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On a free boundary problem related to an irreversible process*

by

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The mathematical model studied here describes the evolution of the thermal field in a material undergoing an irreversible change of structure at a prescribed temperature with latent heat absorption. An existence theorem is proved for the multidimensional case. More results (particularly a uniqueness theorem) are given for the one-dimensional case and some results of numerical computations are presented.

1. Introduction

Intumescent paints when heated above some threshold temperature by external heat sources change their structure, releasing gas and absorbing heat. The process is used e.g. to create a protective coating against fire, owing to the low thermal conductivity of the material produced, which looks swollen and porous. Such a transformation is of course irreversible. In [1], [2] a first approach was considered basically consisting in a Stefan problem. Only the case of monotone processes was taken into account, as suggested by the technical problem dealt with. On the other hand, we shall see that introducing irreversibility in free boundary problems (even in the simple

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Stefan's scheme) leads to non-standard problems which are very interesting and by no means trivial. The resulting class of problems seems to be related to a number of phenomena, presently under investigation, which are characterized by an irreversible change of structure (see also Remark 1.4).

In usual, reversible change of phase processes it is natural to introduce an energy-temperature relationship of the form

$$E = u + \lambda H(u) \tag{1.1}$$

($\lambda > 0$ is the latent heat and the heat capacity has been taken equal to one for simplicity), where $H(u)$ is the Heaviside graph, jumping at the temperature $u = 0$, at which the transition from "state 1" ($u < 0$) to "state 2" ($u > 0$) occurs.

In an irreversible process, once the material is in state 2 it will never pass to state 1, even if the temperature becomes negative. Thus, if at a point P_0 the material is in state 1 (necessarily $u < 0$ in P_0) latent heat will be absorbed only the first time the temperature $u = 0$ is crossed. Let us seek for a modification of (1.1) accounting for irreversibility.

Let Ω be a bounded domain in $\mathbf{R}^n$ occupied by the heat conducting medium. Assume that $\partial\Omega$ is smooth and for $0 < T < \infty$ set $\Omega_T = \Omega \times (0, T]$. Denote by Ω_1, Ω_2 the subsets of Ω_T corresponding to state 1 and to state 2, respectively. The temperature $u(x, t)$ satisfies the heat equation in Ω_1 and in Ω_2 (for simplicity we set all thermal coefficients equal to one in both phases, although this is not needed for the formulation below). For a classical solution the interface $\Gamma \subset Q_T$ can be represented by the equation $\Phi(x, t) = 0$, where Φ is a C^1 function such that $\Phi < 0$ in state 1 and $\Phi > 0$ in state 2.

On Γ we have the continuity of $u(x, t)$ and either Stefan type conditions (brakets denoting jumps, as usual)

$$u = 0, \; [\nabla_x u] \cdot \nabla_x \Phi = \lambda \Phi_t$$

($\lambda =$ latent heat), or just a "diffraction" type thermal balance

$$[\nabla_x u] \cdot \nabla_x \Phi = 0,$$

according to whether state 2 is forming (i.e. $\Phi_t > 0$) or not ($\Phi_t = 0$).

If we want to interpret the above model (complemented with initial and boundary conditions) in a weak sense, the standard procedure of multiplying by a test function from a suitable space and integrating by parts leads to the following differential equation

$$\partial/\partial t \, (u + \lambda\chi) - \Delta u = 0,$$

in the sense of $\mathscr{D}'(\Omega_T)$, where χ is the characteristic function of the region occupied by state 2.

The effect of irreversibility amounts to the formal identification

$$\chi = H\big(\sup_{0<\tau<t} u(x,\tau)\big).$$

Hence it looks natural to substitute (1.1) with

$$E(x,t) = u(x,t) + H\big(\sup_{0<\tau<t} u(x,\tau)\big) \tag{1.2}$$

and to proceed to formulating the problem as follows.

Let $u_0(x), f(x,t)$ be given functions such that

(A$_1$) $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$,

(A$_2$) $f \in C\big(0,T; L^2(\Omega)\big) \cap L^2\big(0,T; H^{\frac{1}{2}}(\partial\Omega)\big)$,

$$\|u_0\|_{\infty,\Omega} + \|f\|_{\infty,\partial\Omega\times(0,T)} \leqslant M \tag{1.3}$$

for some constant $M>0$.

The functions u_0 and f will play the role of the initial temperature and of the boundary data respectively.

We assume that the support of u_0^+ does not coincide with $\bar{\Omega}$, since otherwise the problem is trivial.

The initial state of the system is described by a function $\zeta_0(x) \in H\big(u_0(x)\big)$, $x \in \Omega$.

Problem (P) Find a pair (u,ξ) such that

$$u \in C\big(0,T; L^2(\Omega)\big) \cap L^2\big(0,T; H^1(\Omega)\big) \cap L^\infty(\Omega_T), \tag{1.4}$$

$$u = f \text{ on } \partial\Omega \times (0,T) \text{ in the sense of the traces,} \tag{1.5}$$

$$\xi(x,t) \text{ is included in } H\big(\operatorname*{ess\,sup}_{0<\tau<t} u(x,t)\big) \text{ in the sense of the graphs,} \tag{1.6}$$

and such that the equation

$$\iint_{\Omega_T} \{-(u+\xi)\,\varphi_t + \nabla_x u \nabla_x \varphi\}\,dx\,dt = \int_\Omega (u_0+\xi_0)\,\varphi(\cdot,0)\,dx \tag{1.7}$$

is satisfied for all $\varphi \in H^1(\Omega_T) \cap L^2\big(0,T; H_0^1(\Omega)\big)$, which vanish for $t=T$.

Here and in the following we assume $\lambda = 1$

Remark 1.1. The above scheme includes the possibility of mushy regions evolving in the system, although it does not specify their degree of irreversibility. For complete irreversibility one should replace $\xi(x,t)$ in (1.7) with $\operatorname*{ess\,sup}_{0<\tau<t} \xi(x,\tau)$, with $\xi(x,\tau)$ included in $H\big(u(x,\tau)\big)$. Of course this remark is irrelevant when the measure of the set of zero temperature is zero. It can be observed that this kind of artificial monotonicity of energy in mushy regions represents a limiting case of hysteresis with "open cycles" (the closing

side being moved to infinity). Change of phase problems with hysteresis have been studied in [3] (see also the literature quoted therein). The present treatment contains substantial differences.

REMARK 1.2. In stating Problem (P) we assumed implicitly that at any point where $u_0(x) < 0$ the material is in state 1 for $t = 0$. However it is not difficult to take into account the possibility that in some subset of $u_0 < 0$ the material is in state 2. It suffices to introduce a suitable non-negative function $\alpha_0(x)$ and to replace $H(u_0(x))$ by $H(u_0(x) + \alpha_0(x))$, and $H(\operatorname{ess\,sup}_{0<\tau<t} u(x, \tau))$ by $H(\operatorname{ess\,sup}_{0<\tau<t} u(x, \tau) + \alpha_0(x))$. Such a substitution leaves all the analysis presented here unaffected.

REMARK 1.3. In the model considered here all thermal coefficients are taken equal to one. However we can expect they depend in general on u and on $\sup_{0<\tau<t} u(x, \tau)$, being different in state 1 and in state 2. Another simplification introduced consists in neglecting the deformation undergone by the material. These simplifications are not crucial for the one-dimensional case. Finally the gas dynamics is neglected as well as its interaction with heat transport. The motivation for considering such a simplified model is to focus our attention on the difficulties involved by irreversibility.

REMARK 1.4. Some other examples of free boundary problems with irreversibility can be found in the literature. An oxygen diffusion-consuption model studied in [4] divides the tissue in three zones: the alive zone (diffusing and absorbing; oxygen above some threshold concentration $\lambda > 0$) a sort of reservoir zone (reversible, absorbing but not diffusing; oxygen concentration $u \in (0, \lambda)$), and the dead zone (irreversible, diffusing but not absorbing; $\inf_{0<\tau<t} u(x, \tau) = 0$).

Change of phase with removal of the formed phase (see [5]) is a one-phase problem with artificial monotonicity of the free boundary. The direct accessibility of the phase front makes this problem substantially different from the one studied here.

Another class of irreversible phenomena with free boundaries is represented by pyrolysis and combustion of intumescent polymers as described e.g. in [6] (see also the literature quoted therein).

In this paper the following existence theorem will be proved.

THEOREM 1.5. *Under assumptions (A_1)–(A_2), Problem P possesses at least one solution.*

In addition some comments will be made concerning the one-dimensional case (Thm. 8.1). In particular, uniqueness can be shown to hold in this case, although in a special class.

The plan of the paper is as follows. In Sec. 2 the problem is reformulated in a way which is more suitable for the use of compactness arguments. In Sections 3, 4 and 5 a regularized problem is introduced (by smoothing H) and solved. Further a priori estimates on the approximating solutions are obtained in Sec. 6, while in Sec. 7 it will be concluded that there exists a sequence of approximating solutions converging to the solution of the reformulated problem. A basic lemma will concern the equicontinuity of the negative part of the regularized solutions.

Finally, it will be shown that the solution obtained for the reformulated problem is actually a solution of problem (P) and that its negative part is continuous.

One-dimensional problems will be discussed in Sec. 8 along with some numerical computation.

2. An auxiliary formulation

In order to construct a solution to Problem (P) we will use a compactness argument on a sequence of regularized solutions. However in such a setting it is hard to recover $\xi(x, t)$ in $H\big(\operatorname{ess\,sup}_{0<\tau<t} u(x, \tau)\big)$ as a limit of a convergent sequence. For this reason it is convenient to replace $H\big(\operatorname{ess\,sup}_{0<\tau<t} u(x, \tau)\big)$ in (1.6) by $H\big(\int_0^t u^+(x, \tau)\,d\tau + u_0^+(x)\big)$ and to note that $H(u_0(x)) \subset H(u_0^+(x))$. Of course we cannot say a priori that the two formulations are equivalent. Indeed, if for some (x, t) we have $\operatorname{ess\,supp}_{0<\tau<t} u(x, \tau) < 0$, the first formulation implies $\xi(x, t) = 0$, but the same conclusion is not generally true in the second one.

Therefore, once a solution to the reformulated problem will be obtained, we will have to show that $\xi(x, t)$ actually vanishes whenever $\operatorname{ess\,sup}_{0<\tau<t} u(x, \tau) < 0$.

3. The approximating problem (P_ε)

For each $\varepsilon > 0$ set

$$H_\varepsilon(s) = \begin{cases} 1 & \text{if } s \geqslant \varepsilon \\ s/\varepsilon & \text{if } 0 \leqslant s < \varepsilon, \\ 0 & \text{if } s < 0 \end{cases} \tag{3.1}$$

and

$$\alpha_\varepsilon(x) = \begin{cases} u_0^+(x), & \text{if } u_0^+(x) \geqslant \varepsilon, \\ \varepsilon\xi_0(x), & \text{if } u_0^+(x) \in [0, \varepsilon). \end{cases} \tag{3.2}$$

In such a way

$$H_\varepsilon(\alpha_\varepsilon(x)) = \xi_0(x), \qquad x \in \Omega, \tag{3.3}$$

for all $\varepsilon > 0$.

We look for a solution $u_\varepsilon(x, t)$ of the following problem.

Problem (P_ε) Find $u_\varepsilon \in C(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, $u_\varepsilon \in H^1_{\text{loc}}(\Omega_T)$, such that the equation

$$\frac{\partial}{\partial t}\left[u_\varepsilon + H_\varepsilon\left(\int_0^t u_\varepsilon^+(x, \tau)\, d\tau + \alpha_\varepsilon(x)\right)\right] - \Delta u_\varepsilon = 0 \tag{3.4}$$

is satisfied in $\mathscr{D}'(\Omega_T)$ and

$$u_\varepsilon(x, t) = f(x, t),\ (x, t) \in \partial\Omega \times (0, T], \tag{3.5}$$

$$u_\varepsilon(x, 0) = u_0(x),\ x \in \Omega \tag{3.6}$$

in the sense of the traces.

More explicitly, u_ε satisfies (3.5) and

$$\iint_{\Omega_T} \{-[u_\varepsilon + H_\varepsilon(\int_0^t u_\varepsilon^+(x, \tau)\, d\tau + \alpha_\varepsilon(x))]\, \varphi_t +$$

$$+ \nabla_x u \cdot \nabla_x \varphi\}\, dx\, dt = \int_\Omega (u_0 + \xi_0)\, \varphi(x, 0)\, dx \tag{3.7}$$

for all φ in the space of test functions specified in Sec. 1.

Let us first prove uniqueness.

Proposition 3.1. *(P_ε) has at most one solution.*

The proof is standard. Let u_1, u_2 be two solutions and set $w = u_1 - u_2$. Choosing $\varphi(x, t) = \int_t^T w(x, \tau)\, d\tau$ as a test function, it is not difficult to get the inequality

$$\iint_\Omega w^2\, dx\, dt \leqslant (T/\varepsilon) \iint_\Omega w^2\, dx\, dt, \tag{3.8}$$

which implies $w \equiv 0$ in $\Omega \times (0, \varepsilon/2]$. An iteration process yields uniqueness in Ω_T for arbitrary $T > 0$.

To prove existence, we use a time discretization procedure motivated by the following formal calculation.

Let us introduce the function

$$v(x, t) = \int_0^t u(x, \tau)\, d\tau. \tag{3.9}$$

It is easily seen that (3.4)–(3.6) reduces to

$$v_t - \Delta v = -H_\varepsilon\Big(\int_0^t v_t^+ \, d\tau + \alpha_\varepsilon(x)\Big) + u_0 + \xi_0 \tag{3.10}$$

$$v(x,t) = \int_0^t f(x,\tau)\, d\tau, \quad (x,t) \in \partial\Omega \times (0,T]. \tag{3.11}$$

$$v(x,0) = 0, \quad x \in \Omega. \tag{3.12}$$

The term $H_\varepsilon\big(\int_0^t v_t^+ \, d\tau + \alpha_\varepsilon\big)$ can be regarded as a nonlinear source containing the "history" of the process. Similarly, our discretized problem will contain at each time step the "memory" of the values of the solution at every previous time.

Next section will be devoted to prove existence of a solution to the discretized problems. In Section 5 we will prove the existence of u_ε.

4. Time discretized approximation to (P_ε)

For any $n \in \mathbf{N}$ set $h = T/n$, $t_i = ih$, $i = 0, 1, \dots, n$, and consider the discretized problem

$$v(x, t_0) = 0, \quad x \in \Omega, \tag{4.1}$$

$$\frac{v(x,t_{i+1}) - v(x,t_i)}{h} - \Delta v(x, t_{i+1}) =$$

$$= -H_\varepsilon\Big(\sum_{j=0}^{i} [v(x,t_{j+1}) - v(x,t_j)]^+ + \alpha_\varepsilon(x)\Big) + u_0(x) + \xi_0(x) \ \text{ in } \mathscr{D}'(\Omega), \tag{4.2}$$

$$v(x, t_{i+1}) = h \sum_{j=0}^{i} f(x, t_{j+1}), \quad x \in \partial\Omega, \ i = 0, \dots, n-1. \tag{4.3}$$

PROPOSITION 4.1. *The above recursive scheme uniquely defines*

$$v(x, t_i) \in H^2(\Omega) \cap L^\infty(\Omega) \cap C^{1+\alpha}(\Omega)$$

for $i = 0, \dots, n$.

Proof. The existence is proved by the Schauder-Leray fixed point theorem, via standard elliptic estimates, since $H_\varepsilon(\cdot)$ is Lipschitz continuous. Uniqueness follows from the monotonicity of $H_\varepsilon(\cdot)$.

Now we define

$$u(x, t_0) = u_0(x), \tag{4.4}$$

$$u(x, t_{i+1}) = \frac{v(x, t_{i+1}) - v(x, t_i)}{h}, \tag{4.5}$$

$i = 0, 1, \ldots, n-1$, $x \in \Omega$ and we consider the time-piecewise constant function defined by

$$\tilde{u}(x, 0) = u_0(x), \tag{4.6}$$

$$\tilde{u}(x, t) = u(x, t_{i+1}), \text{ if } t_i < t \leqslant t_{i+1},\ i = 0, 1, \ldots, n-1. \tag{4.7}$$

It is easy to verify that

$$\sum_{j=0}^{i} [v(x, t_{j+1}) - v(x, t_j)]^+ = \int_0^{t_{i+1}} \tilde{u}^+(x, \tau)\, d\tau \tag{4.8}$$

and that

$$\frac{u(x, t_{i+1}) - u(x, t_i)}{h} - \Delta u(x, t_{i+1}) =$$

$$= -\frac{1}{h}\left\{H_\varepsilon\left(\int_0^{t_{i+1}} \tilde{u}^+(x, \tau)\, d\tau + \alpha_\varepsilon(x)\right) - H_\varepsilon\left(\int_0^{t_i} \tilde{u}^+(x, \tau)\, d\tau + \alpha_\varepsilon(x)\right)\right\}$$

$$\text{a.e. in } \Omega, \tag{4.9}$$

$$u(x, t_{i+1}) = f(x, t_{i+1}) \text{ on } \partial\Omega,\ i = 0, 1, \ldots, n-1. \tag{4.10}$$

We proceed to derive a priori estimates independent of h.

LEMMA 4.1. *There exists a constant γ independent of h and of ε, such that*

$$\int_\Omega u^2(x, t_i)\, dx \leqslant \gamma,\ 0 \leqslant i \leqslant n, \tag{4.11}$$

$$\iint_{\Omega_T} |\nabla_x \tilde{u}|^2\, dx\, d\tau \leqslant \gamma. \tag{4.12}$$

Proof. Let $w(x, t_i)$ be recursively defined by

$$\frac{w(x, t_{i+1}) - w(x, t_i)}{h} - \Delta w(x, t_{i+1}) = 0 \text{ in } \Omega, \tag{4.13}$$

$$w(x, t_0) = u_0(x), \tag{4.14}$$

$$w(x, t_i) = f(x, t_i) \text{ on } \partial\Omega,\ i = 0, 1, \ldots, n. \tag{4.15}$$

It is well known that such a w exists and is unique, and that the following estimate holds

$$|||w||| \leqslant C, \tag{4.16}$$

where

$$|||w||| = \sum_{i=0}^{n-1} h^{-1} \int_\Omega (w(x, t_{i+1}) - w(x, t_i))^2 \, dx + \\ + \sum_{i=0}^{n-1} h \int_\Omega [\nabla_x w(x, t_{i+1})]^2 \, dx + \max_{0 \leqslant i \leqslant n} \|w(\cdot, t_i)\|_{\infty,\Omega} \tag{4.17}$$

and C is a constant depending on the data, but not on h. The function $U = \tilde{u} - w$ satisfies

$$\frac{U(x, t_{i+1}) - U(x, t_i)}{h} - \Delta U(x, t_{i+1}) = \\ = -\frac{1}{h}\left\{H_\varepsilon\left(\int_0^{t_{i+1}} \tilde{u}^+(x, \tau)\, d\tau + \alpha_\varepsilon(x)\right) - H_\varepsilon\left(\int_0^{t_i} \tilde{u}^+(x, \tau)\, d\tau + \right.\right. \\ \left.\left. + \alpha_\varepsilon(x)\right)\right\}, \quad i = 0, 1, \ldots, n-1, \tag{4.18}$$

with zero initial and boundary values.

Multiplying (4.18) by $U(x, t_{i+1})$, integrating over Ω and adding over $i = 0, 1, \ldots, j-1, 0 < j \leqslant n$, after some algebraic operations one obtains

$$\frac{1}{2} \int_\Omega U^2(x, t_j)\, dx + \int_0^{t_j} \int_\Omega |\nabla_x U|^2 \, dx\, dt \leqslant \\ \leqslant \sum_{i=0}^{j-1} \int_\Omega \left\{H_\varepsilon\left(\int_0^{t_{i+1}} \tilde{u}^+(x, \tau)\, d\tau + \alpha_\varepsilon(x)\right) - H_\varepsilon\left(\int_0^{t_i} \tilde{u}^+(x, \tau)\, d\tau + \right.\right. \\ \left.\left. + \alpha_\varepsilon(x)\right)\right\} w(x, t_{i+1})\, dx. \tag{4.19}$$

Performing a discrete integration by parts on the right hand side, the latter is estimated in terms of T, $|\Omega|$, and of $|||w|||$, i.e. of a constant independent of h. At this point the proof of (4.11), (4.12) follows immediately.

LEMMA 4.2. *For every compact set $\mathscr{K} \subset \Omega$ there exists a constant $\gamma(\mathscr{K})$ dependent on dist $(\mathscr{K}, \partial\Omega)$, but neither on h nor on ε, such that*

$$\sum_{i=0}^{n-1} h \int_{\mathscr{K}} \left|\frac{u(x, t_{i+1}) - u(x, t_i)}{h}\right|^2 dx \leqslant \gamma(\mathscr{K})/\varepsilon^2. \tag{4.20}$$

Proof. Let $\zeta(x)$ be a smooth function such that $\zeta \equiv 1$ on $\mathscr{K}$, $\zeta = 0$ on $\partial\Omega$, $|\nabla_x \zeta|$ bounded in terms of dist $(\mathscr{K}, \partial\Omega)$. Multiply (4.13) by $[u(x, t_{i+1}) - u(x, t_i)]\,\zeta^2(x)$, integrate over $\partial\Omega$ and add over $i = 0, 1, \dots, n-1$ to get

$$\sum_{i=0}^{n-1} h^{-1} \int_\Omega [u(x, t_{i+1}) - u(x, t_i)]\,\zeta^2\,dx +$$

$$+ \sum_{i=0}^{n-1} \int_\Omega \nabla_x u(x, t_{i+1})\,\nabla_x [u(x, t_{i+1}) - u(x, t_i)]\,\zeta^2\,dx =$$

$$= -2 \sum_{i=0}^{n-1} \int_\Omega [u(x, t_{i+1}) - u(x, t_i)]\,\zeta\nabla_x \zeta \cdot \nabla_x u(x, t_{i+1})\,dx -$$

$$- \sum_{i=0}^{n-1} h^{-1} \int_\Omega \{H_\varepsilon(\int_0^{t_{i+1}} \tilde{u}^+(x, \tau)\,d\tau + \alpha_\varepsilon(x)) -$$

$$- H_\varepsilon(\int_0^{t_i} \tilde{u}^+(x, \tau)\,d\tau + \alpha_\varepsilon(x))\}\,[u(x, t_{i+1}) - u(x, t_i)]\,\zeta^2\,dx. \quad (4.21)$$

Using Cauchy's inequality and Lemma 4.2, the absolute value of the right hand side of (4.21) is found to be less than

$$\frac{1}{2} \sum_{i=0}^{n-1} h^{-1} \int_\Omega [u(x, t_{i+1}) - u(x, t_i)]\,\zeta^2\,dx + \gamma(\mathscr{K})/\varepsilon^2.$$

where the dependence on $\mathscr{K}$ results from $|\nabla_x \zeta|$.

The first term on the left side can be shown to be greater than $-\|u_0\|^2_{H^1(\Omega)}$. Hence (4.20) follows by proper redefinition of $\gamma(\mathscr{K})$.

5. Existence of a solution to $(\mathbf{P}_\varepsilon)$

PROPOSITION 5.1. *A solution to* (P_ε) *exists.*

Proof. If we introduce the function

$$u_h(x, t) = u(x, t_{i+1}) + h^{-1}(t - t_{i+1})\,[u(x, t_{i+1}) - u(x, t_i)],$$
$$t \in [t_i, t_{i+1}],\ i = 0, 1, \dots, n-1, \quad (5.1)$$

as a consequence of Lemmas 4.1 and 4.2 we have that

$$\|\nabla_x u_h\|_{2,\Omega_T} \leqslant \gamma \quad (5.2)$$

for some γ independent of h and of ε,

$$\int_0^T \left\|\frac{\partial}{\partial t} u_h\right\|^2_{2,\mathscr{K}} d\tau \leqslant \gamma(\mathscr{K})/\varepsilon^2, \quad (5.3)$$

where $\gamma(\mathscr{K})$ is the constant appearing in (4.20).

Hence we have proved the following.

LEMMA 5.2. *There exists a subsequence of* $\{u_n\}$ *(which we relabel with* h*) such that*

$$u_h \to u_\varepsilon \text{ strongly in } L^2(\Omega_T), \tag{5.4}$$

$$u_h \to u_\varepsilon \text{ a.e. in } \Omega_T, \tag{5.5}$$

$$\nabla_x u_h \to \nabla_x u_\varepsilon \text{ weakly in } L^2(\Omega_T), \tag{5.6}$$

$$\partial u_h/\partial t \to \partial u_\varepsilon/\partial t \text{ weakly over } L^2(\mathscr{K}\times(0,T)), \quad \text{implying } \partial u_\varepsilon/\partial t \in L^2_{\text{loc}}(\Omega_T). \tag{5.7}$$

Now we prove that u_ε coincides with f on the boundary.

LEMMA 5.3. $u_\varepsilon(x,t)=f(x,t)$ *in the sense of the traces for a.e.* $t\in[0,T)$.

Proof. If $f_h(x,t)$ denotes the piecewise linear function interpolating f in the intervals (t_i, t_{i+1}), it is easy to show that $\|f_h\text{-trace } u_\varepsilon\|_{2,\partial\Omega\times(0,T)}$ tends to zero as $h\to 0$ (use (5.4)). Since $\|f_h-f\|_{2,\partial\Omega\times(0,T)}$ also tends to zero, the lemma follows. ∎

To complete the existence proof, we have to show that u_ε satisfies (3.7). Omitting the details, we confine ourselves to sketching the main steps. Take a test function φ with a compact support in $\mathscr{K}$, multiply (4.9) by $\varphi(x,t_{i+1})\,h$, integrate over Ω and add over $i=0,1,\dots,n-1$, finally perform a discrete integration by parts in time. The resulting equation is

$$\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_\Omega\left\{-u(x,t_i)\frac{\partial\varphi_h}{\partial t}+\nabla_x u(x,t_{i+1})\cdot\nabla_x\varphi(x,t_{i+1})\right\}dx\,d\tau=$$

$$=\int_\Omega[u_0(x)+\xi_0(x)]\,\varphi(x,0)\,dx+\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_\Omega H_\varepsilon\left(\int_0^{t_i}\tilde{u}^+(x,\tau)\,d\tau+\alpha_\varepsilon(x)\right)\frac{\partial\varphi_h}{\partial t}\,dx\,d\tau,$$

where φ_h is defined in the same way as u_h.

At this point the subscript h can be transferred from φ_h to u in the terms where it appears, with the addition of terms, which are easily shown to go to zero as $h\to 0$ (use Lemma 4.3). By virtue of Lemma 5.2 the limit $h\to 0$ can now be performed in (5.8), leading to (3.7).

REMARK 5.4. From (5.7) it follows that u_ε satisfies (3.4) a.e. in Ω_T.

6. A priori estimates on the solution of (P_ε)

On the basis of Remark 5.4 we can rewrite equation (3.4) in the form

$$\partial u_\varepsilon/\partial t - \Delta u_\varepsilon = -H'_\varepsilon\Big(\int_0^t u_\varepsilon^+(x,\tau)\,d\tau + \alpha_\varepsilon(x)\Big)\,u_\varepsilon^+(x,t) \tag{6.1}$$

and prove that u_ε is bounded uniformly

LEMMA 6.1. *For all* $\varepsilon > 0$

$$\|u_\varepsilon\|_{\infty,\Omega_T} \leqslant M, \tag{6.2}$$

where M *is the constant appearing in* (1.3).

Proof. Multiply (6.1) by the functions $\pm(u \mp M)^\pm$, which vanish on the parabolic boundary of Ω_T, and integrate over $\Omega\times(0,t)$ to obtain

$$\int_{\Omega\times\{t\}} |(u_\varepsilon \mp M)^\pm|^2\,dx + \int_0^t\int_\Omega |\nabla_x(u_\varepsilon \mp M)^\pm|^2\,dx\,d\tau =$$

$$\mp\int_0^t\int_\Omega H'_\varepsilon\Big(\int_0^t u_\varepsilon^+(x,s)\,ds + \alpha_\varepsilon(x)\Big)\,u_\varepsilon^+(x,\tau)\,(u_\varepsilon \mp M)^\pm\,dx\,d\tau. \tag{6.3}$$

The right hand side of (6.3) is non-positive when we choose the upper sign and it is zero when we choose the lower sign. Hence the proof of (6.2) has been completed. ■

REMARK 6.2. By Lemma 6.1, the right hand side of (6.1) is bounded and therefore u_ε is Hölder continuous in Ω_T, (see e.g. [10] Thm. 10.1 p. 204) although non-uniformly in ε.

A crucial role in performing in (3.7) the limit passage as $\varepsilon \to 0$ will be played by the equicontinuity of the family $\{u_\varepsilon^-\}$. Let K be a compact subset of Ω and let $\mathscr{K}$ be a compact subset of Ω_T of the form

$$\mathscr{K} = K \times [t_1, t_2],\ 0 < t_1 < t_2 \leqslant T.$$

By $\partial_P\,\Omega_T$ and $\partial_p\,\mathscr{K}$ we will denote the parabolic boundary of Ω_T and of $\mathscr{K}$, respectively.

PROPOSITION 6.3. *For every compact subset* $\mathscr{K} \subset \Omega_T$ *and for every* $\varepsilon > 0$ *the functions* $u_\varepsilon^-(x,t)$ *are continuous in* $\mathscr{K}$ *with uniform modulus of continuity*

$$\omega(\varrho) = [\log\log(A/\varrho)]^{-B}, \tag{6.4}$$

where A, B *are two positive constants dependent upon* M, *on the data, and on* dist$(\mathscr{K}, \partial_p\,\Omega_T)$, *but independent of* ε. *For a pair of points* $(x_i, t_i) \in \mathscr{K}$,

$i=1,2$, ϱ *denotes the parabolic distance*

$$\varrho = |x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}.$$

If in addition $u_0^-(x)$ *is continuous in* $\bar{\Omega}$ *with modulus of continuity* $\omega_0(\cdot)$, *then for every compact set* $\mathscr{K} \subset \Omega$ *the above equicontinuity extends to* $\mathscr{K} \times [0, T]$ *with modulus of continuity*

$$\tilde{\omega}(\varrho) = \max\{\omega(\varrho), \omega_0(\varrho)\}.$$

In such a case the constants A, B *depend on* $\operatorname{dist}(\mathscr{K}, \partial\Omega)$.

Proof. The proof of the proposition follows the same arguments as [7] starting from basic inequalities which we derive next. Let $(x_0, t_0) \in \Omega_T$ be fixed and let $B(R) \equiv \{x \in \Omega : |x - x_0| < R\}$. Denote by Q_R^θ the cylinder

$$Q_R^\theta \equiv B(R) \times (t_0 - \theta R^2, t_0), \ \theta > 0.$$

If $\sigma_1, \sigma_2 \in (0, 1)$ consider also the coaxial cylinder

$$Q_R^\theta(\sigma_1, \sigma_2) \equiv B(R - \sigma_1 R) \times (t_0 - (1-\sigma_2)\theta R^2, t_0).$$

We let R be so small that $Q_R^\theta \subset \Omega_T$. With $(x, t) \to \zeta(x, t)$ we denote a piecewise smooth cutoff function in Q_R^θ which equals one on $Q_R^\theta(\sigma_1, \sigma_2)$, vanishes on the parabolic boundary of Q_R^θ and satisfies

$$|\nabla_x \zeta| \leqslant (\sigma_1 R)^{-1}; \ |\Delta\zeta| \leqslant c(\sigma_1 R)^{-2}; \ 0 \leqslant \zeta_t \leqslant (\sigma_2 \theta R^2)^{-1}.$$

For notational simplicity we set

$$v = u_\varepsilon^-,$$

and denote with k a positive number.

Following the notation of [10] set

$$V^{1,0}(Q_R^\theta) = C[t_0 - \theta R^2, t_0; L^2(B(R))] \cap \\ \cap L^2[t_0 - \theta R^2, t_0; H^1(B(R))],$$

and if $w \in V^{1,0}(Q_R^\theta)$,

$$\|w\|^2_{V^{1,0}(Q_R^\theta)} = \max_{t_0 - \theta R^2 \leqslant t \leqslant t_0} \|w(\cdot, t)\|^2_{2,B(R)} + \|\nabla_x w\|^2_{2,Q_R^\theta}.$$

LEMMA 6.4. *There exists a constant* γ *independent of* ε, *such that for every* $k > 0$

$$\|(v-k)^+\|^2_{V^{1,0}(Q_R^\theta(\sigma_1,\sigma_2))} \leqslant \gamma\,[(\sigma_1 R)^{-2} + (\sigma_2 \theta R^2)^{-1}]\,\|(v-k)^+\|^2_{2,Q_R^\theta}. \tag{6.4}$$

Proof. Multiply (6.1) by $\varphi = -(-u_\varepsilon^- + k)^- \zeta^2$ and integrate over $Q_R^{\theta,t} \equiv B(R) \times (t_0 - QR^2, t)$, where $t \in [t_0 - (1-\sigma_2)\theta R^2, t_0]$. Since φ vanishes except when $u_\varepsilon < -k$, and $k > 0$, the right hand side does not give any contribution.

As to the left hand side, it equals

$$-\iint_{Q_R^{\theta,t}} [(v-k)^+]^2 \zeta\zeta_t \, dx \, d\tau + \frac{1}{2} \int_{B(R)\times\{t\}} [(v-k)^+]^2 \zeta^2 \, dx +$$

$$+\iint_{Q_R^{\theta,t}} |\nabla_x (v-k)^+|^2 \zeta^2 \, dx \, d\tau + 2 \iint_{Q_R^{\theta,t}} (v-k)^+ \nabla_x (v-k)^+ \zeta \nabla_x \zeta \, dx \, d\tau.$$

The last term is not less than

$$-\frac{1}{2} \iint_{Q_R^{\theta,t}} |\nabla_x (v-k)^+|^2 \zeta^2 \, dx \, d\tau - 2 \iint_{Q_R^{\theta}} [(v-k)^+]^2 |\nabla_x \zeta|^2 \, dx \, d\tau.$$

Hence recalling the definition of ζ, for arbitrary $t \in [t_0-(1-\sigma_2)\theta R^2, t_0]$ and for some real constant γ the following inequality holds:

$$\int_{B(R-\sigma_1 R)\times\{t\}} [(v-k)^+]^2 \, dx + \iint_{Q_R^{\theta,t}} |\nabla_x (v-k)^+|^2 \zeta^2 \, dx \, d\tau \leqslant$$

$$\leqslant \gamma \, [(\sigma_1 R)^{-2} + (\sigma_2 \theta R^2)^{-1}] \, \|(v-k)^+\|^2_{2,Q_R^\theta}$$

and the lemma is proved.

For the next lemma we set the following definition

$$A^-_{k,R} \equiv \{(x,t) \in Q_R^\theta : v(x,t) \leqslant k\},$$

LEMMA 6.5. *There exists a constant γ depending upon the data and independent of ε, such that for every $k>0$*

$$\|(v-k)^-\|^2_{V^{1,0}(Q_R^\theta(\sigma_1,\sigma_2))} \leqslant \gamma \, [(\sigma_1 R)^{-2} + (\sigma_2 \theta R^2)^{-1}] \times$$

$$\times \{\|(v-k)^-\|^2_{2,Q_R^\theta} + k \text{ meas } A^-_{k,R}\}. \tag{6.5}$$

Proof. We multiply (6.1) by the function

$$\varphi = (-u_\varepsilon^- + k)^+ \zeta^2$$

and integrate over $Q_R^{\theta,t}$, where $Q_R^{\theta,t}$ is defined as before. We observe that $\varphi \geqslant 0$ and therefore the product on the right hand side gives a non-positive contribution and it is dropped.

We treat the remaining terms as follows. First

$$\frac{\partial}{\partial t} u_\varepsilon (-u_\varepsilon^- + k)^+ = \frac{\partial}{\partial t} (u_\varepsilon^+ - u_\varepsilon^-)(-u_\varepsilon^- + k)^+ =$$

$$= \frac{1}{2} \frac{\partial}{\partial t} [(v-k)^-]^2 + k \frac{\partial}{\partial t} u_\varepsilon^+ .$$

Therefore

$$\iint\limits_{Q_R^{\theta,t}} \frac{\partial}{\partial t} u_\varepsilon (-u_\varepsilon^- + k)^+ \zeta^2 = \frac{1}{2} \int\limits_{B(R)\times\{t\}} [(v-k)^-]^2 \zeta^2 (x,t)\, dx -$$

$$- \iint\limits_{Q_R^{\theta,t}} [(v-k)^-]^2 \zeta\zeta_t \, dx\, d\tau + k \int\limits_{B(R)\times\{t\}} u_\varepsilon^+ \zeta^2 \, dx -$$

$$-2k \iint\limits_{Q_R^{\theta,t}} u_\varepsilon^+ \zeta\zeta_t \, dx\, d\tau.$$

Recalling the properties of the cutoff function ζ we have

$$\iint\limits_{Q_R^{\theta,t}} \frac{\partial}{\partial t} u_\varepsilon (-u_\varepsilon^- + k)^+ \zeta^2 \, dx\, d\tau \geqslant \frac{1}{2} \|(v-k)^- (\cdot, t)\|^2_{2B(R-\sigma,R)} -$$

$$-(\sigma_2 \theta R^2)^{-1} \|(v-k)^-\|^2_{2,Q_R^\theta} - 2k (\sigma_2 \theta R^2)^{-1} \iint\limits_{Q_R^\theta} u_\varepsilon^+ \, dx\, d\tau.$$

For the second term we have

$$J = \iint\limits_{Q_R^{\theta,t}} \nabla u_\varepsilon \cdot \nabla (-u_\varepsilon^- + k)^+ \zeta^2 \, dx\, d\tau + 2 \iint\limits_{Q_R^{\theta,t}} \nabla_x u_\varepsilon (-u_\varepsilon^- + k)^+ \zeta \nabla_x \, dx\, d\tau =$$

$$= J_1 + J_2.$$

The first integral is extended only to the set where $u_\varepsilon \equiv -u_\varepsilon^-$ and therefore

$$J_1 = \iint\limits_{Q_R^{\theta,t}} |\nabla_x (v-k)^-|^2 \zeta^2 \, dx\, d\tau.$$

In the second integral we perform an integration by parts to obtain

$$J_2 = -2 \iint\limits_{Q_R^{\theta,t}} u_\varepsilon \nabla_x (-u_\varepsilon^- + k)^+ \zeta \nabla_x \zeta \, dx\, d\tau =$$

$$-2 \iint\limits_{Q_R^{\theta,t}} u_\varepsilon (-u_\varepsilon^- + k)^+ (|\nabla_x \zeta|^2 + \zeta \nabla \zeta)\, dx\, d\tau = J_2^{(1)} + J_2^{(2)}.$$

We observe that the integral in $J_2^{(1)}$ is extended only to the set $\{-k < -u_\varepsilon^- \leqslant 0\}$ and such a set is included in the set $A_{k,R}^-$. Therefore

$$J_2^{(1)} \geqslant -\frac{1}{2} \iint\limits_{Q_R^{\theta,t}} |\nabla_x (v-k)^-|^2 \zeta^2 \, dx\, d\tau - 2k^2 (\sigma_1 R)^{-2} \operatorname{meas} A_{k,R}^-.$$

For $J_2^{(2)}$, since $|u_\varepsilon| \leqslant M$, recalling the structure of the cutoff function ζ we have

$$J_2^{(2)} \geqslant -CM(\sigma_1 R)^{-2} \iint_{Q_R^\theta} (v-k)^- \, dx \, d\tau .$$

Combining these estimates we find

$$\frac{1}{2} \|(v-k)^-\|^2_{2,B(R-\sigma,R)\times\{t\}} + \frac{1}{2} \iint_{Q_R^{\theta,t}} |\nabla_x (v-k)^-|^2 \zeta^2 \, dx \, d\tau \leqslant$$

$$\leqslant C(\sigma_2 \theta R^2)^{-1} \|(v-k)^-\|^2_{2,Q_R^\theta} + CM[(\sigma_1 R)^{-2} + (\sigma_2 \theta R^2)^{-1}] \times$$

$$\times \left\{ \iint_{Q_R^\theta} (v-k)^- \, dx \, d\tau + k \iint_{Q_R^\theta} u_\varepsilon^+ \, dx \, dt + k^2 \text{ meas } A^-_{k,R} \right\}, \tag{6.6}$$

with C properly redefined.

Since $v \geqslant 0$ we obviously have $\|(v-k)^-\|_\infty \leqslant k$ and therefore

$$\iint_{Q_R^\theta} (v-k)^- \, dx \, d\tau \leqslant k \text{ meas } A^-_{k,R} .$$

Also the integral $\iint_{Q_R^\theta} u_\varepsilon^+ \, dx \, d\tau$ is extended only to the set $\{u_\varepsilon > 0\} \cap Q_R^\theta$ and such a set is included in the set $\{-u_\varepsilon^- > -k\} \cap Q_R^\theta \equiv A^-_{k,R}$. Hence

$$k \iint_{Q_R^\theta} u_\varepsilon^+ \, dx \, d\tau \leqslant kM \text{ meas } A^-_{k,R} .$$

Substituting these estimates in (6.6), since $t \in [t_0 - (1-\sigma_2)\theta R^2, t_0]$ is arbitrary, we deduce

$$\|(v-k)^-\|^2_{V^{1,0}(Q_R^\theta(\sigma_1,\sigma_2))} \leqslant \gamma [(\sigma_1 R)^{-2} + (\sigma_2 \theta R^2)^{-1}] \times$$
$$\times \{ \|(v-k)^-\|^2_{2,Q_R^\theta} + k \text{ meas } A^-_{k,R} \}$$

and the lemma is proved.

Let $k > 0$ and let μ, η be positive numbers satisfying

$$\mu \geqslant \operatorname*{ess\,sup}_{Q_R^\theta} (v-k)^+ ; 0 < \eta < \mu . \tag{6.7}$$

Set

$$\Psi(x), t) = \log^+ \left[\frac{\mu}{\mu - (v-k)^+ + \eta} \right] . \tag{6.8}$$

LEMMA 6.6. *There exists a constant $C = C(\theta)$ such that for all $t \in [t_0 - \theta R^2, t_0]$*

$$\int_{B(R-\sigma_1 R)} \Psi^2(x,t)\,dx \leqslant \int_{B(R)} \Psi^2(x, t_0 - \theta R^2)\,dx + \\ + C(\theta)\,\sigma_1^{-2} \log(\mu/\eta) \operatorname{meas} B(R). \tag{6.9}$$

Proof. Let $x \to \zeta(x)$ be a cutoff function in $B(R)$ which equals one on $B(R - \sigma_1 R)$, vanishes on $\partial B(R)$ and $|\nabla\zeta| \leqslant (\sigma_1 R)^{-1}$. We multiply (6.1) by $-(\Psi^2)'\zeta^2$, the prime denoting differentiation w.r.t. v, and integrate over $Q_R^{\theta,t}$.

We observe that $(\Psi^2)' = 2\Psi\Psi_v$ vanishes on the set $\{v < k\}$. Such a set includes the support of u_ε^+ and therefore the product on the right hand side will give a zero contribution. The various integrals will be extended to the set $\{v > k\}$, i.e. $\{-u_\varepsilon^- < -k\}$.

Therefore we have

$$-\iint_{Q_R^{\theta,t}} \frac{\partial}{\partial t} u_\varepsilon (\Psi^2)' \zeta^2(x)\,dx\,d\tau = \int_{Q_R^{\theta,t}} \frac{\partial}{\partial t}(v-k)^+ (\Psi^2)'\zeta^2\,dx\,d\tau =$$

$$= \iint_{Q_R^{\theta,t}} \frac{\partial}{\partial t} \Psi^2 \zeta^2(x)\,dx\,d\tau = \int_{B(R)} \Psi^2(x,t)\,\zeta^2\,dx - \\ - \int_{B(R)} \Psi^2(x, t_0 - \theta R^2)\,\zeta^2\,dx \geqslant \\ \geqslant \int_{B(R-\sigma_1 R)} \Psi^2(x,t)\,dx - \int_{B(R)} \Psi^2(x, t_0 - \theta R^2)\,dx.$$

In estimating the second term we first observe that

$$(\Psi^2)'' = 2(1+\Psi)(\Psi')^2.$$

Hence, arguing as before,

$$-\iint_{Q_R^{\theta,t}} \nabla u_\varepsilon \nabla (\Psi^2)' \zeta^2\,dx\,d\tau = 2\iint_{Q_R^{\theta,t}} \nabla u_\varepsilon (\Psi^2)' \zeta\nabla\zeta\,dx\,d\tau =$$

$$= 2\iint_{Q_R^{\theta,t}} (1+\Psi)(\Psi')^2 |\nabla v|^2 \zeta^2\,dx\,d\tau + 4\iint_{Q_R^{\theta,t}} \nabla v \Psi\Psi' \zeta\nabla\zeta\,dx\,d\tau \geqslant$$

$$\geqslant \iint_{Q_R^{\theta,t}} (1+\Psi)(\Psi')^2 |\nabla v|^2 \zeta^2\,dx\,d\tau - 8(\sigma_1 R)^{-2} \iint_{Q_R^{\theta}} \Psi\,dx\,d\tau.$$

Combining these estimates we have for all $t \in [t_0 - \theta R^2, t_0]$

$$\int_{B(R-\sigma_1 R)} \Psi^2(x,t)\,dx \leqslant \int_{B(R)} \Psi^2(x, t_0 - \theta R^2)\,dx +$$

$$+ C(\sigma_1 R)^{-2} \iint_{Q_R} \Psi\,dx\,d\tau. \quad (6.10)$$

From the definition (6.8) it follows that

$$|\Psi(x,t)| \leqslant \log(\mu/\eta)$$

and therefore

$$C(\sigma_1 R)^{-2} \iint_{Q_R^\theta} \Psi\,dx\,d\tau \leqslant C(\sigma_1 R)^{-2} \ln(\mu/\eta) \text{ meas } Q_R^\theta \leqslant$$

$$\leqslant C\theta\sigma_1^{-2} \ln(\mu/\eta) \text{ meas } B(R).$$

Substituting this in (6.10), the lemma follows.

Remark 6.7. Lemma 6.6 is analogous to Lemma 2.2 of [7], page 140.

The conclusion of the Proposition 6.2 now follows from inequalities (6.4), (6.5) and (6.9) via the arguments of [7], pp. 143–160. In this connection see also remarks on page 160 and 175. The continuity up to $t = 0$ can be proved as in Theorem 5.1, page 161. The specific modulus of continuity $\omega(\varrho)$ was estimated in [11] (see Remark 3.1 page 101).

7. The limit as $\varepsilon \to 0$

From the results of sections 4, 5 we have that

$$\|u_\varepsilon\|^2_{2,\Omega_T} + \|\nabla_x u_\varepsilon\|^2_{2,\Omega_T} \leqslant C, \quad (7.1)$$

where C is independent of ε. By Proposition 6.3, the sequence $\{u_\varepsilon^-\}$ is equibounded and equicontinuous over compact subsets of Ω_T.

Therefore a subsequence out of $\{u_\varepsilon\}$ can be selected (and relabelled with ε) such that

$u_\varepsilon \to u$ weakly in $L^2(\Omega_T)$
$\nabla_x u_\varepsilon \to \nabla_x u$ " " "
$u_\varepsilon^+ \to \Phi$ " " "
$u_\varepsilon^- \to w$ uniformly over compact subsets of Ω_T.

Lemma 7.1. $u^+ = \Phi$, $u^- = w$.

Proof. Obviously $\Phi \geqslant 0$. Since for every $\varphi \in L^2(\Omega_T)$

$$\iint_{\Omega_T} u\varphi \, dx\, dt = \lim_{\varepsilon\to 0} \iint_{\Omega_T} (u_\varepsilon^+ - u_\varepsilon^-)\, \varphi \, dx\, dt = \iint_{\Omega_T} (\Phi - w)\, \varphi \, dx\, dt$$

and $-w < 0$, to show the assertion it will suffice to prove that

$$(\text{supp})\, \Phi \overset{\circ}{\cap} \text{supp}\, w) = \emptyset .$$

If K is compact non-empty and contained in $(\text{supp}\, \Phi \overset{\circ}{\cap} \text{supp}\, w)$, we have $w > \eta$ on K and $u_\varepsilon^- > \eta/2$ on K, for all ε sufficiently small. Because of the uniform convergence of u_ε^- to w in K, if $\varphi \in L^2(\Omega_T)$ and $\text{supp}\, \varphi \subset K$,

$$\iint_{\Omega_T} (\Phi - w)\, \varphi \, dx\, dt = \lim_{\varepsilon\to 0} \iint_{\Omega_T} (u_\varepsilon^+ - u_\varepsilon^-)\, \varphi \, dx\, dt = -\iint_{\Omega_T} w\varphi \, dx\, dt .$$

Hence

$$\iint_{\Omega_T} \Phi\varphi \, dx\, dt = 0, \; \forall \varphi \in L^2(K)$$

and hence $\Phi = 0$ a.e. in K. A contradiction. ∎

Remark. If neither $\{u_\varepsilon^+\}$ nor $\{u_\varepsilon^-\}$ converges uniformly it is not clear that the weak limits of $\{u_\varepsilon^+\}$ and $\{u_\varepsilon^-\}$ have disjoint support and hence it is not clear that if $u_\varepsilon \to u$ we also have $u_\varepsilon^+ \to u$ and $u_\varepsilon^- \to u^-$.

Lemma 7.2. $\int_0^t u_\varepsilon^+(x,\tau)\, d\tau \to \int_0^t u^+(x,\tau)\, d\tau$ a.e. in Ω_T.

Proof. Since obviously

$$\left\| \frac{\partial}{\partial t} \int_0^t u_\varepsilon^+(x,\tau)\, d\tau \right\|_{2,\Omega_T} + \left\| \nabla_x \int_0^t u_\varepsilon^+(x,\tau)\, d\tau \right\|_{2,\Omega_T} \leqslant C,$$

where C is independent of ε, for a subsequence (again relabelled with ε)

$$\int_0^t u_\varepsilon^+(x,\tau)\, d\tau \to \Psi(x,\tau) \text{ strongly in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T. \tag{7.2}$$

We want to identify Ψ as $\int_0^t u^+(x,\tau)\, d\tau$. Such an identification is not immediate since $u_\varepsilon^+ \to u^+$ only weakly in $L^2(\Omega_T)$.

Let $g \in L^2(\Omega_T)$. Then

$$\iint_{\Omega_T} \left(\Psi - \int_0^t u^+(x,\tau)\,d\tau\right) g\,dx\,dt = \iint_{\Omega_T} \left(\Psi - \int_0^t u_\varepsilon^+(x,\tau)\,d\tau\right) g\,dx\,dt + \\ + \iint_{\Omega_T} g\left(\int_0^t [u_\varepsilon^+(x,\tau) - u^+(x,\tau)]\right) d\tau.$$

The first integral converges to zero as $\varepsilon \to 0$ because of (7.2) and the second tends to zero since $u_\varepsilon^+ \rightharpoonup u^+$ weakly in $L^2(\Omega_T)$. Therefore, letting $\varepsilon \to 0$,

$$\iint_{\Omega_T} \left(\Psi - \int_0^t u^+(x,\tau)\,d\tau\right) g\,dx\,dt = 0$$

for all $g \in L^2(\Omega_T)$ and hence

$$\Psi = \int_0^t u^+(x,\tau)\,d\tau \text{ a.e. in } \Omega_T.$$

We may now conclude the proof of existence.

P r o o f o f T h e o r e m 1.5. Consider (3.8) and let $\varepsilon \to 0$ in the subsequence chosen above. We obtain

$$\iint_{\Omega_T} \{-u\varphi_t + \nabla_x u \cdot \nabla_x \varphi\}\,dx\,d\tau = \int_\Omega (u_0 + \xi_0)\,\varphi(x,0)\,dx + \\ + \lim_{\varepsilon \to 0} \iint_{\Omega_T} \left(H_\varepsilon\left(\int_0^t u_\varepsilon^+(x,\tau)\,d\tau + \alpha_\varepsilon(x)\right)\right) \varphi_t\,dx\,d\tau. \tag{7.3}$$

As $\varepsilon \to 0$ we have

$$H_\varepsilon\left(\int_0^t u_\varepsilon^+(x,\tau)\,d\tau + \alpha_\varepsilon(x)\right) \to \xi \subset H\left(\int_0^t u^+(x,\tau) + u_0^+(x)\right).$$

As discussed in section 2, such a selection coincides with a selection out of $H\left(\sup\limits_{0 \leqslant \tau \leqslant t} u(x,\tau)\right)$, except possibly at those (x,τ) for which

(i) $x \in \Omega \backslash \operatorname{supp}(u_0^+)$

(ii) $\sup\limits_{0 \leqslant \tau \leqslant t} u(x,\tau) < 0$.

At such points we must have $u(x,\tau) < 0, 0 \leqslant \tau \leqslant t$ and

$$\sup_{0 \leqslant \tau \leqslant t} u_\varepsilon(x,\tau) < 0$$

for all ε sufficiently small, because of the uniform convergence of u_ε^- to u^-. Hence

$$\int_0^t u_\varepsilon^+(x,\tau)\,d\tau = 0 \quad \text{for all } \varepsilon \text{ sufficiently small.}$$

and

$$H_\varepsilon\Big(\int_0^t u_\varepsilon^+(x,\tau)\,d\tau + \alpha_\varepsilon(x)\Big) = 0 \quad \text{for all } \varepsilon \text{ sufficiently small.}$$

It follows that

$$\xi(x,t) \subset H\Big(\sup_{0\leqslant\tau\leqslant t} u(x,t)\Big), \quad \text{a.e. in } \Omega_T$$

and the theorem is proved. ∎

8. Remarks on the one-dimensional case

In order to be specific, assume $\Omega = (0,1)$, $u_0 \in C([0,1])$ and $u_0(x) < 0$ for $0 \leqslant x \leqslant b$, where the material is in state 1, $u_0(x) > 0$ for $b < x \leqslant 1$, where the material is in state 2. Moreover the boundary data are $u(0,t) = f_0(t) < 0$, $u(1,t) = f_1(t) > 0$.

In the classical statement of the one-dimensional problem the free boundary is a C^1 curve $x = s(t)$, separating state 1 on the left from state 2 on the right (if mushy regions are absent).

Assuming a classical solution exists, we can say that in any time interval where $\dot s(t) < 0$ the classical solution coincides with the solution (σ, u) of the Stefan problem (SP) with free boundary conditions

$$u\big(\sigma(t)\pm, t\big) = 0, \; u_x\big(\sigma(t)-, t\big) - u_x\big(\sigma(t)+, t\big) = \dot\sigma(t).$$

In the opposite situation, i.e. if $\dot s(t) \equiv 0$ in some interval, the classical solution coincides with the solution of a usual initial boundary value problem(*) (IBP) and is characterized by the fact that the zero level curve $x = \alpha(t)$ may enter the region occupied by state 2.

In the following we will assume that

(i) the functions $f_0(t), f_1(t)$ are analytic for $t > 0$ and continuous for $t = 0$.

It is known that (SP) is uniquely solvable for $t > 0$ and that the free boundary $x = \sigma(t)$ is analytic for $t > 0$ ([8], [9]).

Assume that $u_0(x)$ is such that either

(ii) $\lim\limits_{t\to 0+} \dot\sigma(t) < 0$

or

(ii′) $\underset{t\to 0+}{}\ \dot\alpha(t) > 0$ (**).

(*) We recall that thermal coefficients are set equal to one throughout the system. However this simplification is not crucial for the case we want to study.

(**) In the stationary case $\sigma \equiv b$ or $\alpha \equiv b$ there is no difference among problems (P), (SP) and (IBP).

We want to prove the following.

THEOREM 8.1. *Under assumptions* (i) *and* (ii) *or* (ii′) *Problem (P) has a classical solution having piecewise analytic free boundary and zero level curve. Such a solution is unique in the class specified and can be constructed by solving (SP) and (IBP) in successive time intervals.*

Proof. We confine ourselves to the case (ii), since the proof concerning (ii′) is completely analogous.

First we note that under the conditions guaranteeing (ii) also the level curve $x = \alpha(t)$ lies on the left of $x = b$ for t sufficiently small. This implies that the Stefan solution is the only solution of our problem with monotonic zero level curve up to the time $T = \sup\{t : \dot{\sigma}(t) \leqslant 0\}$. From the analyticity of $\sigma(t)$ we infer that $\sigma(t) > \sigma(T)$ in some interval $(T, T+\delta)$.

Let us solve (IBP) in $(T, T+\delta)$, assuming $u(x, T) \equiv f(x)$ as the initial datum. We show that the curve $u(x, t) = 0$ lies on the right of $x = \sigma(T)$ for a non-zero interval.

Let us define

$$g(x) = \begin{cases} f(x), & x > \sigma(T), \\ -f(-x), & x \leqslant \sigma(T). \end{cases}$$

The difference $f(x) - g(x)$ is zero for $x > \sigma(T)$ and coincides with the analytic function $f(x) + f(-x)$ for $x < \sigma(T)$ (remember $u(x, T)$ is analytic both for $x \geqslant \sigma(T)$ and for $x \leqslant \sigma(T)$).

Hence there exists some positive constant η such that $f(x) - g(x)$ has a given sign in $(\sigma(T) - \eta, \sigma(T))$. Indeed if this conclusion were false, the function f should be odd implying that all derivatives of $\sigma(t)$ vanish for $t = T$, thus contradicting the definition of T.

On the other hand, comparing the solution of (SP) for $t > T$ with the stationary solution (corresponding to an odd datum for $t = T$) we conclude that $f(x) - g(x) > 0$ in $(\sigma(T) - \eta, \sigma(T))$ by monotone dependence.

At this point the monotone dependence of the solution of (IBP) on the datum $f(x)$ yields $\alpha(t) > \sigma(T)$ in a right neighbourhood of $t = T$ Hence the solution of (IBP) is also a solution of (P) and no other solution exists with monotone zero level curve.

Next we define $\bar{T} = \sup\{t : t > T, \alpha(t) \geqslant \sigma(T)\}$.

The function $\alpha(t)$ must have at least one non-zero derivative for $t = \bar{T}$, otherwise $\partial^n u / \partial t^n = 0$, for all n at $(s(\bar{T}), \bar{T})$, which is not permitted by the definition of $\bar{T}$.

Let $m > 0$ be the order of he first non-vanishing derivative of α. Then

$$\partial^m u / \partial t^m = \partial^{2m} u / \partial x^{2m} < 0, \tag{8.1}$$

$$\partial^k u / \partial t^k = \partial^{2k} u / \partial x^{2k} = 0 \text{ for all } k < m \text{ at } (s(\bar{T}), \bar{T}). \tag{8.2}$$

We will use (8.1), (8.2) to show that solving the Stefan problem for $t > \bar{T}$, the corresponding free boundary is found to lie on the left of $x = \alpha(\bar{T})$ in the vicinity of $t = \bar{T}$.

Put $f(x) = u(x, \bar{T})$ and note that if

$$\lim_{x \to s(\bar{T})} d^{2l} f/dx^{2l} = 0, \; l = 1;2, \ldots, k$$

then

$$\lim_{t \to \bar{T}^+} d^l \sigma/dt^l = 0, \; l = 1, 2, \ldots, k+1.$$

This result is proved recursively by considering that the derivatives $d^l \sigma/dt^l$ solve integral equations of the form

$$\sigma^{(l)}(t) = F_l(t) + \int_{\bar{T}}^{t} \int_{\sigma(\bar{T})}^{1} K_l(x, t, \tau)\, \sigma^{(l)}(\tau)\, dx\, d\tau,$$

where the free term tends to zero as $t \to \bar{T}^+$ and the kernel is weakly singular.

As a consequence, by differentiating $u(\sigma(t), t) = 0$ $k+1$ times one finds that $(\partial^{k+1} u/\partial t^{k+1})_{x=\sigma(t)\pm}$ tends to zero as $t \to \bar{T}^+$.

Recalling (8.2) we have

$$\lim_{t \to \bar{T}^+} (\partial^m u/\partial t^m)_{x=\sigma(t)\pm} = 0 \tag{8.3}$$

(take $k = m-1$). Therefore $\partial^m u/\partial t^m$ is discontinuous for $t = \bar{T}, x = \sigma(\bar{T})$

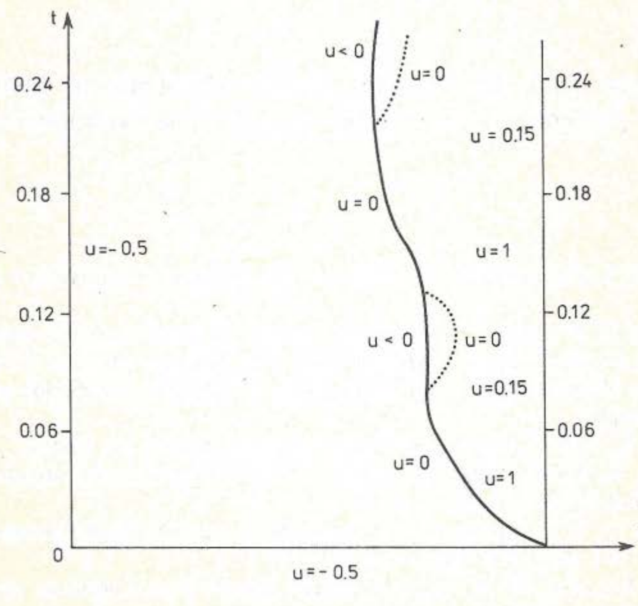

Fig. 1. Free boundary solid line and zero level curve (dotted line) for initial and boundary data (8.6), (8.7)

(remember (8.1)) and it is easy to see that

$$\lim_{t \to T^+} (\partial^{m+1} u / \partial t^{m+1} x)_{x=\sigma(t)\pm} = \mp \infty . \tag{8.4}$$

Thus

$$\lim_{t \to T^+} d^{m+1} s / dt^{m+1} = -\infty . \tag{8.5}$$

This analysis can be iterated showing that the solution can be constructed in a unique way in successive time intervals either by solving (SP) or by solving (IBP). The shift from one problem to the other is marked by the change of slope of the zero temperature curve.

The picture below shows the computed free boundary (solid line) and zero level curve (dotted line) corresponding to the initial datum

$$u_0(x) = -0.5 \tag{8.6}$$

and to the boundary data

$$f_0(t) = -0.5, f_1(t) = \begin{cases} 1 & \text{for } t \in (2n\theta, (2n+1)\theta), \\ 0.15 & \text{for } t \in ((2n+1)\theta, 2(n+1)\theta), \theta = 0.06. \end{cases} \tag{8.7}$$

A finite difference explicit method was used to compute the discretized version of $\partial (u+\xi)/\partial t - \Delta u = 0$ with ξ specified by (1.8).

The data were chosen is such a way to create a non-monotone zero level curve.

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O zagadnieniu ze swobodną granicą związanym z pewnym procesem nieodwracalnym

Rozważany model matematyczny opisuje ewolucję pola cieplnego w materiale, który podlega nieodwracalnej przemianie struktury, zachodzącej przy zadanej temperaturze z absorpcją ciepła utajonego. Dowodzi się istnienia rozwiązań w przypadku wielowymiarowym. Inne wyniki (w szczególności twierdzenie o jednoznaczności) dotyczą przypadku jednowymiarowego. Przedstawione zostają w tym przypadku również pewne wyniki eksperymentów numerycznych.

Об одной проблеме со свободной границей относящейся к необратимому процессу

Рассматривается математическую модель эволюции теплового поля в материале который подлежит необратимому изменению структуры, произходящему при заданной температуре с абсорбцией скрытой теплоты. Доказывается существование решений в многомерном случае. В одномерной формулировке даны тоже другие результаты (в частности, теорема о единственности решений). В этом случае представлены некоторые результаты численных экспериментов.