*

This is an easy consequence of the fact that the trace of x on I and $\partial^+ B$ is in $H^{1,1}(I), H^{1,1}(\partial^+ B)$ resp. and that $x\in H^{1,2}(B)$.

8. COROLLARY. *Let S satisfy an R-sphere condition, i.e. S is the boundary of some open set G in $\mathbf{R}^3$ with $\partial G\in C^2$, and for every $x_0\in S$ there are two balls of radius R, tangent to S at x_0, which do not contain any points of S, then d may be taken as R in the Theorem.*

The two-sided R-sphere condition implies that the principal curvatures of S are bounded by $1/R$ and globally it garantees that a strip of width $d=R$ exists where the decomposition

$$x=a(x)+\xi(x)\,n(x)$$

is unique.

Proof of the Theorem.

Let δ be some small positive number. The length of

$$T_\delta=\{x(u, 0)\,|\,|u|<1-\delta\}$$

is given by

$$L(T_\delta)=\int_{I_\delta}|x_u|\,du,\ \ I_\delta=(-1+\delta, 1-\delta)$$

and because of the conformality relations in (2) this equals

$$\int_{I_\delta}|x_v|\,du.$$

Since x meets S transversally (see 4. Lemma) we have

$$L(T_\delta)=\int_{I_\delta}|(\xi\cdot x)_v|\,du. \tag{5}$$

We now construct some function η which is smooth on B such that

$$\eta_v\geqslant|(\xi\cdot x)_v| \tag{6}$$

on I. Since $\xi\cdot x$ is defined in some neighbourhood of I only, we take a function $\psi\in C^2(\mathbf{R})$ with $\psi(t)=\psi(d)\,(t\geqq d)$, $\psi(t)=-\psi(d)\,(t\leqq -d)$, $\psi(0)=0$, $\psi'(0)=1$, $\psi'\in C_0^2(-d,d)$, $|\psi'(t)|\leqslant(1+\delta)(1-|t|/d)$ and $|\psi''|\leqq(1+\delta)/d$. We set

$$\zeta(x)=\begin{cases}\psi(\xi(x))^2 & \text{for } x\in U_d\\ \psi(d)^2 & \text{for } x\in\mathbf{R}^3\setminus U_d\end{cases}$$

and

$$\eta=(\alpha^2+\xi\cdot x)^{1/2}$$

on $\bar{B}$, where $\alpha(u,v)=\delta v$. We see that in B

$$\nabla\eta=(\alpha^2+\zeta\cdot x)^{-1/2}\left(\alpha\,\nabla\alpha+\nabla(\zeta\cdot x)/2\right).$$

Thus near I,

$$\eta_v=\frac{\delta\alpha+\psi'(\xi\cdot x)\,\psi(\xi\cdot x)\,(\xi\cdot x)_v}{\left(\alpha^2+\psi(\xi\cdot x)^2\right)^{1/2}}$$

which tends to

$$\frac{\delta^2+\left((\xi\cdot x)_v\right)^2}{\left(\delta^2+\left((\xi\cdot x)_v\right)^2\right)^{1/2}}=\left(\delta^2+\left((\xi\cdot x)_v\right)^2\right)^{1/2}\geqq|(\xi\cdot x)_v|$$

as $v\to 0,\ v>0$.

We shall have to estimate $-\Delta\eta$ from above. So, let us calculate this expression

$$\begin{aligned}-\Delta\eta=(\alpha^2+\zeta\cdot x)^{-3/2}\,|\alpha\,\nabla\alpha+\nabla(\zeta\cdot x)|^2-&\\ -(\alpha^2+\zeta\cdot x)^{-1/2}&\left(|\nabla\alpha|^2+\Delta(\zeta\cdot x)/2\right),\\ \nabla(\zeta\cdot x)=2\psi'(\xi\cdot x)\,\psi(\xi\cdot x)\,\nabla(\xi\cdot x)&\end{aligned}$$

and abbreviating $\gamma=\psi\psi'$ we get

$$\Delta(\zeta\cdot x)=2\gamma'(\xi\cdot x)\,|\nabla(\xi\cdot x)|^2+2\gamma(\xi\cdot x)\,\Delta(\xi\cdot x).$$

Thus

$$\begin{aligned}-\Delta\eta&=(\alpha^2+\zeta\cdot x)^{-3/2}\left(|\alpha\,\nabla\alpha+\gamma(\xi\cdot x)\,\nabla(\xi\cdot x)|^2-\right.\\ &\qquad\left.-(\alpha^2+\zeta\cdot x)\left(|\nabla\alpha|^2+\gamma'(\xi\cdot x)\,\nabla(\xi\cdot x)|^2\right)\right)-\\ &\qquad-(\alpha^2+\zeta\cdot x)^{-1/2}\,\gamma(\xi\cdot x)\,\Delta(\xi\cdot x)=\\ &=\alpha^2+\zeta\cdot x)^{-3/2}\left(-|\alpha\psi'(\xi\cdot x)\,\nabla(\xi\cdot x)-\psi(\xi\cdot x)\,\nabla\alpha|^2\right)-\\ &\qquad-\left(\alpha^2+\psi(\xi\cdot x)^2\right)^{-1/2}\left(\psi(\xi\cdot x)\,\psi''(\xi\cdot x)\,|\nabla(\xi\cdot x)|^2+\right.\\ &\qquad\left.+\gamma(\xi\cdot x)\,\Delta(\xi\cdot x)\right).\end{aligned}$$

From this is follows that

$$-\Delta\eta\leqq|\psi''(\xi\cdot x)|\,|\nabla(\xi\cdot x)|^2+|\psi'(\xi\cdot\xi x)|\,|\Delta(\xi\cdot x)|. \tag{7}$$

With $B_\delta = B \cap \{w=(u,v) \mid |w \pm 1| > \delta\}$ formulas (5), (6) and (7) yield

$$L(T_\delta) \leqq \int_{I_\delta} \eta_v \, du = -\int_{I_\delta} \frac{\partial \eta}{\partial v} \, du = \int_{B_\delta} -\Delta \eta \, du \, dv + \int_{\partial B_\delta \setminus I_\delta} \frac{\partial \eta}{\partial v} \, ds \tag{8}$$

and since

$$|\Delta (\xi \cdot x)| = |H \cdot x x_u \cdot x_u + H \cdot x x_v \cdot x_v| \leqq$$

$$\leqq \frac{k}{1-k|\xi \cdot x|} \left(|\nabla x|^2 - |\nabla (\xi \cdot x)|^2 \right)$$

for $|\xi \cdot x| \leqq d$,

$$|\psi' (\xi \cdot x)| \, |\nabla (\xi \cdot x)| \leqq$$

$$\leqq (1+\delta) \left(1 - \frac{|\xi \cdot x|}{d} \right) \frac{k}{1-k|\xi \cdot x|} \left(|\nabla x|^2 - |\nabla (\xi \cdot x)|^2 \right)$$

$$\leqq (1+\delta) \frac{1}{d} \left(|\nabla x|^2 - |\nabla (\xi \cdot x)|^2 \right). \tag{9}$$

Because of

$$|\psi'' (\xi \cdot x)| \, |\nabla (\xi \cdot x)|^2 \leqq (1+\delta) \frac{1}{d} |\nabla (\xi \cdot x)|^2,$$

(7) and (9) together give us the following bound for the first integral in (8):

$$\int_{B_\delta} -\Delta \eta \, du \, dv \leqslant (1+\delta) \frac{1}{d} D(x). \tag{10}$$

The second integral in (8) consists of two boundary integrals. The first one ranges over $\partial^+ B_\delta = \partial^+ B \cap \bar{B}_\delta$ and can be estimated like this:

$$\left| \int_{\partial_1 B_\delta} \frac{\partial \eta}{\partial v} \, ds \right| \leqq \delta \pi + (1+\delta) L(C), \tag{11}$$

because

$$\left| \frac{\partial \eta}{\partial v} \right| \leqq (\alpha^2 + \zeta \cdot x)^{-1/2} \, |\alpha \, \alpha_r + (\zeta \cdot x)_r / 2| \leqq \delta + (1+\delta) \, |(\xi \cdot x)_r|$$

and

$$|(\xi \cdot x)_r| \leqq |x_r| = |x_s|,$$

where $r = |w|$. Here we have used conformal parameters (2).

For given $\delta_1 > 0$, the Lemma of Courant-Lebesgue gives us some number δ from the interval (δ_1^2, δ_1) such that

$$\int_{\partial_1 B_\delta} |x_s| \, ds \leqq \left(\pi D(x) / \log \frac{1}{\delta_1} \right)^{1/2}, \tag{12}$$

where $\partial_1 B_\delta = \{ w \big| |w-1| = \delta, w \in B \}$. The same is true for the remaining integral, for another δ perhaps. But this creates technical difficulties only. If we estimate as in the proof of (11), then collect our inequalities (8), (10), (11) and (12), we arrive at

$$L(T_\delta) \leqq (1+\delta) \frac{1}{d} D(x) + (1+\delta) L(C) + o(1)$$

as $\delta \to 0$, and the Theorem is proved. ■

References

[1] COURANT R. Dirichlet's principle, conformal mapping, and minimal surfaces. New York, Interscience, 1950.

[2] DZIUK G. Über die Stetigkeit teilweise freier Minimalflächen. *Man. math.* **36** (1981), 241–251.

[3] DZIUKG. C^2 — Regularity for partially free minimal surfaces. *Math Ž.* **189** (1985), 71–79.

[4] GILBARG D., TRUDINGER N. S. Elliptic partial differential equations of second order Berlin–Heidelberg–New York, 1977.

[5] GOLDHORN K. H., HILDEBRANDT S. Zum Randverhalten der Lösungen gewisser zweidimensionaler Variationsprobleme mit freien Randbedingungen. *Math. Z.* **118** (1970), 241–253.

[6] GRÜTER M., HILDEBRANDT S., NITSCHE J. C. C. On the boundary behavior of minimal surfaces with a free boundary which are not minima of the area. *Man. math.* **35** (1981), 387–410.

[7] HILDEBRANDT S., NITSCHE J. C. C. Minimal surfaces with free boundaries. *Acta Math.* **143** (1979), 251–272.

[8] HILDEBRANDT S., NITSCHE J. C. C. A uniqueness theorem for surfaces of least area with partially free boundaries on obstacles. *Arch. Rat. Mech. Anal.* **79** (1982), 189–218.

[9] HILDEBRANDT S., NITSCHE J. C. C. Geometric properties of minimal surfaces with free boundaries. *Math. Z.* **184** (1983), 497–509.

[10] JÄGER W. Behaviour of minimal surfaces with free boundaries. *Comm. Pure Appl. Math.* **23** (1970), 803–818.

[11] KÜSTER A. An optimal estimate of the free boundary of a minimal surface. *J. f. d. Reine u. Angew. Math.* **349** (1984), 55–62.

[12] NITSCHE J. C. C. Minimal surfaces with partially free boundary. Least area property and Hölder continuity for boundaries satisfying a chord-arc condition. *Arch. Rat. Mech. Anal.* **39** (1970), 131–145.

[13] NITSCHE J. C. C. Vorlesungen über Minimalflächen Berlin-Heidelberg-New York, Springer, 1975.

[14] NITSCHE J. C. C. The regularity of the trace for minimal surfaces. *Annali Scuola Norm. Sup. Pisa*, Ser. IV, **3** (1976), 139–155.

O długości swobodnej granicy powierzchni minimalnej

Niech S będzie gładką powierzchnią w euklidesowej przestrzeni 3-wymiarowej, a C — gładką krzywą mającą końce na S. Rozważana jest powierzchnia x, ograniczona przez S i C, będąca punktem stacjonarnym całki Dirichleta. Podane zostaje optymalne oszacowanie długości śladu x na S. Dowód S. Hildebrandta i J. C.C. Nitsche zostaje uogólniony na powierzchnie x z mającymi nieparzysty rząd punktami rozgałęzienia na swobodnej granicy.

О длине свободной границы минимальной поверхности

Пусть S будет гладкой поверхностью в трехмерном эвклидовом пространстве и C обозначает гладкую кривую с концами на S. Рассмативается поверхность x ограниченную S и C, которая является стационарной точкой интеграла Дирихле. Подана оптимальная оценка длины следа x на S. Доказательство принадлежащее C. Хильдебрандту и Й.С.С. Ниче обобщается на случай поверхностей x, у которых на свободной границе возникают точки разветвления нечетного порядка.