Control
and Cybernetics
VOL. 14 (1985) No. 1—3

On the thermostat problem

by

HANS WILHELM ALT

Institut für Angewandte Mathematik
Universität Bonn
D-5300 Bonn 1
Wegelerstr. 6
BRD

The thermostat problem for ordinary differential equations and the heat operator is considered, and the existence of a solution is proved. Nonuniqueness is discussed in several examples.

1. Introduction

The thermostat problem can be formulated as $L(u, s) = 0$, where L is a differential operator in u, and s the thermostat or switch variable. In general $s = s(Mu)$ with an observable $w := Mu$. We have to distinguish wether Mu ranges in a finite dimensional space or not.

In the simplest case w is real valued and then $s(w(t))$ is described as follows. Assume that at a given time $s = 0$ and $w < 1$.. Then if with increasing time w crosses the value 1, the function s switches to 1. Conversely, if $s = 1$ and $w > 0$ at some given time, and if with increasing time w crosses 0, then s switches to 0. In other words, (w, s) must lie in what we shall call switch configuration. In the situation of Fig. 1 we define $S_0 :=]-\infty, 1[$ and $S_1 :=]0, \infty[$. The arrows in Fig. 1 indicate that at the switch values 0 and 1 only jumps in certain directions are allowed. Of course, more general switch configurations are possible (Fig. 2), and in the vector case it may look as in Fig. 3. If w is $\mathbf{R}^m$ valued and if the thermostat has the two states 0 and 1, we define the switch configuration by two open sets S_0 and S_1 in $\mathbf{R}^m$ with $S_0 \cup S_1 = \mathbf{R}^m$ and let

$$\begin{aligned} \Sigma &:= \{(w, s) \in \mathbf{R}^{m+1};\, s = 0 \text{ and } w \in \bar{S}_0, \text{ or } s = 1 \text{ and } w \in \bar{S}_1\}, \\ \hat{\Sigma} &:= \{(w, s) \in \mathbf{R}^{m+1};\, (w, s) \in \Sigma, \text{ or } 0 \leqslant s \leqslant 1 \text{ and } w \in \partial S_0 \cup \partial S_1\}. \end{aligned} \tag{1.1}$$

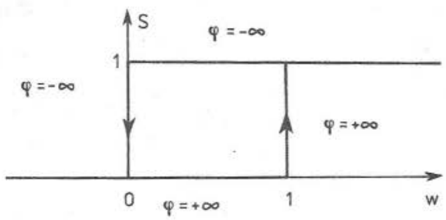

Fig. 1

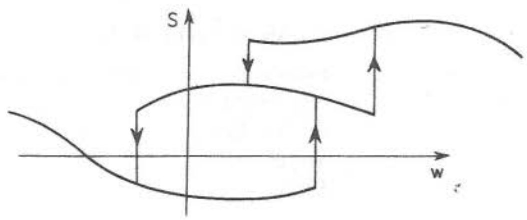

Fig. 2

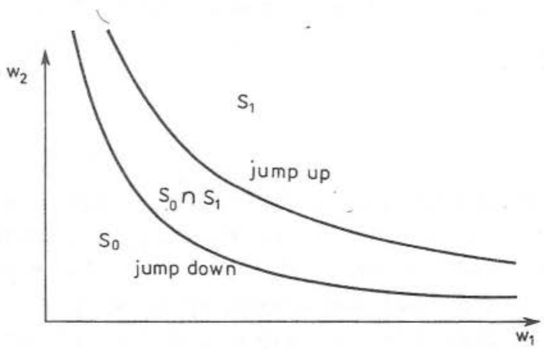

Fig. 3

The operator L will be such that for each s there is a unique solution u_s of the corresponding initial value problem on the time interval $[0, T]$ with $L(u_s, s) = 0$.

In section 2 we consider the case that L is an ordinary differential operator. But, of course, the results in section 2 apply to any situation, in which M maps into $\mathbf{R}^m$ and the functions Mu_s are compact in $C^0([0,T];\mathbf{R}^m)$. This is, for example, the case when L is the heat operator and M the evaluation at a finite number of points, see e.g. the problem in [1].

In section 3 we consider the heat equation with $Mu = u$. This means that every space point is an atomic thermostat. It applies to any physical quantity u, whose spatial distribution is governed by diffusion with a production rate, which behaves like a thermostat at every space point (see [2]). The difference to the situation in section 2 is, that switching at a point at a certain time is influenced via diffusion by previous switching in a whole neighborhood. In order to handle this effect we apply the maximum principle. Therefore the proof in its present version is restricted to a scalar u and one space dimension. The optimality of the result is shown in 3.3–3.5.

To construct a solution of the thermostat problem one has to give a mathematical formulation of the jump condition at the switch values, which is not contained in the definition of Σ. An adequate formulation seems to be

$$\dot{s} = \varphi(w, s)$$

with φ as in Fig. 1. To solve this one has to approximate the problem. One possibility is

$$\dot{s} = \varphi_\delta(w, s)$$

with Lipschitz functions φ_δ converging to φ. Example 2.1 shows that φ_δ has to converge to $+\infty$ also in a neighborhood of the critical line $\{1\} \times]0, 1[$, and similarly to $-\infty$ near $\{0\} \times]0, 1[$. This means that the switch configuration is approximated by the zero set of smooth functions as in Fig. 4. But even then there are cases (see Example 2.2), where δ dependent small perturbation of the initial data do not lead to a solution of the thermostat problem, for (w, s) is not allowed to enter the interior of the loop. Therefore the thermostat problem in principle is the limit of regular hysteresis problems (see [3]), however, approximations of this type not always are consistent.

Another way to interprete this is to understand the character of the thermostat problem by postulating that the switch reacts faster than w has time to change its values. In other words, the time scale for the thermostat is much smaller then, for example, the time scale for diffusion. To realize this we use time discretization as approximation and set

$$\frac{s(t+h) - s(t)}{h} = \varphi_\delta(w(t), s(t))$$

with $|\varphi_\delta| = O\left(\frac{1}{\delta}\right)$ outside the switch configuration. Here δ should tend to zero fast enough compared with h. The best choice is $\delta = h$ and φ_δ as in Fig. 5. Then, assuming that $0 \leqslant s(0) \leqslant 1$,

$$s(t+h) = \begin{cases} 1 & \text{if } w(t) \in S_1 \setminus S_0, \\ s(t) & \text{if } w(t) \in S_0 \cap S_1, \\ 0 & \text{if } w(t) \in S_0 \setminus S_1. \end{cases} \tag{1.2}$$

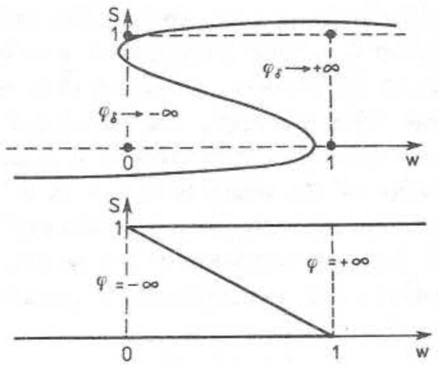

Fig. 4

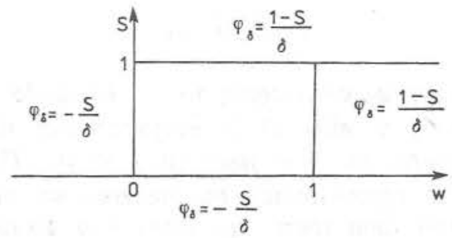

Fig. 5

In section 3 we shall use the modification

$$s(t+h) = \begin{cases} 1 & \text{if } w(t+h) \in S_1 \setminus \bar{S}_0, \\ s(t) & \text{if } w(t+h) \in \bar{S}_0 \cap \bar{S}_1, \\ 0 & \text{if } w(t+h) \in S_0 \setminus \bar{S}_1. \end{cases} \tag{1.3}$$

The advantage is, that already the approximations have thermostat character. The fact that the operation in (1.2) and (1.3) is not continuous in w, is no disadvantage, because in general the solution of the thermostat problem does not depend continuously on the data.

This is connected to nonuniqueness. If we consider all limits of stable solutions, then for certain data several solutions exist. In other words, there is no stable algorithm for solving the thermostat problem. As a consequence, in order to avoid that the special numerical solution obtained by a given algorithm depends on the algorithm itself, one could apply random perturbations to the numerical solution at every time step. But then it becomes unlikely to reach certain solutions, for example, u_- in Example 2.3. Another example for nonuniqueness is given in 2.4.

Finally, for the solution of the thermostat problem the side condition on (w, s) can be formulated as follows. We must have $(w(t), s(t)) \in \Sigma$ (or at least $\hat{\Sigma}$) and on $\{t; w(t) \in S_0\}$ the switch function s must be nonincreasing, and nondecreasing on $\{t; w(t) \in S_1\}$. An equivalent formulation is given in (2.3).

2. Existence for ODE

First let us prove the existence of a solution, if the underlying operator L is an ordinary differential operator and $Mu = u$. As pointed out in the introduction, the proof applies also to a general class of problems.

We begin with three examples related to the switch configuration in Fig. 1. The first two use approximations $\dot{s} = \varphi_\delta(u, s)$ of the switch condition.

2.1. Example. For given $\varkappa \in]0, \frac{1}{2}[$ we consider the problem

$$\dot{u} = \varkappa - s, \quad u(0) = 1, \quad s(0) = 0.$$

If

$$\varphi_\delta(u, s) := \begin{cases} \frac{1}{\delta} & \text{for } u > 1, 0 < s < 1, \\ 0 & \text{for } 0 < u < 1, 0 < s < 1, \\ -\frac{1}{\delta} & \text{for } u < 0, 0 < s < 1, \end{cases}$$

then $\dot{s}_\delta(t) = \frac{1}{\delta}$ for small t, and we calculate

$$u_\delta(t) = 1 + \varkappa t - \frac{t^2}{2\delta}, \quad s_\delta(t) = \frac{t}{\delta} \quad \text{for } 0 < t < 2\delta\varkappa,$$

and

$$u_\delta(t) = 1 - \varkappa(t - 2\delta\varkappa), \quad s_\delta(t) = 2\varkappa \quad \text{for } 0 < t - 2\delta\varkappa < \frac{1}{\varkappa}.$$

After that the solution behaves symmetrically on the left side of the loop. Thus as $\delta \downarrow 0$ the approximation (u_δ, s_δ) converges to a $\frac{2}{\varkappa}$-periodic solution (u, s) with (see Fig. 6)

$$u(t) = 1 - \varkappa t, \; s(t) = 2\varkappa \quad \text{for } 0 < t < \frac{1}{\varkappa},$$

$$u(t) = \varkappa t - 1, \quad s(t) = 0 \quad \text{for } \frac{1}{\varkappa} < t < \frac{2}{\varkappa}.$$

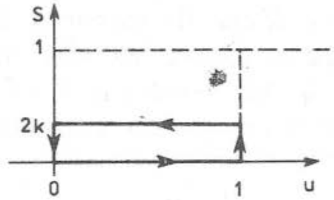

Fig. 6

This is not a solution of the thermostat problem, although the initial value lies on the switch configuration. On the other hand, if we set

$$\varphi_\delta(u,s)=\begin{cases}\dfrac{1}{\delta} & \text{if } u+s>1, 0<s<1,\\ -\dfrac{1}{\delta} & \text{if } u+s<1, 0<s<1,\end{cases}$$

then the solution is given by

$$u_\delta(t)=1+\varkappa t-\frac{t^2}{2\delta},\ s_\delta(t)=\frac{t}{\delta}\ \text{for}\ 0<t<\delta.$$

Thus $|u_\delta-1|\leqslant\dfrac{\delta}{2}$ on $[0,\delta]$ and in the limit we obtain a solution of the thermostat problem with period $(\varkappa(1-\varkappa))^{-1}$ given by

$$u(t)=1-(1-\varkappa)t,\ s(t)=1\ \text{for}\ 0<t<\frac{1}{1-\varkappa},$$

$$u(t)=\varkappa t-\frac{\varkappa}{1-\varkappa},\ s(t)=0\ \text{for}\ 0<t-\frac{1}{1-\varkappa}<\frac{1}{\varkappa}.$$

This solution is also reached by time discretization

$$\frac{1}{h}(u_h(t+h)-u_h(t))=\varkappa-s_h(t)$$

with $u_h(0)=1$, $s_h(0)=0$ using formula (1.2) or (1.3) for s_h.

2.2. Example. Next we consider the initial value problem

$$\dot{u}=-u-s,\ u(0)=1,\ s(0)=0$$

using the approximation

$$\varphi_\delta(u,s)=\frac{1+\delta}{\delta}(u+s-1)\ \text{for}\ u+s>1, 0\leqslant s<1.$$

If we perturbe the initial data by the order of δ, say, $u_\delta(0)=1+\delta$, we obtain as solution

$$u_\delta(t)=(1+\delta)(1-t),\ s_\delta(t)=(1+\delta)\,t \text{ for } 0<t<\frac{1}{1+\delta}.$$

Setting $\varphi_\delta(u,s):=-\frac{1+\delta}{\delta}(1-u-s)$ for $u+s<1$, $0<s\leqslant 1$, after time $\frac{1}{1+\delta}$ the solution behaves similar as in Example 2.1, that is,

$$u_\delta(t_\delta)=0,\ s_\delta(t_\delta)=1 \text{ for } t_\delta:=\frac{1}{1+\delta}+\log(1+\delta),$$

and then

$$1-(s_\delta+u_\delta)(t_\delta+t)=\delta(e^{t/\delta}-1),$$
$$1-s_\delta(t_\delta+t)=(1+\delta)\left(\delta(e^{t/\delta}-1)-t\right),$$

in particular $u^\delta(\tilde{t}_\delta), s_\delta(\tilde{t}_\delta)=O\left(\delta\log\frac{1}{\delta}\right)$ for $\tilde{t}_\delta:=t_\delta+\delta\log\frac{1}{\delta}$. Therefore as $\delta\downarrow 0$ the limit solution is

$$u(t)=1-t,\ s(t)=t \text{ for } 0<t<1,$$
$$u(t)=0,\ s(t)=0 \text{ for } t>1.$$

We should remark that the initial state never can be reached again.

On the other hand, if we use time discretization for approximation as in Example 2.1, and the initial value $u_h(0)=1+h$, we obtain in the limit the following solution of the thermostat problem.

$$u(t)=-1+2e^{-t},\ s(t)=1 \text{ for } 0<t<\log 2,$$
$$u(t)=0,\ s(t)=0 \text{ for } t>\log 2.$$

The next two examples deal with the nongeneric nonuniqueness of the thermostat problem.

2.3. Example. The initial value problem here is

$$\dot{u}(t)=\max(0,1-t)-s(t),\qquad u(0)=\frac{1}{2}+\varepsilon,\ s(0)=0.$$

For $\varepsilon<0$ the solution $(u_\varepsilon, s_\varepsilon)$ is

$$u_\varepsilon(t)=\begin{cases}\frac{1}{2}+\varepsilon+t-\frac{t^2}{2} & \text{for } 0<t<1,\\ 1+\varepsilon & \text{for } t>1,\end{cases}$$
$$s_\varepsilon(t)=0 \text{ for } t>0,$$

whereas for $\varepsilon > 0$

$$s_\varepsilon(t) = 0, u_\varepsilon(t) = \frac{1}{2} + \varepsilon + t - \frac{t^2}{2} \text{ for } 0 < t < t_\varepsilon := 1 - \sqrt{2\varepsilon},$$

$$s_\varepsilon(t) = 1, u_\varepsilon(t) = 1 - \int_{t_\varepsilon}^{t} \min(\xi, 1)\, d\xi \text{ for } t_\varepsilon < t < \tilde{t}_\varepsilon,$$

$$s_\varepsilon(t) = 0, u_\varepsilon(t) = 0 \text{ for } t > \tilde{t}_\varepsilon,$$

where $\tilde{t}_\varepsilon$ is given by

$$\int_{t_\varepsilon}^{\tilde{t}_\varepsilon} \min(\xi, 1)\, d\xi = 1.$$

As $\varepsilon \to 0$ on $[0, 1]$ they converge to the same solution (u, s), but on $]1, \infty[$ the limits

$$u_- := \lim_{\varepsilon \uparrow 0} u_\varepsilon \text{ and } u_+ := \lim_{\varepsilon \downarrow 0} u_\varepsilon$$

are different (see Fig. 7). The reason is that $u(1) = 1$, but $\dot{u}(1) = 0$. Therefore

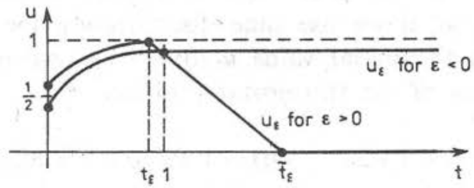

Fig. 7

it is not clear wether the thermostat should decide to switch at time 1 or not. Both solutions u_+ and u_- are limits of stable solutions. Consequently if we want the solution set to be closed we have to allow nonuniqueness. The same situation arises, if we fix the initial value $u(0) = \frac{1}{2}$, but perturb the switch value 1.

The functions $u_\pm$ are not the only solutions. For any $\varkappa > 1$ the function

$$u_\varkappa(t) := \begin{cases} \frac{1}{2} + t - \frac{t^2}{2} & \text{for } 0 \leqslant t \leqslant 1, \\ 1 & \text{for } 1 \leqslant t \leqslant \varkappa, \\ 1 + \varkappa - t & \text{for } \varkappa \leqslant t \leqslant \varkappa + 1, \\ 0 & \text{for } t > \varkappa + 1, \end{cases}$$

is a solution in the sense of (2.1)–(2.3), if

$$s_\varkappa(t) := \begin{cases} 0 \text{ for } t < \varkappa, \\ 1 \text{ for } \varkappa < t < \varkappa + 1, \\ 0 \text{ for } t > \varkappa + 1. \end{cases}$$

This solution derives from u_- by introducing a positive perturbation of u_- at time $\varkappa$. If we consider random perturbations of u_- over the whole time interval, then the probability for such a perturbation to have a sign on a given interval is 0. Consequently we will obtain the solution u_+ with probability 1. In this sense u_+ is the stable and u_- the unstable solution. Consider also the problem $\dot{u}(t) = 1 - t - s(t)$ with $u(0) = \frac{1}{2}$, $s(0) = 0$.

A vector version of this example is

2.4. Example. Let

$$S_0 := \{u \in \mathbf{R}^2; u_1 < 1\},\ S_1 := \{u \in \mathbf{R}^2; u_1 > 0\},$$

and for $\varepsilon, \delta > 0$ consider the initial value problem

$$\begin{aligned} \dot{u}_1 &= \varepsilon,\ u_1(0) = 1 - \delta, \\ \dot{u}_2 &= f(u, s),\ u_2(0) = 0 \end{aligned}$$

with $s(0) = 0$. The solution is

$$u_1(t) = 1 - \delta + \varepsilon t \text{ for } t > 0,$$

$$s(t) = \begin{cases} 0 \text{ for } t < \dfrac{\delta}{\varepsilon}, \\ \\ 1 \text{ for } t > \dfrac{\delta}{\varepsilon}. \end{cases}$$

If $\varepsilon, \delta \to 0$ the limit problem is

$$\begin{aligned} \dot{u}_1 &= 0,\ u_1(0) = 1, \\ \dot{u}_2 &= f(u, s),\ u_2(0) = 0 \end{aligned}$$

with $s(0) = 0$. For the partial limit $\delta = \varkappa\varepsilon \to 0$, $\varkappa > 0$ given, the above solutions converge to

$$u_1(t) = 1 \text{ for } t > 0,$$

$$s(t) = \begin{cases} 0 \text{ for } t < \varkappa, \\ 1 \text{ for } t > \varkappa, \end{cases}$$

$$\dot{u}_2(t) = \begin{cases} f(1, u_2, 0) \text{ for } t < \varkappa, \\ f(1, u_2, 1) \text{ for } t > \varkappa \end{cases}$$

with $u_2(0)=0$. If, for example, $f(u,s):=1-s$ then

$$u_2(t)=\begin{cases} t & \text{for } t\leqslant \varkappa, \\ \varkappa & \text{for } t\geqslant \varkappa. \end{cases}$$

Thus we get a whole family of solutions. The reason is that for $u\in\partial S_0$ the right side $(0, f(u,s))$ is a tangent vector of ∂S_0 at u.

Now we treat the thermostat problem for a system of ordinary differential equations

$$\dot{u}(t)=f\big(t,u(t),s(t)\big),\; u(0)=u_0,$$

where $u:[0,T]\to\mathbf{R}^m$.

2.5. Assumptions. $f(t,u,s)$ and $g(t,v)$ are Caratheodory functions with

$$|f(t,u,s)|\leqslant g(t,|u|),$$
$$g(t,v) \text{ increasing in } v.$$

We assume that the initial value $u_0\in\mathbf{R}^m$ allows a solution of

$$\dot{v}(t)=g\big(t,v(t)\big),\; v(0)=|u_0|,$$

which is absolute continuous in $[0,T]$.

The switch configuration (see Fig. 3) is given by two open sets $S_0, S_1\subset\mathbf{R}^m$ with $S_0\cup S_1=\mathbf{R}^m$, and Σ is defined as in (1.1). Also $s_0\in\mathbf{R}$ is given. We call (u,s) a solution of the thermostat problem, if $u\in H^{1,1}([0,T];\mathbf{R}^m)$, $s\in L^\infty(]0,T[;\mathbf{R})$ with

$$\dot{u}(t)=f\big(t,u(t),s(t)\big) \text{ for almost all } t,\; u(0)=u_0. \tag{2.1}$$

$$\big(u(t),s(t)\big)\in\Sigma \text{ for almost all } t. \tag{2.2}$$

For all $\eta\in C_0^\infty([0,T[)$ with $\eta\geqslant 0$ in $\{t; u(t)\in\partial S_0\}$ and $\eta\leqslant 0$ in $\{t; u(t)\in\partial S_1\}$

$$\int_0^T (s-s_0)\,\dot{\eta}\leqslant 0. \tag{2.3}$$

The last condition determines the direction of jumps of s, and contains the initial condition for s as well.

We prove

2.6. Theorem. *Under the assumptions in 2.5, and if $(u_0,s_0)\in\Sigma$, there is a solution (u,s) of the thermostat problem* (2.1)–(2.3) *in the time interval* $[0,T]$. *In addition, s has a finite number of switch times, if any.*

Proof: For $h>0$ we define approximations (u_h,s_h) by $u_h(0):=u_0$ and

$$u_h(t+h):=u_h(t)+\int_t^{t+h} f\big(\xi,u_h(t),s_h(t)\big)\,d\xi.$$

$s_h(t+h)$ is given by (1.2) with $w(t) = u_h(t)$, and $s_h(0) := s_0$. First we see that u_h converges. For this let v as in 2.5. Then $|u_h(t)| \leqslant v(t)$ implies

$$|u_h(t+h)| \leqslant |u_h(t)| + \int_t^{t+h} g(\xi, |u_h(t)|)\, d\xi \leqslant v(t+h).$$

Therefore $|u_h| \leqslant v$. Similarly

$$\begin{aligned} |u_h(t+kh) - u_h(t)| &\leqslant \sum_{j=0}^{k-1} \int_{t+jh}^{t+(j+1)h} g(\xi, |u_h(t+jh)|)\, d\xi \\ &\leqslant v(t+kh) - v(t). \end{aligned}$$

Interpolating u_h, for example, by

$$u_h(t+\tau) := u_h(t) + \int_t^{t+\tau} f(\xi, u_h(t), s_h(t))\, d\xi$$

for $0 \leqslant \tau \leqslant h$, we conclude that for a subsequence $u_h \to u$ uniformly with $u \in C^0([0, T]; \mathbf{R}^m)$. Next we interpolate s_h piecewise constant. Since $0 \leqslant a_h \leqslant 1$ we get that again for a subsequence

$$s_h \to s \text{ weakly star in } L^\infty(]0, T[).$$

We have to show (2.2). If $u(t_0) \in S_0 \backslash \bar{S}_1$, by the uniform convergence for some $\varepsilon > 0$ (not depending on t_0)

$$u_h(t) \in S_0 \backslash \bar{S}_1 \text{ for } |t - t_0| \leqslant \varepsilon \text{ (and } t \geqslant 0),$$

provided h is small enough. Then by (1.2)

$$s_h(t) = 0 \text{ for } -\varepsilon + h \leqslant t - t_0 \leqslant \varepsilon,$$

consequently by the weak convergence $s = 0$ in $B_\varepsilon(t_0)$, that is, $(u(t_0), s(t_0)) \in \Sigma$. The same if $u(t_0) \in S_1 \backslash \bar{S}_0$. If $u(t_0) \in S_0 \cap S_1$ again for some $\varepsilon > 0$ and small h

$$u_h(t) \in S_0 \cap S_1 \text{ for } |t - t_0| \leqslant \varepsilon.$$

Now (1.2) yields that there are numbers c_h with

$$s_h(t) = c_h \text{ for } -\varepsilon + h \leqslant t - t_0 \leqslant \varepsilon.$$

Let i_h the smallest integer with $s_h(ih) = c_h$ for all $i_h h \leqslant ih \leqslant t_0$. If $i_h > 0$ then $s_h(i_h h) \neq s_h((i_h - 1)h)$, and therefore (1.2) implies that $c_h = 0$ or $c_h = 1$. If $i_h = 0$ we use the information $(u_0, s_0) \in \Sigma$. In case $u_0 \notin \partial S_0 \cup \partial S_1$ this initial condition implies $c_h = s_0 = 0$ or 1, and if $u_0 \in \mathbf{R}^m \backslash (S_0 \cap S_1)$ then $c_h = s_h(h) = 0$ or 1 by (1.2). Thus $c_h = 0$ or 1 in any case, consequently by weak convergence $s = 0$ in $B_\varepsilon(t_0)$ or $s = 1$ in $B_\varepsilon(t_0)$. This proof also shows that $s_h \to s$ pointwise in

$$\{u \notin \partial S_0 \cup \partial S_1\} := \{t; u(t) \notin \partial S_0 \cup \partial S_1\}.$$

Now let us consider the critical case $u(t_0) \in \partial S_0$ (for $u(t_0) \in \partial S_1$ the argument is the same). Then $u(t_0) \in S_1$ and as above for some $\varepsilon > 0$ and small h

$$u_h(t) \in S_1 \text{ for } |t - t_0| \leqslant \varepsilon.$$

Now the algorithm in (1.2) tells us that s_h cannot jump to 0 in the time interval $[t_0 - \varepsilon + h, t_0 + \varepsilon]$. Hence there is a constant c_h and an integer i_h, such that for all ih in that interval

$$s_h(ih) = \begin{cases} c_h & \text{for } i \leqslant i_h, \\ 1 & \text{for } i > i_h. \end{cases}$$

Here $i_h h > t_0 - \varepsilon$, but possibly $i_h h > t_0 + \varepsilon$. By the above argument $c_h = 0$ or $c_h = 1$. Choose a subsequence with $c_h \to c \in \{0, 1\}$ and, if necessary, $i_h h \to t_1$. Then for $|t - t_0| \leqslant \varepsilon$

$$s_h(t) = \begin{cases} c & \text{if } t < t_1, \\ 1 & \text{if } t > t_1. \end{cases}$$

Therefore $(u(t_0), s(t_0)) \in \Sigma$, if $c = 1$ or $t_1 \neq t_0$. If $c = 0$, then t_1 is a switch time. Since ε did not depend on t_0 the proof shows that those time values are isolated in $[0, T]$. Also the weak jump condition (2.3) is a byproduct of the proof, and (2.1) follows from the pointwise convergence of s_h. ■

Finally we prove that nonuniqueness always is caused by the same effect as in the examples given in 2.3–2.4.

2.7. Lemma. *Every solution of* (2.1)–(2.3) *has a finite number of switch times. For two different solutions* (u_1, s_1) *and* (u_2, s_2) *there is a* $t_0 \in [0, T[$ *with* $u_1 = u_2$ *on* $[0, t_0]$, *such that* t_0 *is a switch time, say, for* s_1 *but not for* s_2. *In addition, if* $t_0 > 0$, *then*

$$\pm f(t_0, u_i(t_0), s_2(t_0))$$

ly in the tangent cone of S_0 *at* $u_i(t_0)$ *in case* $s_2(t_0) = 0$, *and* S_1 *in case* $s_2(t_0) = 1$. *If* $t_0 = 0$ *then this holds for* $+f(0, u_0, s_0)$.

Proof. Pick any solution (u, s) and any $t_0 \geqslant 0$. Due to the continuity of u and (2.3) the function s is constant in $B_\varepsilon(t_0) \cup]0, T[$ for some $\varepsilon > 0$, provided $u(t_0) \in \mathbf{R}^m \setminus \partial S_0 \setminus \partial S_1$. By (2.2) this constant must be 0 or 1. If $u(t_0) \in \partial S_0$ then by (2.3) the function $s - s_0$ (with $s(t) := s_0$ for $t < 0$) is monotone nondecreasing in $B_\varepsilon(t_0)$ for some $\varepsilon > 0$. By (2.2) the function s is 0,1 valued. Therefore, if s is not constant in $B_\varepsilon(t_0) \cap]0, T[$, there is a $t_1 \geqslant 0$ such that for $|t - t_0| < \varepsilon, t \geqslant 0$,

$$s(t) = 0 \text{ if } t < t_1, s(t) = 1 \text{ if } t > t_1.$$

Since u is uniformly continous, ε does not depend on t_0. Consequently the number of switch times of s is finite.

Now consider two solutions and let $[0, t_1]$ the maximum interval on which $u_1 = u_2$. Then $s_1(t) \neq s_2(t)$ for a sequence of times $t \downarrow t_1$, hence the maximum interval $[0, t_0]$ on which $s_1 = s_2$ is contained in $[0, t_1]$. The case $t_0 < t_1$ occurs only if $f(t, u_i(t), 0) = f(t, u_i(t), 1)$ for $t_0 < t < t_1$. Anyway t_0 must be a switch time for s_1 or s_2, hence $u_i(t_0) \in \partial S_0 \cup \partial S_1$. For definiteness assume $u_i(t_0) \in \partial S_0$. Then by (2.2) and (2.3) the switch functions s_i must be con-decreasing near t_0. Therefore for t near t_0, say,

$$s_2(t) = 0, \text{ and } s_1(t) = \begin{cases} 0 & \text{for } t < t_0, \\ 1 & \text{for } t > t_0. \end{cases}$$

In particular $u_2(t) \in \bar{S}_0$ for t near t_0, which implies the last statement of the lemma. If $t_0 = 0$ then the above monotonicity yields $s_0 = 0$. ■

3. Existence for the heat operator

In this section we demonstrate, that the existence procedure also works, if the underlying operator L defines a partial differential equation. For simplicity we consider the easiest parabolic initial boundary value problem

$$\dot{u} - u'' + f(s) = 0 \text{ in }]0, T[\times \Omega.$$

In contrast to the previous section here we treat the situation, where every space point consists of a thermostat. We postulate that they work independent to each other, their relationship is governed by diffusion only. Therefore s is a function in time and space and with $Mu = u$ we require that $(u(t, x), s(t, x))$ lies in the switch configuration for almost all (t, x). We assume that u is a scalar and Ω an interval. This will be used in the local part of the existence proof.

3.1 Assumptions. Ω is an open bounded interval, $T > 0$, and $\Omega_T :=]0, T[\times \Omega$. The given Dirichlet data u_0 are regular, say in $H^{1,\infty}(\Omega_T)$ with $u_0'' \in L^\infty(\Omega_T)$ and $u_0' \in L^2(\Omega_T)$. We take $u_0(0, \cdot)$ as initial data. We can assume that f is linear in the switch variable s, that is,

$$f(t, x, s) := f_0(t, x)(1 - s) + f_1(t, x)\, s,$$

where f_0 and f_1 are bounded measurable functions. The monotonicity condition is

$$f_0(t, x) \leqslant 0 \leqslant f_1(t, x),$$

that is, switching of each thermostat changes sign of the production rate.

The initial state of the thermostat is given by a measurable function s_0 such that $(u_0(0, x), s_0(x)) \in \Sigma$ for almost all x. Here Σ is the switch configuration in (1.1) with $m = 1$, $S_0 =]-\infty, 1[$, and $S_1 =]0, \infty[$. We also assume that $u_0(0, x) \notin \{0, 1\}$ for almost all x.

In the following existence theorem we see that in general we cannot solve the diffusion equation with $f(s)$. However, as shown in Example 3.3, the existence result is optimal.

3.2 THEOREM. *Under the above assumptions there is a solution* (u, s) *of the thermostat problem with* $u - u_0 \in L^2(0, T; \mathring{H}^{1,2}(\Omega))$, $\partial_t u \in L^2(\Omega_T)$. *This means that* $u(0) = u_0$ *and*

$$\dot{u} = u'' + f(\sigma) = 0 \text{ in } L^2(\Omega_T) \text{ with } \sigma = s \text{ in } \{u \neq 0, 1\},$$

and in the remainder $f(\sigma) = 0$ *and* $0 \leqslant \sigma \leqslant s$ *on* $\{u = 0\}$,

$$s \leqslant \sigma \leqslant 1 \text{ on } \{u = 1\}. \tag{3.1}$$

$$(u(t, x), s(t, x)) \in \hat{\Sigma} \text{ for almost all } (t, x). \tag{3.2}$$

For all $\eta \in C_0^\infty([0, T[\times \Omega)$ *with* $\eta \geqslant 0$ *on* $\bar{\Omega}_T \cap \{u = 1\}$ *and* $\eta \leqslant 0$ *on* $\bar{\Omega}_T \cap \{u = 0\}$

$$\int_0^T \int_\Omega (s - s_0)\, \dot{\eta} \leqslant 0. \tag{3.3}$$

Proof: First we construct approximations (u_h, s_h) by time discretization. For this we approximate the Dirchlet data by

$$u_{0h}(ih, x) := \frac{1}{h} \int_{(i-1)h}^{ih} u_0(\tau, x)\, d\tau.$$

For any step function v in time we identify $v(ih)$ with its value on the time interval $](i-1)h, ih[$. We start with $u_h(0, x) := u_0(0, x)$ and $s_h(0, x) := s_0(x)$. Then, if $u_h(t-h)$ and $s_h(t-h)$ are already known, $u_h(t) \in H^{1,2}(\Omega)$ is defined as the solution of the Dirichlet problem

$$\frac{1}{h}(u_h(t) - u_h(t-h)) - u''(t) + \gamma_h(t) = 0 \text{ in } \Omega,\ u_h(t) = u_{0h}(t) \text{ on } \partial\Omega \tag{3.4}$$

with $\gamma_h(t, x) \in F_h(t, x, u_h(t, x))$. The monotone graph F_h is defined by

$$F_h(t, x, z) := f_{0h}(t, x) + (f_{1h}(t, x) - f_{0h}(t, x))\, G_h(t, x, z).$$

Here f_{0h} and f_{1h} are defined in the same manner as u_{0h}, and G_h by

$$G_h(t, x, z) := \begin{cases} 0 & \text{if } z < 0, \\ [0, s_h(t-h, x)] & \text{if } z = 0, \\ s_h(t-h, x) & \text{if } 0 < z < 1, \\ [s_h(t-h, x), 1] & \text{if } z = 1, \\ 1 & \text{if } z > 1. \end{cases}$$

This is possible since below $s_h(t)$ will be defined so that inductively $0 \leqslant s_h \leqslant 1$. Approximating F_h pointwise in (t, x) by continuous monotone functions in z we obtain in a standard manner a unique solution $u_h(t)$ of (3.4). Then we define (see (1.3))

$$s_h(t, x) := \begin{cases} 0 & \text{if } u_h(t, x) < 0, \\ s_h(t-h, x) & \text{if } 0 \leqslant u_h(t, x) \leqslant 1, \\ 1 & \text{if } u_h(t, x) > 1. \end{cases}$$

With

$$f_h(t, x, z) := f_{0h}(t, x)(1-z) + f_{1h}(t, x)\, z$$

we can write $\gamma_h(t, x) = f_h(t, x, \sigma_h(t, x))$, where $0 \leqslant \sigma_h(t, x) \leqslant 1$ and

$$\sigma_h(t, x) = s_h(t, x) \text{ if } u_h(t, x) \neq 0, 1; \tag{3.5}$$

and

$$\begin{aligned} 0 \leqslant \sigma_h(t, x) \leqslant s_h(t, x) & \text{ if } u_h(t, x) = 0, \\ s_h(t, x) \leqslant \sigma_h(t, x) \leqslant 1 & \text{ if } u_h(t, x) = 1. \end{aligned} \tag{3.6}$$

To derive convergence of u_h we first multiply the differential equation (3.4) with $u_h(t) - u_{0h}(t)$. Integrating over t we obtain the energy estimate

$$\sup_t \int_\Omega |u_h(t)|^2 + \int_0^T \int_\Omega |u_h'|^2 \leqslant C.$$

Then multiplying with

$$\frac{1}{h}(u_h(t) - u_h(t-h)) - \frac{1}{h}(u_{0h}(t) - u_{0h}(t-h))$$

and integrating over t we obtain after standard manipulations that

$$\int_0^T \int_\Omega \left| \frac{u_h(t) - u_h(t-h)}{h} \right|^2 dt + \sup_t \int_\Omega |u_h'(t)|^2 \leqslant C.$$

Therefore the piecewise linear interpolation

$$\tilde{u}_h(t, x) := \frac{1}{h} \int_t^{t+h} u_h(\tau, x)\, d\tau \text{ for } (t, x) \in \Omega_T$$

is bounded in $H^{1,2}(\Omega_T)$. It follows that there exist $u \in H^{1,2}(\Omega_T)$ and

$s, \gamma \in L^\infty(\Omega_T)$ so that for a subsequence

$$\begin{aligned}&\tilde{u}_h \to u \text{ weakly in } H^{1,2}(\Omega_T),\\&\tilde{u}_h \to u \text{ almost everywhere in } \Omega_T,\\&s_h \to s \text{ weakly star in } L^\infty(\Omega_T),\\&\gamma_h \to \gamma \text{ weakly star in } L^\infty(\Omega_T).\end{aligned}$$

Obviously $0 \leqslant s \leqslant 1$, and u is the weak solution of the initial boundary value problem

$$\begin{gathered}\dot{u} - u'' + \gamma = 0 \text{ in } \Omega_T,\\u(0) = u_0(0),\ u = u_0 \quad \text{on} \quad]0, T[\times \partial\Omega.\end{gathered} \tag{3.7}$$

Since γ is bounded, we can use the DeGiorgi regularity result, that is, u is Hölder continuous in $\bar{\Omega}_T$. Applying the DeGiorgi technique to the approximations, we also obtain that $\tilde{u}_h$ are equicontinuous in $\bar{\Omega}_T$. Therefore $\tilde{u}_h \to u$ uniformly, hence also $u_h \to u$ uniformly.

As a consequence we derive (3.3). In fact, if $\eta \in C_0^\infty([0, T[\times \Omega)$ with $\operatorname{supp} \eta \subset \{u > 0\}$, we have $\operatorname{supp} \eta \subset \{u \geqslant \delta\}$ for some $\delta > 0$. Hence, if h is small enough, $\eta(t, x) \neq 0$ implies $u_h(t, x) > 0$. Therefore by definition $s_h(t, x) \geqslant s_h(t-h, x)$. If in addition $\eta \geqslant 0$ it follows that

$$0 \leqslant \int_0^T \int_\Omega \frac{1}{h}(s_h(t, x) - s_h(t-h, x))\, \eta(t, x)\, dx\, dt =$$

$$= \int_0^T \int_\Omega (s_h(t, x) - s_0(x)) \frac{1}{h}(\eta(t, x) - \eta(t+h, x))\, dx\, dt \to$$

$$\to \int_0^T \int_\Omega (s - s_0)\, \dot{\eta}.$$

A corresponding inequality holds in $\{u < 1\}$, which proves (3.3).

Next we consider γ. σ_h was defined by

$$\gamma_h = f_{0h}(1 - \sigma_h) + f_{1h}\sigma_h,$$

and $0 \leqslant \sigma_h \leqslant 1$. Let σ be a weak star limit of σ_h in $L^\infty(\Omega_T)$. Since $f_{0h} \to f_0$ and $f_{1h} \to f_1$ in $L^1(\Omega_T)$ we derive

$$\gamma = f_0(1 - \sigma) + f_1\sigma = f(\sigma) \text{ and } 0 \leqslant \sigma \leqslant 1.$$

From (3.5), (3.6) and the uniform convergence of u_h we conclude that $0 \leqslant \sigma \leqslant s$ in $\{u < 1\}$ and $s \leqslant \sigma \leqslant 1$ in $\{u > 0\}$, and that $\sigma = s$ in $\{u \neq 0, 1\}$. Now, from (3.7) we see that $u'' \in L^2(\Omega_T)$. Therefore $\dot{u} = 0$ and $u'' = 0$ almost everywhere in $\Omega_T \cap \{u = 1\}$ and $\Omega_T \cap \{u = 0\}$. Then we get from (3.7) that $\gamma = 0$ almost everywhere in these sets. Hence all statements in (3.1) are proved.

It remains to show (3.2). Since u is continuous and $u_h \to u$ uniformly, in $\{u > 1\}$ and in $\{u < 0\}$ condition (3.2) can be proved in the same way as in 2.6.

Now let $(t_0, x_0) \in \Omega_T$ with $0 < u(t_0, x_0) < 1$. We have to show that at almost all such points $s(t_0, x_0) = 0$ or 1. First, by assumption on the initial data, we can choose x_0 with

$$u_0(0, x_0) \neq 0, 1. \tag{3.8}$$

Next, to derive a contradiction, we choose x_0 so that $\{0 < s < 1\}$ has density 1 at (t_0, x_0). Then by continuity we have $0 < u < 1$ in a neighbourhood of (t_0, x_0), and therefore (3.3) implies that $s = 0$ in this neighbourhood. Consequently the function $s(t_0)$ is well defined in a neighbourhood of x_0. and the above choice of (t_0, x_0) means that $\{0 < s(t_0) < 1\} \subset \Omega$ has density 1 at x_0, that is,

$$\frac{1}{2\varepsilon} \mathscr{L}^1 \left(B_\varepsilon(x_0) \cap \{0 < s(t_0) < 1\}\right) \to 1 \text{ as } \varepsilon \to 0. \tag{3.9}$$

First let us consider the case that

$$0 < u(t, x_0) < 1 \text{ for all } 0 \leqslant t \leqslant t_0.$$

Then for some $\varepsilon > 0$ and sufficient small h

$$\varepsilon \leqslant u_h \leqslant 1 - \varepsilon \text{ in } [0, t_0] \times \overline{B_\varepsilon(x_0)}.$$

Hence by construction $s_h(t, x) = s_h(t-h, x)$ for (t, x) in this region. Therefore $s_h(t, x) = s_0(x)$ and then also $s(t, x) = s_0(x)$. Since $(u_0(0, x), s_0(x)) \in \Sigma$ by assumption, we conclude that $s(t_0)$ has only values 0 and 1 almost everywhere in $B_\varepsilon(x_0)$, which is a contradiction to (3.9).

Next we assume that $u(t, x_0) \notin]0, 1[$ for some $t \leqslant t_0$. Since u is continuous and using (3.8) we find a finite number of times

$$t_0 > t_1 > \ldots > t_k = 0$$

with $u(t_j, x_0) \neq 0, 1$, so that u on each interval $[t_{j+1}, t_j]$ ranges in $]0, \infty[$ or $]-\infty, 1[$. By the continuity of u and the uniform convergence of u_h we can choose $\varepsilon > 0$ so that the same is true on $\overline{B_\varepsilon(x_0)}$, that is,

$$u(t_j, x) \neq 0, 1 \text{ for } j = 0, \ldots, k \text{ and } |x - x_0| \leqslant \varepsilon,$$

and for $j = 0, \ldots, k-1$

$$u\left([t_{j+1}, t_j] \times \overline{B_\varepsilon(x_0)}\right) \subset]0, \infty[\text{ or }]-\infty, 1[. \tag{3.10}$$

First let us prove that

$$0 \leqslant u(t, x_0) \leqslant 1 \text{ for } t_1 \leqslant t \leqslant t_0, \tag{3.11}$$

where for definiteness we take the second case in (3.10). Then (3.3) yields

$$\dot{s} \leqslant 0 \text{ in } [t_1, t_0] \times \overline{B_\varepsilon(x_0)}. \tag{3.12}$$

Now assume $u(\tau, x_0) < 0$ for some $\tau \in]t_1, t_0[$. By uniform convergence there is a neighbourhood $B_\delta((\tau, x_0))$ such that the inequality $u_h < 0$ holds uniformly for all small h. Then by definition $s_h = 0$ in that neighbourhood, hence also $s = 0$. But then (3.12) implies that $s = 0$ in $]\tau, t_0[\times B_\delta(x_0)$, in particular $s(t_0) = 0$ in $B_\delta(x_0)$, a contradiction to (3.9).

As a consequence of (3.10) and (3.11) we have $0 < u(t_1, x) < 1$ for $|x - x_0| \leqslant \varepsilon$. (If $t_1 = 0$, we replace t_1 by an arbitrary small time and define $t_2 := 0$.) Therefore as above the function $s(t_1)$ is well defined in $B_\varepsilon(x_0)$. We distinguish between two cases.

The first case is that $\{0 < s(t_1) < 1\}$ has positive measure in $B_{\varepsilon/2}(x_0)$. Then we find a point $x_1 \in B_{\varepsilon/2}(x_0)$ such that $\{0 < s(t_1) < 1\}$ has density 1 at x_1, that is (3.9) is satisfied for $s(t_1)$ at x_1. In that case we start recursively the whole argument following (3.10) with (t_1, x_1) instead of (t_0, x_0) and with $\frac{\varepsilon}{2}$ as new ε. Note that the time values $t_2, \ldots, t_k$ can be left unchanged.

The second case then is, that (calling $\frac{\varepsilon}{2}$ the new ε)

$$s(t_1, x) = 0 \text{ or } 1 \text{ for almost all } x \in B_\varepsilon(x_0).$$

Since the assumptions of the theorem are symmetric with respect to the switch values, we again have to consider only the second case in (3.10), which is

$$u(t, x) < 1 \text{ for } t_1 \leqslant t \leqslant t_0 \text{ and } |x - x_0| \leqslant \varepsilon. \tag{3.13}$$

Now, if the denisty of $\{s(t_1) = 0\}$ at x_0 would be positive, we infer from (3.12) that the same is true for $\{s(t_0) = 0\}$, which by (3.9) is impossible. Thus we are left with the case that

$$\{s(t_1) = 1\} \text{ has density 1 at } x_0. \tag{3.14}$$

We have to show that together with (3.13) this leads to a contradiction to (3.9).

It follows from (3.9) that there are values

$$x_0 - \varepsilon < x_- < x_0 < x_+ < x_0 + \varepsilon$$

such that

$$\{s(t_0) > 0\} \text{ has density 1 at } x_- \text{ and } x_+. \tag{3.15}$$

Since we consider $\{x_-, x_+\}$ as the boundary of $]x_-, x_+[$, here we make essential use of the fact that Ω is one dimensional. As in the proof of (3.11) it follows that $u(t, x_-) \geqslant 0$ and $u(t, x_+) \geqslant 0$ for $t_1 \leqslant t \leqslant t_0$. By (3.10) and (3.11) we also know that $0 < u(t_j, x) < 1$ for $j = 0, 1$ and $|x - x_0| \leqslant \varepsilon$.

Thus

$$u \geqslant 0 \text{ on } \partial D, \text{ where } D :=]t_1, t_0[\times]x_-, x_+[. \tag{3.16}$$

Since $\sigma = s = 0$ in $\{u < 0\}$ by (3.1), we see that

$$\dot{u} - u'' = -f(\sigma) = -f(0) \geqslant 0 \text{ in } D \cap \{u < 0\}.$$

Together with (3.16) the weak maximum principle yields $u \geqslant 0$ in D. We want to have the same for u_h. First for small h

$$0 < u_h(t, x) < 1 \text{ for } t = t_1, t_0 \text{ and } |x - x_0| \leqslant \varepsilon, \tag{3.17}$$

$$u_h(t, x) < 1 \text{ for } t_1 \leqslant t \leqslant t_0 \text{ and } |x - x_0| \leqslant \varepsilon. \tag{3.18}$$

Next we consider the lateral boundary of D. Let $y = x_-$ or $y = x_+$. We know that $0 < u_h < 1$ uniformly for small h in a neighbourhood $B_\delta((t_0, y))$. Therefore by definition s_h is time independent in this neighbourhood. The same holds for s. This implies that the weak star convergence of s_h to s in $L^\infty(B_\delta(t_0, y))$ is equivalent to the weak star convergence of $s_h(t_0)$ to $s(t_0)$ in $L^\infty(B_\delta(y))$. By (3.15) we can choose δ so that

$$\frac{1}{2\delta} \int_{B_\delta(y)} s(t_0) \geqslant \frac{1}{2}.$$

Then for small h

$$\frac{1}{2\delta} \int_{B_\delta(y)} s_h(t_0) \geqslant \frac{1}{4}.$$

Therefore for small h we can pick a point $y_h \in B_\delta(y)$ for which $\{s_h(t_0) > 0\}$ has density 1 at y_h. This implies that

$$u_h(t, y_h) \geqslant 0 \text{ for } t_1 \leqslant t \leqslant t_0. \tag{3.19}$$

If not, then $u_h(\tau, y_h) < 0$ for some $\tau \in]t_1, t_0[$. But then also $u_h(\tau, z) < 0$ for $z \in B_{\delta_h}(y_h)$ for some $\delta_h > 0$. By definition of s_h this means that $s_h(\tau, z) = 0$ for $z \in B_{\delta_h}(y_h)$. If δ and δ_h were chosen small enough then $B_{\delta_h}(y_h) \subset B_\varepsilon(x_0)$. Using (3.18) we then conclude that $s_h(t, z) = 0$ for $\tau \leqslant t \leqslant t_0$ and $|z - y_h| \leqslant \delta_h$. This is a contradiction to the choice of y_h.

Thus replacing y by y_h, that is, x_- and x_+ by certain x_-^h and x_+^h, (3.17) and (3.19) gives

$$u_h \geqslant 0 \text{ on } \partial D_h, \text{ where } D_h :=]t_1, t_0[\times]x_-^h, x_+^h[.$$

Now consider any time $t \in [t_1 + h, t_0]$ such that

$$u_h(t - h) \geqslant 0 \text{ in } \Omega_h :=]x_-^h, x_+^h[,$$

which is true for $t = t_1 + h$. Then multiplying (3.4) by $\zeta := \min(u_h(t), 0)$

we obtain

$$\int_{\Omega_h} \frac{1}{h} |\zeta|^2 + \int_{\Omega_h} \frac{1}{h} u_h(t-h) \cdot (-\zeta) + \int_{\Omega_h} |\zeta'|^2 + \int_{\Omega_h} f_h(\sigma_h(t)) \zeta = 0. \tag{3.20}$$

Clearly $\zeta(x) \neq 0$ implies $u_h(t, x) < 0$ and therefore $\sigma_h(t, x) = s_h(t, x) = 0$ by (3.5). We conclude

$$f_h(t, x, \sigma_h(t, x)) = f_h(t, x, 0) \leqslant 0.$$

Thus the last term in (3.20) is nonnegative, as are the others. So they must be zero, that is, $u_h(t) \geqslant 0$. Together with (3.18) and the definition of s_h we also obtain that $s_h(t) = s_h(t-h)$.

Applying this maximum principle inductively in time, we deduce that $u_h \geqslant 0$ in D_h, and that s_h is time independent in D_h. Since the δ in the proof of (3.19) was arbitrarily small, we obtain in the limit $h \to 0$ that $\dot{s} = 0$ in D. But then (3.14) implies that also $\{s(t_0) = 1\}$ has density 1 at x_0, a contradiction to (3.9). ∎

3.3. Example. Let $\Omega =]-1, 1[$, $f_0 < 0 < f_1 = 1$, and $s_0 = 1$. Define

$$u(t, x) := \begin{cases} v(x-t) & \text{for } x \geqslant t \geqslant 0, \\ 0 & \text{for } 0 \leqslant x \leqslant t \end{cases}$$

with

$$v(\xi) := \xi - 1 + e^{-\xi},$$

and continue u to the left as an even function in x. Then, if

$$\sigma := \begin{cases} 1 & \text{in } \{u > 0\}, \\ -\dfrac{f_0}{f_1 - f_0} & \text{in } \{u = 0\}, \end{cases}$$

(u, σ) satisfies

$$\dot{u} - u'' + f(\sigma) = 0.$$

There are many possibilities to choose s so that one obtains a solution of the thermostat problem in the sense of (3.1)–(3.3). The easiest choice is $s = 1$. But also any function s with

$$s = 1 \text{ in } \{u > 0\},\ s \geqslant \sigma \text{ in } \{u = 0\},$$

which is nondecreasing in time will be a solution. However, since $f_0 < 0$, s is never allowed to take values 0, because (3.1) contains the condition that $0 \leqslant \sigma \leqslant s$ on $\{u = 0\}$. Note also the only s for which (u, s) ranges in Σ (not $\hat{\Sigma}$) is $s = 1$. Another observation is, that if $f_0 =$ const then $s = \sigma$ seems to be the physical solution, since it means that the change in the production rate is due to a certain density of switching atomic thermostats. On the other hand, if $f_0 < 0$ then the choice $s = \sigma$ would contradict (3.3). (We assume that the upper threshold of the thermostat is near infinity.)

The following consideration is more important. Let us consider the above solution only up to a certain small time t_0. Then we start to increase the boundary data to a high level, but still below the upper threshold. By the maximum principle the solution must be strictly positive after some time. There are two possibilities. The first one is

$$s(t, x) = \begin{cases} 1 \text{ for } t > t_0, |x| > t_0, \\ 0 \text{ for } t > t_0, |x| < t_0, \end{cases}$$

and $u(t) > 0$ in Ω for $t > t_0$. For $t_0 = 1$ we have with an appropriate choice of the Dirichlet data $u(t, x) = -f_0 \cdot (t-1)$ for $t > 1$. These solutions correspond to the functions $u_\varkappa$ in 2.3. The second possibility is that $s = 1$ all time, therefore $\sigma \neq s$ in $\{u = 0\}$. These solutions are reached by the approximation in the proof of 3.2. To see this, define

$$u_h(t, x) := v_h(x-t) \text{ for } x \geqslant t \geqslant 0,$$

where

$$v_h(\xi) := \xi - 1 + e^{-\alpha\xi},$$

and α is the unique solution of

$$e^{-\alpha h} = 1 - h\alpha^2 \text{ with } \alpha = 1 - \frac{h}{2} + O(h^2).$$

Then

$$\frac{1}{h}(u_h(t, x) - u_h(t-h, x)) - u_h''(t, x) + 1 = 0 \text{ for } x > t.$$

Furthermore, define

$$u_h(t, x) := 0 \text{ for } 0 \leqslant x \leqslant t,$$

and σ_h by

$$f(t, x, \sigma_h(t)) := \begin{cases} 1 & \text{for } x > t > 0, \\ \frac{1}{h} v_h(x-(t-h)) & \text{for } 0 < t-h < x < t, \\ 0 & \text{for } 0 < x < t-h. \end{cases}$$

Then $0 \leqslant \sigma_h \leqslant 1$, $\sigma_h = 1$ in $\{u_h > 0\}$, and u_h solves

$$\frac{1}{h}(u_h(t) - u_h(t-h)) - u_h''(t) + f(\sigma_h(t)) = 0$$

for each $t = ih$. Since $u_h \geqslant 0$ the algorithm in the proof of Theorem 3.2

says $s_h = 1$.

3.4. EXAMPLE. This example explains why it has been assumed that $[f_0, f_1]$ encloses the value 0. Let $\Omega =]0, 1[$, f_0 a constant with $0 < f_0 < f_1 = 1$, and again $s_0 = 1$. Choose any $\gamma \in [f_0, 1[$ and let

$$u(t, x) := \begin{cases} v_+ (x-t) & \text{for } x \geqslant t, \\ \gamma v_- (t-x) & \text{for } x \leqslant t. \end{cases}$$

Here v_+ is the same function as in Example 3.3 and

$$v_-(\xi) := e^{\xi} - 1 - \xi.$$

Then $u \in C^{1,1}([0, \infty[\times \mathbf{R})$ and

$$\dot{u} - u'' + f(\sigma) = 0 \text{ in }]0, \infty[\times \Omega,$$

if $\sigma := 1$ in $\{x > y\}$ and $0 \leqslant \sigma < 1$ in $\{x < t\}$ with $f(\sigma) = \gamma$. For this example the procedure in the existence proof will not work. Indeed if $s := \sigma$ then conditions (3.1) and (3.3) for a solution are satisfied, but not (3.2) if $\gamma > f_0$. It should be possible to prove that the approximation u_h in the proof of 3.2 with $s_h(0) = 1$ converge to u, since u_h creates small intervals of negativity near $\{u = 0\}$, but each such interval in the next time step immediately disappears.

3.5. REMARK. It might be unreasonable to consider the thermostat problem with the reversed monotonicity $f_1 < f_0$. For example, let $f_1 = 0$, $f_0 = 1$, and $\Omega =]0, \pi[$. Then

$$v(t, x) := e^{-t} \sin x, \ \sigma := s := 1$$

is a solution. But there will be another one, say u, which at a certain time creates a set $\{u < 0\}$ near $\partial\Omega$. Of course, he discrete solutions v_h with $s_h(0) = 1$ converge to v. But it seems that there are discrete solutions u_h with

$$\int_{\Omega} (1 - s_h(0)) \to 0 \text{ as } h \to 0,$$

which converge to u.

References

[1] GLASHOFF K., SPREKELS J. The regulation of temperature by thermostats and set-valued integral equations. *J. Int. Equ.* **4** (1982), 95–112.

[2] JÄGER W. A diffusion-reaction system modelling spatial patterns. Equadiff., in: Teubner Texte zur Math. 47, Bratislava 1981, 1982.

[3] VISINTIN A. Evolution problems with hysteresis in the source term. *SIAM J. Appl. Math.* (to appear).

O zagadnieniu termostatu

W pracy rozważane jest zagadnienie termostatu opisywane przez równanie $L(u, s) = 0$, gdzie L jest operatorem różniczkowym względem u; $s = s(\dot{M}u)$ odpowiada zmiennej przełączania termostatu. Zakładając że L jest operatorem różniczkowym zwyczajnym lub operatorem różniczkowym cząstkowym oraz przyjmując różne postacie M udowodniono istnienie rozwiązań zagadnienia termostatu.

О проблеме термостата

В работе обсуждается проблема термостата описанная уравнением $L(u, s) = 0$ где L является дифференциальным оператором относительно u, $s = s(Mu)$ соответствует переменной переключения термостата. В предположении что L обыкновенный дифференциальный оператор или оператор в частных производных, и принимая разные формы М доказывается существование решений проблемы термостата.