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One phase Stefan problems with a class of nonlinear boundary conditions on the fixed boundary*)

by

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0. Introduction

In this paper we consider the following problem: Find a curve $x = l(t) > 0$ on $[0, \infty)$ and a function $u = u(x, t)$ on $[0, \infty) \times [0, \infty)$, satisfying

$$u_t - u_{xx} = 0 \text{ for } 0 < x < l(t) \text{ and } 0 < t < \infty, \tag{0.1}$$

$$u(x, 0) = u_0(x) \text{ for } 0 < x < \infty, \tag{0.2}$$

$$u_x(0+, t) \in \partial b^t(u(0, t)) \text{ for } 0 < t < \infty, \tag{0.3}$$

$$u(x, t) = 0 \text{ for } l(t) \leqq x < \infty \text{ and } 0 < t < \infty, \tag{0.4}$$

$$\begin{cases} l'(t)\left(= \dfrac{dl(t)}{dt}\right) = -u_x(l(t)-, t) \text{ for } 0 < t < \infty, \\ l(0) = l_0, \end{cases} \tag{0.5}$$

where l_0 is a given positive number, u_0 is a given function on $[0, \infty)$ and ∂b^t stands for the subdifferential of a given proper lower semicontinuous convex function b^t on $\mathbf{R}$, for each $t \geqq 0$. This is a one phase Stefan problem with the flux $u_x(0+, t)$ governed by the subdifferential $\partial b^t(u(0, t))$ on the fixed boundary $x = 0$.

*) Dedicated to Professor I. Miyadera on his 60th birthday.

This type of Stefan problems was earlier studied by Yotsutani [15, 16]; in fact, he treated the case when b^t is independent of t, i.e. $b^t(\cdot) = b(\cdot)$, and employed a difference method to obtain some results on the existence--uniqueness of global solutions and their asymptotic behaviour. However, the treatment for the time-dependent case of b^t seems to be complicated because of the nonlinearity in the boundary condition (0.3). Recently, the author (cf. [10]) has proposed a new method for Stefan problems of the type mentioned above, giving rise to an easy treatment of the boundary condition (0.3), and showed that the problem (0.1)–(0.5) has a global solution. This method exploits techniques of the theory of nonlinear evolution equations in Hilbert spaces, involving time-dependent subdifferential operators.

The purpose of this paper is to discuss the following three subjects:

(a) The monotone dependence of solutions on $\{b^t, l_0, u_0\}$ and the uniqueness of solutions.

(b) The asymptotic behaviour of the free boundary $x = l(t)$; an sufficient conditions on $\{b^t\}$ in order that $\lim_{t\to\infty} l(t)$ is finite.

(c) The asymptotic behaviour of $u = u(x, t)$; and evaluation of $\liminf_{t\to\infty} u(x, t)$ and $\limsup_{t\to\infty} u(x, t)$ in terms of $\{b^t\}$.

In [10; Theorem 1.3], the uniqueness of the solution was verified for a specific class of initial values. In section 4 of this paper we show the uniqueness for a more general class of initial values as an immediate consequence of the monotone dependence of solutions on b^t, l_0, u_0. In [11] the author dealt with a special case of b^t of the form

$$b^t(r) = \begin{cases} 0 & \text{if } r \geqq g(t), \\ \infty & \text{if } r < g(t), \end{cases}$$

for a given non-negative function g on $[0, \infty)$, and gave some results about (a), (b) and (c). In this paper we establish some theorems concerning (a), (b) and (c) for our general problem (0.1)–(0.5) by employing the same techniques as in [11].

Notation. For a general (real) Banach space X we denote by $|\cdot|_X$ the norm. Also, for a Hilbert space H we denote by $(\cdot,\cdot)_H$ the inner product. Given a proper lower semi-continuous (l.s.c.) convex function φ on a Hilbert space, we denote by $\partial\varphi$ the subdifferential operator of φ, by $D(\partial\varphi)$ its domain and by $D(\varphi)$ the effective domain of φ. For these notations and general properties we refer to Brézis [1].

1. Quasi-variational formulation

In this section we formulate a parabolic quasi-variational problem associated with system (0.1)–(0.5).

Existence and uniqueness of a solution to our system are discussed for $\{b^t\}$ in the class $B_T(\beta_0, \beta_1)$ (or $B_\infty(\beta_0, \beta_1)$) given below. For $0 < T < \infty$, $\beta_0 \in W^{1,2}(0, T)$ and $\beta_1 \in W^{1,1}(0, T)$ we denote by $B_T(\beta_0, \beta_1)$ the set of all $\{b^t\} = \{b^t; 0 \leqq t \leqq T\}$ of proper l.s.c. convex functions on $\mathbf{R}$ satisfying the following (b1) and (b2):

(b1) $\partial b^t(r) \subset (-\infty, 0]$ if $0 \leqq t \leqq T$ and $r \in D(\partial b^t) \cap (-\infty, 0)$.

(b2) For each $s, t \in [0, T]$ with $s \leqq t$ and each $r \in D(b^s)$ there is $\tilde{r} \in D(b^t)$, such that

$$|\tilde{r} - r| \leqq |\beta_0(t) - \beta_0(s)| \left(1 + |r| + |b^s(r)|^{1/2}\right)$$

and

$$b^t(\tilde{r}) - b^s(r) \leqq |\beta_1(t) - \beta_1(s)| \left(1 + |r|^2 + |b^s(r)|\right).$$

Also, we denote by $B_\infty(\beta_0, \beta_1)$ with $\beta_0 \in W^{1,2}_{\mathrm{loc}}([0, \infty))$ and $\beta_1 \in W^{1,1}_{\mathrm{loc}}([0, \infty))$ the set of all $\{b^t\} = \{b^t; 0 \leqq t < \infty\}$, such that $\{b^t\} \in B_T(\beta_0, \beta_1)$ for every finite $T > 0$.

For simplicity, we set $H = L^2(0, \infty)$, $X = W^{1,2}(0, \infty)$,

$\Lambda_T = \{l \in C([0, T]);\ l$ is positive and non-decreasing on $[0, T]\}$, $0 < T < \infty$, and

$$\Lambda_\infty = \{l \in C([0, \infty));\ l \text{ is positive and non-decreasing on } [0, \infty)\}.$$

Given a family $\{b^t\}$ in $B_T(\beta_0, \beta_1)$ or $B_\infty(\beta_0, \beta_1)$, we define a function φ_l^t on H for each $l \in \Lambda_T$ or Λ_∞ and each $t \geqq 0$ as follows:

$$\varphi_l^t(z) = \begin{cases} \dfrac{1}{2} |z_x|_H^2 + b^t(z(0)) & \text{if } z \in K_l(t), \\ \infty & \text{otherwise}, \end{cases} \tag{1.1}$$

where $K_l(t) = \{z \in X;\ z = 0 \text{ on } [l(t), \infty),\ z(0) \in D(b^t)\}$. Clearly it is proper, l.s.c. and convex on H and $D(\varphi_l^t) = K_l(t)$. We then consider the Cauchy problem $\mathrm{CP}(\varphi_l^t; u_0)$ $t \in [0, T]$ in H:

$$\mathrm{CP}(\varphi_l^t; u_0): \begin{cases} -u'(t) \in \partial \varphi_l^t(u(t)), & 0 < t < T, \\ u(0) = u_0, \end{cases}$$

where $0 < T < \infty$, $l \in \Lambda_T$ and $u_0 \in H$ are given; the unknown u is an H-valued function on $[0, T]$, which is identified with the function $u = u(x, t)$ on $[0, \infty) \times [0, T]$ by $[u(t)](x) = u(x, t)$, and $u'(t) = (d/dt)\, u(t)$ in H. By a solution of $\mathrm{CP}(\varphi_l^t; u_0)$ on $[0, T]$ we mean an H-valued function u on

$[0, T]$, satisfying

$$\begin{cases} u \in C([0, T]; H) \cap W^{1,2}(\delta, T; H) \text{ for every } 0 < \delta < T, \\ t \to \varphi_l^t(u(t)) \text{ is integrable on } [0, T], \\ t \to \varphi_l^t(u(t)) \text{ is bounded on } [\delta, T] \text{ for every } 0 < \delta < T, \end{cases} \tag{1.2}$$

$$u(0) = u_0 \tag{1.3}$$

and

$$-u'(t) \in \partial\varphi_l^t(u(t)) \text{ for a.e. } t \in [0, T]. \tag{1.4}$$

Also, $u: [0, \infty) \to H$ is called a solution to $\mathrm{CP}(\varphi_l^t; u_0)$ on $[0, \infty)$, if it is a solution to $\mathrm{CP}(\varphi_l^t; u_0)$ on every finite interval $[0, T]$.

We are now in a position to give a quasi-variational formulation corresponding to system (0.1)–(0.5).

Definition 1.1. Let $0 < T < \infty$, $\{b^t\} \in B_T(\beta_0, \beta_1)$, $0 < l_0 < \infty$ and $u_0 \in H$. Then a pair $\{l, u\} \in \Lambda_T \times C([0, T]; H)$ is called a solution to $\mathrm{QV}(b^t; l_0, u_0)$ on $[0, T]$, if the following conditions (QV1) and (QV2) are fulfilled:

(QV1) u is a solution to $\mathrm{CP}(\varphi_l^t; u_0)$ on $[0, T]$.

(QV2) $l \in W^{1,2}(\delta, T)$ for every $0 < \delta < T$, $l(0) = l_0$ and

$$l'(t) = -u_x(l(t)-, t) \text{ for a.e. } t \in [0, T]. \tag{1.5}$$

Also, given $\{b^t\} \in B_\infty(\beta_0, \beta_1)$, $0 < l_0 < \infty$ and $u_0 \in H$, a pair $\{l, u\}$ in $\Lambda_\infty \times C([0, \infty); H)$ is called a solution to $\mathrm{QV}(b^t; l_0, u_0)$ on $[0, \infty)$, when it is a solution to $\mathrm{QV}(b^t; l_0, u_0)$ on every finite interval $[0, T]$.

Remark 1.1. (a) (cf. [10; Lemma 1.2]) For every $l \in \Lambda_T$ and every $t \in [0, T]$, $z^* \in \partial\varphi_l^t(z)$ if and only if the following (1.6) and (1.7) hold:

$$z^* \in H \text{ and } z \in K_l(t). \tag{1.6}$$

$$(z^*, z_1 - z)_H \leqq (z_x, z_{1,x} - z_x)_H + b^t(z_1(0)) - b^t(z(0)) \text{ for all } z_1 \in K_l(t). \tag{1.7}$$

Also, under (1.6), (1.7) is equivalent to the system $\{(1.8), (1.9)\}$ below:

$$z_{xx} \in L^2(0, l(t)) \text{ and } z^* = -z_{xx} \text{ on } (0, l(t)). \tag{1.8}$$

$$z_x(0+) \in \partial b^t(z(0)). \tag{1.9}$$

We note by (1.8) that z_x is absolutely, continuous on $(0, l(t))$, and hence $z_x(0+)$ and $z_x(l(t)-)$ exist.

(b) According to the part (a), under (1.2), system {(1.3), (1.4)} is equivalent to

$$\begin{cases} u_t(\cdot, t) - u_{xx}(\cdot, t) = 0 \text{ in } L^2(0, l(t)) \text{ for a.e. } t \in [0, T]. \\ u(\cdot, 0) = u_0 \text{ in } H, \\ u_x(0+, t) \in \partial b^t(u(0, t)) \text{ for a.e. } t \in [0, T], \\ u(\cdot, t) = 0 \text{ on } [l(t), \infty) \text{ for all } t \in (0, T]. \end{cases} \tag{1.10}$$

Therefore QV $(b^t; l_0, u_0)$ may be regarded as a weak formulation for problem (0.1)–(0.5).

(c) Let $\{l, u\}$ be a solution to QV $(b^t; l_0, u_0)$ on $[0, T]$. Then, by definition, u is continuous in $(x, t) \in [0, \infty) \times (0, T]$; more precisely, $u \in W^{1,2}(\delta, T; H) \cap \cap L^\infty(\delta, T; X) (\subset C([0, \infty) \times [\delta, T]))$ for every $0 < \delta < T$, which is infered from the boundedness of $t \to \varphi_l^t(u(t))$ on $[\delta, T]$ for every $0 < \delta < T$ and (ii) of Lemma 2.1, proved in the next section.

(d) Observe (cf. [2, 3, 5]), that (1.5) of Definition 1.1 is equivalent to each of the following (1.11) and (1.12):

$$l(t) = l(s) + \int_0^{l(s)} u(x, s)\, dx - \int_0^{l(t)} u(x, t)\, dx - \int_s^t u_x(0+, \tau)\, d\tau, \tag{1.11}$$

for every $0 < s \leqq t \leqq T$.

$$l(t)^2 = l(s)^2 + 2\int_0^{l(s)} xu(x, s)\, dx - 2\int_0^{l(t)} xu(x, t)\, dx + 2\int_s^t u(0, \tau)\, d\tau, \tag{1.12}$$

for every $0 \leqq s \leqq t \leqq T$.

These representations of the free boundary $x = l(t)$ appear useful in the sequel.

We recall an existence result for QV $(b^t; l_0, u_0)$.

THEOREM 1.1 (cf. [10; Theorem 1.1]). *Let $0 < T < \infty$, $\{b^t\} \in B_T(\beta_0, \beta_1)$, $0 < l_0 < \infty$ and $u_0 \in H$ such that $u_0 \geqq 0$ a.e. on $[0, \infty)$ and $u_0 = 0$ a.e. on $[l_0, \infty)$. Then there exists at least one solution $\{l, u\}$ to* QV $(b^t; l_0, u_0)$ *on $[0, T]$, such that*

$$\sqrt{t}\, l' \in L^2(0, T),$$
$$t \to t\varphi_l^t(u(t)) \text{ is bounded on } (0, T],$$
$$\sqrt{t}\, u' \in L^2(0, T; H)$$

and

$$u \geqq 0 \text{ on } [0, \infty) \times (0, T].$$

In addition, if $u_0 \in X$ and $u_0(0) \in D(b^0)$, then $l \in W^{1,2}(0, T)$, $t \to \varphi_l^t(u(t))$ is bounded on $[0, T]$ and $u \in W^{1,2}(0, T; H)$.

REMARK 1.2. Let $\{l, u\}$ be the solution to QV $(b^t; l_0, u_0)$ obtained by Theorem 1.1. and denote $u(0, t)$ by $f(t)$ for $0 < t < T$. Then $\{l, u\}$ is the solution to the usual Stefan problem which is described as a system with the boundary condition $u(0,t)=f(t)$ instead of (0.3). Therefore the solution $\{l, u\}$ has the following properties (i) and (ii) (cf. [4, 14]): (i) u_t and u_{xx} are continuous on $\{(x,t); 0<x<l(t), 0<t\leqq T\}$, and (ii) $l\in C^\infty((0,T])$ and $l'(t)=-u_x(l(t)-,t)$ for all $t\in(0,T]$. For the systematic study of the usual Stefan problem, see [6, 13].

2. Some lemmas on $\{\varphi_l^t\}$

Lemma 2.1. *Let* $\{b^t\}\in B_\infty(\beta_0,\beta_1)$ *and suppose*

$$b^t\to b^\infty \text{ on } R \text{ as } t\to\infty \text{ in the sense of Mosco (see the Appendix)} \tag{2.1}$$

for a proper l.s.c. convex function b^∞ *on* $\mathbf{R}$. *Then we have:*

(i) *There is a constant* $C_1\geqq 0$, *depending only on* β_0, β_1 *and* b^∞, *such that*

$$b^t(r)+C_1|r|+C_1\geqq 0 \text{ for any } t\in[0,\infty) \text{ and } r\in\mathbf{R}. \tag{2.2}$$

(ii) *There is a constant* $C_2\geqq 0$, *depending only on the constant* C_1 *of* (i), *such that*

$$|b^t(z(0))|\leqq \varphi_l^t(z)+C_2|z|_H+C_2, \text{ and} \tag{2.3}$$

$$\frac{1}{4}|z_x|_H^2\leqq \varphi_l^t(z)+C_2|z|_H+C_2 \tag{2.4}$$

for all $l\in\Lambda_\infty$, $t\in[0,\infty)$ *and* $z\in K_l(t)$.

(iii) *Let* L *be a positive number. Then there are constants* $C_3\geqq 0$, $C_3'\geqq 0$, $C_4\geqq 0$ *and* $C_4'\geqq 0$, *depending only on* L *and* C_2 *of* (ii), *such that*

$$|z|_H^2\leqq C_3\varphi_l^t(z)+C_3', \tag{2.5}$$

$$|\varphi_l^t(z)|\leqq C_4\varphi_l^t(z)+C_4' \tag{2.6}$$

for all $l\in\Lambda_\infty$ *with* $\lim_{t\to\infty} l(t)\leqq L$, $t\in[0,\infty)$ *and* $z\in K_l(t)$.

Proof. According to [9; §1.5]), there is a constant $c\geqq 0$ corresponding to given $T>0$, β_0, β_1 such that

$$b^t(r)+c|r|+c\geqq 0 \text{ for any } t\in[0,T] \text{ and } r\in\mathbf{R}.$$

Besides, by using a result in [7; Lemma 3.1] we can find reals $T>0$ and $c'\geqq 0$, depending only on β_0, β_1 and b^∞, such that

$$b^t(t)+c'|r|+c'\geqq 0 \text{ for any } t\in[T,\infty) \text{ and } r\in\mathbf{R}.$$

Therefore (i) holds.

Now, let $z\in K_l(t)$. Then we observe

$$|z(0)|\leqq\int_0^1|\{(1-x)z\}_x|\,dx\leqq|z_x|_H+|z|_H.$$

Hence, by (2.2),

$$\begin{aligned}|b^t(z(0))| &\leqq b^t(z(0)) + 2C_1|z(0)| + 2C_1 \leqq \\ &\leqq b^t(z(0)) + \frac{1}{2}|z_x|_H^2 + 2C_1|z|_H + 2C_1^2 + 2C_1 \leqq \\ &\leqq \varphi_l^t(z) + C_2|z|_H + C_2\end{aligned}$$

with $C_2 = 2C_1(1+C_1)$, and

$$\frac{1}{4}|z_x|_H^2 \leqq \frac{1}{2}\{\varphi_l^t(z) + |b^t(z(0))|\} \leqq \varphi_l^t(z) + C_2|z|_H + C_2.$$

Thus (2.3) and (2.4) hold, and (ii) is proved.

Next, let $l \in \Lambda_\infty$ with $\lim_{t\to\infty} l(t) \leqq L$. Then, since

$$|z|_H \leqq L|z_x|_H \text{ for any } t \in [0, \infty) \text{ and } z \in K_l(t),$$

it follows from (2.4) that (2.5) and (2.6) hold for some non-negative constants C_3, C_3', C_4, C_4' depending only on L and C_2. ■

LEMMA 2.2. *Let $l \in \Lambda_\infty$ and $\{b^t\} \in B_\infty(\beta_0, \beta_1)$, and suppose (2.1) holds. Then $\varphi_l^t \to \varphi^\infty$ on H as $t \to \infty$ in the sense of Mosco (see the Apendix), where*

$$\varphi^\infty(z) = \begin{cases} \frac{1}{2}|z_x|_H^2 + b^\infty(z(0)) \text{ if } z \in X,\ z(0) \in D(b^\infty) \text{ and } z = 0 \text{ on } [l_\infty, \infty), \\ \infty \quad \text{otherwise}, \end{cases} \tag{2.7}$$

with $l_\infty = \lim_{t\to\infty} l(t)$; note in (2.7) that the restriction $z = 0$ on $[l_\infty, \infty)$ is to be deleted if $l_\infty = \infty$.

Proof. Let $\{t_n\}$ be any sequence with $t_n \to \infty$ (as $n \to \infty$), and $\{z_n\}$ be any sequence in H such that $z_n \to z$ weakly in H and $A \equiv \liminf_{t\to\infty} \varphi_l^{t_n}(z_n) < \infty$. Then we see from (2.4) of Lemma 2.1 that there is a subsequence $\{n'\}$ of $\{n\}$ such that $\varphi_l^{t_{n'}}(z_{n'}) \to A$ and $z_{n'} \to z$ weakly in X, hence $z_{n'}(0) \to z(0)$ as $n' \to \infty$. Therefore $A \geqq \varphi^\infty(z)$. Next, let z be any element of $D(\varphi^\infty)$. Then, by our assumptions, there is a sequence $\{r_n\}$ such that $r_n \to z(0)$ and $b^{t_n}(r_n) \to b^\infty(z(0))$. Here, using a smooth function ζ on $\mathbf{R}$ such that $0 \leqq \zeta \leqq 1$ on $\mathbf{R}$, $\zeta = 1$ on $(-\infty, -1]$, $\zeta = 0$ on $[0, \infty)$, we define

$$z_n(x) = \begin{cases} z\left(\dfrac{l_\infty}{l(t_n)} x\right) + (r_n - z(0))\, z^0(x) \text{ if } l_\infty < \infty, \\ \zeta(x - l(t_n))\, z(x) + (r_n - z(0))\, z^0(x) \text{ if } l_\infty = \infty \end{cases}$$

with a smooth function z^0 on $[0, \infty)$, satisfying $z^0(0) = 1$ and $z^0 = 0$ on $[l(0), \infty)$. It is easy to see that $z_n \in D(\varphi_l^{t_n})$, $z_n \to z$ in X and $\varphi_l^{t_n}(z_n) \to \varphi^\infty(z)$. Thus we have the conclusion of the lemma.

LEMMA 2.3. *Let $l \in \Lambda_\infty$ and $l_n \in \Lambda_\infty$, $n = 1, 2, \ldots$, such that $l_n \to l$ pointwise on $[0, \infty)$ as $n \to \infty$. Also, let $\{b^t\} \in B_\infty(\beta_0, \beta_1)$. Then for each $t \geqq 0$,*

$$\varphi^t_{l_n} \to \varphi^t_l \text{ on } H \text{ as } n \to \infty \text{ in the sense of Mosco.}$$

We omit the proof of this lemma, as it can be shown by a modification of that of Lemma 2.2.

Given numbers $0 < \delta < L \leqq \infty$, we denote by $\Lambda_\infty(\delta, L)$ the subclass $\{l \in \Lambda_\infty; \delta \leqq l(0), \lim_{t\to\infty} l(t) \leqq L\}$ of Λ_∞. We also consider the class $\Phi(\{\alpha_{0,r}\}, \{\alpha_{1,r}\})$ of families $\{\varphi^t\}$ of proper l.s.c. convex functions on H (see the Appendix for the definition of $\Phi(\{\alpha_{0,r}\}, \{\alpha_{1,r}\})$.

LEMMA 2.4. (i) *Let $\{b^t\} \in B_\infty(\beta_0, \beta_1)$ with $\beta'_0 \in L^1(0, \infty)$ and $0 < \delta \leqq 1$. Then there is a constant $C_5 \geqq 0$, depending only on δ, β_0 and β_1, such that*

$$\{\varphi^t_l\} \in \Phi(\{\alpha_{0,r}\}, \{\alpha_{1,r}\}) \text{ for all } l \in \Lambda_\infty(\delta, \infty),$$

where

$$\alpha_{0,r}(t) = C_5(1+r)\int_0^t |\beta'_0(\tau)|\, d\tau,\ \ \alpha_{1,r}(t) = C_5(1+r^2)\int_0^t \{|\beta'_0(\tau)| + |\beta'_1(\tau)|\}\, d\tau$$

for all $r \geqq 0$.

(ii) *Let $\{b^t\} \in B_\infty(\beta_0, \beta_1)$ with $\beta'_0 \in L^1(0, \infty)$ and $0 < \delta < 1 < L < \infty$. Then there is a constant $C_6 \geqq 0$, depending only on δ, L, β_0 and β_1, such that*

$$\{\varphi^t_l\} \in \Phi(\{\tilde\alpha_{0,r}\}, \{\tilde\alpha_{1,r}\}) \text{ for all } l \in \Lambda_\infty(\delta, L),$$

where

$$\tilde\alpha_{0,r}(t) = C_6\int_0^t |\beta'_0(\tau)|\, d\tau, \quad \tilde\alpha_{1,r}(t) = C_6\int_0^t \{|\beta'_0(\tau)| + |\beta'_1(\tau)|\}\, d\tau$$

for all $r \geqq 0$; note in this case that $\tilde\alpha_{0,r}$ and $\tilde\alpha_{1,r}$ are independent of $r \geqq 0$.

Proof. Let $l \in \Lambda_\infty(\delta, \infty)$, $0 \leqq s \leqq t < \infty$ and $z \in K_l(s)$. Since $z(0) \in D(b^s)$, using condition (b2), we can find $\tilde r \in D(b^t)$ such that

$$|\tilde r - z(0)| \leqq \Big(\int_s^t |\beta'_0(\tau)|\, d\tau\Big)\big(1 + |z(0)| + |b^s(z(0))|^{1/2}\big)$$

and

$$b^t(\tilde r) - b^s(z(0)) \leqq \Big(\int_s^t |\beta'_1(\tau)|\, d\tau\Big)\big(1 + |z(0)|^2 + |b^s(z(0))|\big).$$

We then consider the function

$$\tilde z(x) = z(x) + (\tilde r - z(0))\, z^\delta(x),$$

where

$$z^{\delta}(x) = \begin{cases} 1 - \dfrac{x}{\delta} & \text{for } 0 \leqq x \leqq \delta, \\ 0 & \text{for } \delta < x < \infty. \end{cases}$$

Clearly, $\tilde{z} \in K_l(t)$ with $\tilde{z}(0) = \tilde{r} \in D(b^t)$. We also observe that

$$|\tilde{z} - z|_H = |\tilde{r} - z(0)|\,|z^{\delta}|_H \leqq \Big(\int_s^t |\beta_0'(\tau)|\,d\tau\Big)\Big(1 + |z(0)| + \big|b^s(z(0))\big|^{1/2}\Big)$$

and

$$\varphi_l^t(\tilde{z}) - \varphi_l^s(z) = \frac{1}{2}|\tilde{z}_x|_H^2 - \frac{1}{2}|z_x|_H^2 + b^t(\tilde{r}) - b^s(z(0)) \leqslant$$

$$\leqq \int_0^{\delta} \Big\{ |\tilde{r} - z(0)|\,|z_x^{\delta}(x)|\,|z_x(x)| + \frac{1}{2}|\tilde{r} - z(0)|^2\,|z_x^{\delta}(x)|^2 \Big\}\,dx + b^t(\tilde{r}) - b^s(z(0)) \leqq$$

$$\leqq \frac{1}{\delta}\Big(\int_s^t |\beta_0'(\tau)|\,d\tau\Big)\Big(1 + |z(0)| + \big|b^s(z(0))\big|^{1/2}\Big)\int_0^{\delta} |z_x(x)|\,dx +$$

$$+ \frac{1}{2\delta}\Big(\int_s^t |\beta_0'(\tau)|\,d\tau\Big)|\beta_0'|_{L^1(0,\infty)}\Big(1 + |z(0)| + \big|b^s(z(0))\big|^{1/2}\Big)^2 +$$

$$+ \Big(\int_s^t |\beta_1'(\tau)|\,d\tau\Big)\Big(1 + |z(0)|^2 + \big|b^s(z(0))\big|\Big).$$

By making use of the inequalities in Lemma 2.1, we derive from the above inequalities that

$$|\tilde{z} - z|_H \leqq c\,\Big(\int_s^t |\beta_0'(\tau)|\,d\tau\Big)\big(1 + |z|_H + |\varphi_l^s(z)|^{1/2}\big)$$

and

$$\varphi_l^t(\tilde{z}) - \varphi_l^s(z) \leqq c\,\Big(\int_s^t \{|\beta_0'(\tau)| + |\beta_1'(\tau)|\}\,d\tau\Big)\big(1 + |z|_H^2 + |\varphi_l^s(z)|\big) \tag{2.8}$$

for some constant $c \geqq 0$ depending only on δ, β_0 and β_1. Therefore we can take this c as C_5. Also it is not difficult to derive the conclusion of (ii) of the lemma from (2.8) and (2.5) of Lemma 2.1. ■

3. Some lemmas on CP $(\varphi_l^t; u_0)$

The lemmas, which have been proved in the previous section, allow us to apply the abstract results of the Appendix to problem CP $(\varphi_l^t; u_0)$.

The following comparison lemma is useful.

LEMMA 3.1 (cf. [11; Lemma 2.1]). *Let* $0 < T < \infty$, k *be a constant*, $l \in C([0, T])$ *with* $l > 0$ *on* $[0, T]$, *and* v, w *be functions in* $C([0, T]; H) \cap \cap W^{1,2}(\delta, T; H) \cap L^\infty(\delta, T; X)$ *with* v_{xx}, w_{xx} *in* $L^2(D_\delta)$, $D_\delta = \{(x, t); 0 < x < < l(t), \delta < t < T\}$, *for every* $0 < \delta < T$, *such that*

$$w_t - w_{xx} \leqq v_t - v_{xx} \text{ a.e. on } \{(x, t); 0 < x < l(t), 0 < t < T\},$$
$$w(x, 0) \leqq v(x, 0) + k \text{ for a.e. } x \geqq 0,$$
$$w \leqq v + k \text{ on } \{(x, t); l(t) \leqq x < \infty, 0 < t \leqq T\}, \text{ and}$$
$$(w_x(0+, t) - v_x(0+, t))(w(0, t) - v(0, t) - k)^+ \geqq 0 \text{ for a.e. } t \in [0, T].$$

Then

$$w \leqq v + k \text{ on } [0, \infty) \times (0, T].$$

COROLLARY 1. *Let* $0 < T < \infty$, $l \in \Lambda_T$, $\{b^t\} \in B_T(\beta_0, \beta_1)$, *and let* u_0 *be a non-negative function in* H. *Then the solution* u *to* $CP(\varphi_l^t; u_0)$ *on* $[0, T]$ *is non-negative on* $[0, \infty) \times (0, T]$.

This corollary is a direct consequence of Lemma 3.1 with $w = 0$, $v = u$ and $k = 0$.

COROLLARY 2. *Let* $0 < T < \infty$, $l \in \Lambda_T$, $\hat{l} \in \Lambda_T$, $u_0 \in H$ *with* $u_0 \geqq 0$ *a.e. on* $[0, \infty)$ *and* $u_0 = 0$ *a.e. on* $[l(0), \infty)$, $\hat{u}_0 \in H$ *with* $\hat{u}_0 \geqq 0$ *a.e. on* $[0, \infty)$ *and* $\hat{u}_0 = 0$ *a.e. on* $[\hat{l}(0), \infty)$, $\{b^t\} \in B_T(\beta_0, \beta_1)$ *and* $\{\hat{b}^t\} \in B_T(\hat{\beta}_0, \hat{\beta}_1)$. *Further let* u *and* $\hat{u}$ *be the solutions to* $CP(\varphi_l^t; u_0)$ *and* $CP(\psi_{\hat{l}}^t; \hat{u}_0)$ *on* $[0, T]$, *respectively, where* $\psi_{\hat{l}}^t$ *is the function on* H *given by* (1.1) *with* l *and* b^t *replaced by* $\hat{l}$ *and* $\hat{b}^t$. *Suppose*

$$l \leqq \hat{l} \text{ on } [0, T],\ u_0 \leqq \hat{u}_0 \text{ a.e. on } [0, \infty)$$

and

$$b^t(r_1 \wedge r_2) + \hat{b}^t(r_1 \vee r_2) \leqq b^t(r_2) \text{ for any } r \in [0, T] \text{ and } r_1, r_2 \in \mathbf{R}, \tag{3.1}$$

where $r_1 \vee r_2 = \max\{r_1, r_2\}$ *and* $r_1 \wedge r_2 = \min\{r_1, r_2\}$. *Then we have*

$$u \leqq \hat{u} \text{ on } [0, \infty) \times (0, T]. \tag{3.2}$$

Proof. From (b) of Remark 1.1 and Corollary 1, we see that

$$u_t - u_{xx} = 0 = \hat{u}_t - \hat{u}_{xx} \quad \text{a.e. on } \{(x, t); 0 < x < l(t), 0 < t < T\},$$
$$u(x, 0) = u_0(x) \leqq \hat{u}_0(x) = \hat{u}(x, 0) \text{ for a.e. } x \geqq 0, \text{ and}$$
$$u = 0 \leqq \hat{u} \text{ on } \{(x, t); l(t) \leqq x < \infty, 0 < t \leqq T\}.$$

Also, as it is easily seen from (3.1), $(r_1^* - r_2^*)(r_1 - r_2)^+ \geqq 0$ for any $t \in [0, T]$, $r_1^* \in \partial b^t(r_1)$ and $r_2^* \in \partial b^t(r_2)$, from which it follows that

$$(u_x(0+, t) - \hat{u}_x(0+, t))(u(0, t) - \hat{u}(0, t))^+ \geqq 0 \text{ for a.e. } t \in [0, T],$$

because $u_x(0+, t) \in \partial b^t(u(0, t))$ and $\hat{u}_x(0+, t) \in \partial \hat{b}^t(\hat{u}(0, t))$ for a.e. $t \in [0, T]$. Thus all the assumptions of Lemma 3.1 are satisfied for the case where $w = u$, $v = \hat{u}$ and $k = 0$, so that we get (3.2).

LEMMA 3.2. *Let $l \in \Lambda_\infty$ with $l_\infty \equiv \lim_{t \to \infty} l(t) < \infty$, and $\{b^t\} \in B_\infty(\beta_0, \beta_1)$ with $\beta_0' \in L^1(0, \infty) \cap L^2(0, \infty)$ and $\beta_1' \in L^1(0, \infty)$ such that $b^t \to b^\infty$ on* **R** *as $t \to \infty$ in the sense of Mosco for a proper l.s.c. convex function b^∞ on* **R**. *Let u_0 be a non-negative function in H such that $u_0 = 0$ a.e. on $[l(0), \infty)$. Then* CP$(\varphi_l^t; u_0)$ *has one and only one solution u on $[0, \infty)$ and $u(t) \to u_\infty$ in X as $t \to \infty$, where u_∞ is the function given by*

$$u_\infty(x) = \begin{cases} c\left(1 - \dfrac{x}{l_\infty}\right) & \text{for } 0 \leqq x \leqq l_\infty, \\ 0 & \text{for } l_\infty < x < \infty, \end{cases} \tag{3.3}$$

with the constant c satisfying

$$-\frac{c}{l_\infty} \in \partial b^\infty(c). \tag{3.4}$$

Proof. First we show $u \in L^\infty(0, \infty; H)$. By (iii) of Lemma 2.1 and (ii) of Lemma 2.4, we can apply Theorem A.1 of the Appendix to problem CP$(\varphi_l^t; u_0)$, and obtain that CP$(\varphi_l^t; u_0)$ has one and only one solution u on $[0, \infty)$, and

$$\varphi_l^t(u(t)) - \varphi_l^s(u(s)) \leqq \int_s^t k(\tau)\left(M\varphi_l^\tau(u(\tau)) + M'\right) d\tau$$

for every $0 < s \leqq t < \infty$, where M and M' are constants, and

$$k(\tau) = |\beta_0'(\tau)|^2 + |\beta_0'(\tau)| + |\beta_1'(\tau)|.$$

Therefore, by Gronwall's inequality, $t \to \varphi_l^t(u(t))$ is bounded on $[1, \infty)$, so that (iii) of Lemma 2.1 implies $u \in L^\infty(1, \infty; H)$. Since $u \in C([0, 1]; H)$, it follows that $u \in L^\infty(0, \infty; H)$. Next, on account of Lemma 2.2, $\varphi_l^t \to \varphi^\infty$ on H as $t \to \infty$ in the sense of Mosco, where φ^∞ is as in Lemma 2.2. Accordingly, applying Theorem A.3 in the Appendix to problem CP$(\varphi_l^t; u_0)$ on $[0, \infty)$, we obtain that

$$u(t) \to u_\infty \text{ weakly in } H$$

and

$$\varphi_l^t(u(t)) \to \varphi^\infty(u_\infty) = \min \varphi^\infty \quad (\text{hence } 0 \in \partial \varphi^\infty(u_\infty))$$

for some $u_\infty \in X$. From these convergences we conclude that $u(t) \to u_\infty$ in X, and also, due to the relation $0 \in \partial\varphi^\infty(u_\infty)$ (cf. (*a*) of Remark 1.1), that (3.3) holds with (3.4). ■

LEMMA 3.3. *Let* $\{b^t\} \in B_\infty(\beta_0, \beta_1)$, $l \in \Lambda_\infty$ *and* $l_n \in \Lambda_\infty$, $n = 1, 2, \ldots,$ *such that* $l_n \to l$ *pointwise on* $[0, \infty)$ *as* $n \to \infty$. *Further, let* $u_0 \in H$ *with* $u_0 \geqq 0$ *a.e. on* $[0, \infty)$ *and* $u_0 = 0$ *a.e. on* $[l(0), \infty)$, *and* $u_{0,n} \in H$ *with* $u_{0,n} \geqq 0$ *a.e. on* $[0, \infty)$ *and* $u_{0,n} = 0$ *a.e. on* $[l_n(0), \infty)$, $n = 1, 2, \ldots,$ *such that* $u_{0,n} \to u_0$ *in* H *as* $n \to \infty$. *Then, denoting by* u *and* u_n *the solutions to* $CP(\varphi_l^t; u_0)$ *and* $CP(\varphi_{l_n}^t; u_{0,n})$ *on* $[0, \infty)$, *respectively, we have*

$$u_n \to u \text{ in } C([0, T]; H) \text{ and in } L^2(0, T; X)$$

as $n \to \infty$ *for every finite* $T > 0$.

Proof. Let $0 < T < \infty$. Then, by Lemma 2.3 and (i) of Lemma 2.4, we can apply Theorem A.2 in the Appendix to obtain

$$u_n \to u \text{ in } C([0, T]; H) \text{ and } \int_0^T \varphi_{l_n}^t(u_n(t))\, dt \to \int_0^T \varphi_l^t(u(t))\, dt.$$

From this we get the conclusion of the lemma. ■

4. Monotone dependence

In this section we prove

THEOREM 4.1. *Let* $0 < T < \infty$, $0 < l_0 < \infty$, $0 < \hat{l}_0 < \infty$, $u_0 \in H$ *with* $u_0 \geqq 0$ *a.e. on* $[0, \infty)$ *and* $u_0 = 0$ *a.e. on* $[l_0, \infty)$, *and* $\hat{u}_0 \in H$ *with* $\hat{u}_0 \geqq 0$ *a.e. on* $[0, \infty)$ *and* $\hat{u}_0 = 0$ *a.e. on* $[\hat{l}_0, \infty)$. *Further let* $\{b^t\} \in B_T(\beta_0, \beta_1)$ *and* $\{\hat{b}^t\} \in$ $\in B_T(\hat{\beta}_0, \hat{\beta}_1)$ *such that*

$$b^t(r_1 \wedge r_2) + \hat{b}^t(r_1 \vee r_2) \leqq b^t(r_1) + \hat{b}^t(r_2) \text{ for any } t \in [0, T] \text{ and } r_1, r_2 \in \mathbf{R}.$$

If $t_0 \leqq \hat{l}_0$ *and* $u_0 \leqq \hat{u}_0$ *a.e. on* $[0, \infty)$, *then*

$$l \leqq \hat{l} \text{ on } [0, T] \text{ and } u \leqq \hat{u} \text{ on } [0, \infty) \times (0, T], \tag{4.1}$$

where $\{l, u\}$ *and* $\{\hat{l}, \hat{u}\}$ *are respectively the solutions to* $\mathrm{QV}(b^t; l_0, u_0)$ *and* $\mathrm{QV}(\hat{b}^t; \hat{l}_0, \hat{u}_0)$ *on* $[0, T]$.

Proof. First, assuming $l_0 < \hat{l}_0$, we show that $l < \hat{l}$ on $[0, T]$ and $u \leqq \hat{u}$ on $[0, \infty) \times (0, T]$. To get a contradiction, suppose there is $0 < t_0 \leqq T$ such that

$$l(t_0) = \hat{l}(t_0), \text{ and } l < \hat{l} \text{ on } [0, t_0).$$

Then, on account of Corollary 2 to Lemma 3.1, we have

$$u \leqq \hat{u} \text{ on } [0, \infty) \times (0, t_0]. \tag{4.2}$$

Now, denote $u(0, t)$ and $\hat{u}(0, t)$ by $f(t)$ and $\hat{f}(t)$, respectively. As it has been noticed in Remark 1.2, $\{l, u\}$ (resp. $\{\hat{l}, \hat{u}\}$) is the solution to the usual Stefan problem with the boundary condition $u(0, t) = f(t)$ (resp. $\hat{u}(0, t) = \hat{f}(t)$). Since $f \leqq \hat{f}$ by (4.2), it follows from the result on the monotone dependence (cf. [2; Theorem 6]) that $l < \hat{l}$ on $[0, t_0]$, which is a contradiction. Thus we get

$$l < \hat{l} \text{ on } [0, T], \; u \leqq \hat{u} \text{ on } [0, \infty) \times (0, T].$$

Next, assume $l_0 = \hat{l}_0$, and take a sequence $\{\hat{l}_{0,n}\}$ so that $\hat{l}_{0,n} > \hat{l}_0$ and $\hat{l}_{0,n} \downarrow \hat{l}_0$ (as $n \to \infty$). By virtue of Theorem 1.1, QV $(\hat{b}^t; \hat{l}_{0,n}, \hat{u}_0)$ has a solution $\{\hat{l}_n, \hat{u}_n\}$ on $[0, T]$. Also, from the above argument it follows that

$$l < \hat{l}_n \text{ on } [0, T], \; u \leqq \hat{u}_n \text{ on } [0, \infty) \times (0, T]$$

and

$$\hat{l} < \hat{l}_n \text{ on } [0, T], \; \hat{u} \leqq \hat{u}_n \text{ on } [0, \infty) \times (0, T].$$

Furthermore, on account of (1.11) in (d) of Remark 1.1,

$$0 < \hat{l}_n(t) - \hat{l}(t) \leqq$$
$$\leqq \hat{l}_n(\delta) - \hat{l}(\delta) + \int_0^\infty \{\hat{u}_n(x, \delta) - \hat{u}(x, \delta)\}\, dx - \int_\delta^t \{\hat{u}_{n,x}(0+, \tau) - \hat{u}_x(0+, \tau)\}\, d\tau$$

for every $0 < \delta \leqq t \leqq T$. Here we note that

$$\hat{u}_{n,x}(0+, \tau) \geqq \hat{u}_x(0+, \tau) \text{ for a.e. } \tau \in [0, T]. \tag{4.3}$$

In fact, it follows from the monotonicity of $\partial \hat{b}^\tau$ that $\hat{u}_{n,x}(0+, \tau) \geqq \hat{u}_x(0+, \tau)$ for a.e. $\tau \in [0, T]$ with $\hat{u}_n(0, \tau) > \hat{u}(0, \tau)$. Also, if $\hat{u}_n(0, \tau) = \hat{u}(0, \tau)$ and $\hat{u}_{n,x}(0+, \tau)$ and $\hat{u}_x(0, \tau)$ exist, then

$$\hat{u}_{n,x}(0+, \tau) = \lim_{x \downarrow 0} \frac{\hat{u}_n(x, \tau) - \hat{u}_n(0, \tau)}{x} \geqq \lim_{x \downarrow 0} \frac{\hat{u}(x, \tau) - \hat{u}(0, \tau)}{x} = \hat{u}_x(0+, \tau).$$

Therefore we obtain (4.3) and for every $t \in [\delta, T]$

$$0 < \hat{l}_n(t) - \hat{l}(t) \leqq \hat{l}_n(\delta) - \hat{l}(\delta) + \int_0^\infty \{\hat{u}_n(x, \delta) - \hat{u}(x, \delta)\}\, dx.$$

Letting $\delta \downarrow 0$ in this inequality, we get

$$0 < \hat{l}_n(t) - \hat{l}(t) \leqq \hat{l}_{0,n} - \hat{l}_0 \text{ for any } t \in [0, T].$$

This implies that $\hat{l}_n \to \hat{l}$ in $C([0, T])$, and, by Lemma 3.3, that $\hat{u}_n \to \hat{u}$ in $C([0, T]; H)$. Consequently we get (4.1). ■

COROLLARY. *Let $0 < l_0 < \infty$ and $u_0 \in H$ such that $u_0 \geqq 0$ a.e. on $[0, \infty)$ and $u_0 = 0$ a.e. on $[l_0, \infty)$. Then $\mathrm{QV}(b^t; l_0, u_0)$ has at most one solution on $[0, T]$ for each $\{b^t\} \in B_T(\beta_0, \beta_1)$.*

5. Asymptotic behaviour

In this section we investigate the asymptotic behaviour of the solution to $\mathrm{QV}(b^t; l_0, u_0)$ on $[0, \infty)$.

THEOREM 5.1. *Let $\{b^t\} \in B_\infty(\beta_0, \beta_1)$ and suppose there are two functions g and g^* on $[0, \infty)$, such that*

$$g \text{ is non-negative and non-increasing on } [0, \infty),$$
$$g \in L^1(0, \infty), \; g^* \in L^\infty(0, \infty) \cap L^1(0, \infty)$$

and

$$g^*(t) \in \partial b^t(g(t)) \text{ for all } t \in [0, \infty). \tag{5.1}$$

Let $0 < l_0 < \infty$ and $u_0 \in H$ such that $u_0 \geqq 0$ a.e. on $[0, \infty)$ and $u_0 = 0$ a.e. on $[l_0, \infty)$, and let $\{l, u\}$ be the solution to $\mathrm{QV}(b^t; l_0, u_0)$ on $[0, \infty)$. Then

$$l_\infty \equiv \lim_{t \to \infty} l(t) < \infty.$$

In order to show this theorem we prepare two lemmas. Let $\{b^t\}$ and g be as in Theorem 5.1, and define a function $\hat{b}^t$ on $\mathbf{R}$ by

$$\hat{b}^t(r) = \begin{cases} b^t(r) & \text{if } r \in D(b^t) \text{ and } r \geqq g(t), \\ \infty & \text{if } r < g(t), \end{cases} \tag{5.2}$$

for each $t \in [0, \infty)$. Evidently, $\hat{b}^t$ is proper, l.s.c. and convex on $\mathbf{R}$. Besides, we have the following lemma.

LEMMA 5.1. *Let $\{b^t\}$, g and g^* be as in Theorem 5.1, and $\{\hat{b}^t\}$ be as given by (5.2). Then we have:*

(i) *$b^t(r_1 \wedge r_2) + \hat{b}^t(r_1 \vee r_2) \leqq b^t(r_1) + \hat{b}^t(r_2)$ for any $t \in [0, \infty)$ and $r_1, r_2 \in \mathbf{R}$.*
(ii) *If $r \in D(b^t)$ and $r > g(t)$, then $\partial \hat{b}^t(r) = \partial b^t(r)$.*
(iii) *If we put*

$$\hat{\beta}_1(t) = \int_0^t \{3\,|g^*|_{L^\infty(0,\infty)}\,|\beta_0'(\tau)| + |\beta_1'(\tau)|\}\,d\tau \text{ for } t \geqq 0,$$

then $\{\hat{b}^t\} \in B_\infty(\beta_0, \hat{\beta}_1)$.

Proof. (i) and (ii) can be immediately derived from the definition of $\hat{b}^t$, and clearly $\partial \hat{b}^t(r) = \emptyset$ for $r < 0$. Now, let $0 < s \leqq t < \infty$ and $r \in D(\hat{b}^s)$, i.e. $r \in D(b^s)$ with $r \geqq g(s)$. Then, by assumption, there is $\tilde{r} \in D(b^t)$ such that

$$|\tilde{r}-r| \leqq |\beta_0(t)-\beta_0(s)| \left(1+|r|+|\hat{b}^s(r)|^{1/2}\right) \tag{5.3}$$

and

$$b^t(\tilde{r})-\hat{b}^s(r) \leqq |\beta_1(t)-\beta_1(s)| \left(1+|r|^2+|\hat{b}^s(r)|\right). \tag{5.4}$$

Putting $r_1 = \tilde{r} \vee g(t)$, we are going to show

$$|r_1-r| \leqq |\beta_0(t)-\beta_0(s)| \left(1+|r|+|\hat{b}^s(r)|^{1/2}\right) \tag{5.5}$$

and

$$\hat{b}^t(r_1)-\hat{b}^s(r) \leqq |\hat{\beta}_1(t)-\hat{\beta}_1(s)| \left(1+|r|^2+|\hat{b}^s(r)|\right). \tag{5.6}$$

Indeed, in case $\tilde{r} \geqq g(t)$, (5.5) and (5.6) obviously hold by (5.3) and (5.4). Next, assume $\tilde{r} < g(t)$, i.e. $r_1 = g(t)$. Then, since $\tilde{r} < g(t) \leqq g(s) \leqq r$, (5.5) follows immediately from (5.3), and

$$|\tilde{r}-g(t)| \leqq |\tilde{r}-r| \leqq |\beta_0(t)-\beta_0(s)| \left(1+|r|+|\hat{b}^s(r)|^{1/2}\right).$$

Also, by (5.1),

$$g^*(t)\left(\tilde{r}-g(t)\right) \leqq b^t(\tilde{r})-b^t\left(g(t)\right)$$

and hence it follows from (5.4) that

$$\begin{aligned} \hat{b}^t(r_1)-\hat{b}^s(r)\left(=\hat{b}^t\left(g(t)\right)-\hat{b}^s(r)\right) \leqq \\ \leqq \{3\,|g^*(t)|\,|\beta_0(t)-\beta_0(s)|+|\beta_1(t)-\beta_1(s)|\}\left(1+|r|^2+|\hat{b}^s(r)|\right). \end{aligned}$$

Thus (5.5) and (5.6) hold, and $\{\hat{b}^t\} \in B_\infty(\beta_0, \hat{\beta}_1)$.

For the moment we postulate all the assumptions of Theorem 5.1. Now, choose a number $\hat{l}_0$ satisfying $\hat{l}_0 > l_0$. Then, on account of Lemma 5.1, we see by applying Theorems 1.1 and 4.1 that $\mathrm{QV}(\hat{b}^t; \hat{l}_0, u_0)$ has a unique solution $\{\hat{l}, \hat{u}\}$ on $[0, \infty)$ and

$$l \leqq \hat{l} \text{ on } [0, \infty),\ u \leqq \hat{u} \text{ on } [0, \infty)\times(0, \infty). \tag{5.7}$$

Moreover, we have

LEMMA 5.2. *If $\hat{u}(0, t) > g(t)$ for t in a set with positive linear measure, then there are numbers $T > 0$ and $0 < \delta < \hat{l}_0$, such that*

$$\hat{u}(x, t) \geqq \left(1-\frac{x}{\delta}\right) g(t) \textit{ for } (x, t) \in [0, \delta]\times[T, \infty). \tag{5.8}$$

Proof. First, take $T > 0$ so that $\hat{u}(0, T) > g(T)$ and $\hat{u}_x(x, T)$ is absolutely continuous in $x \in (0, \hat{l}(T))$, and choose $0 < \delta < \hat{l}_0$ so that

$$\hat{u}(x, T) \geqq \left(1-\frac{x}{\delta}\right) g(T) \text{ for } x \in [0, \delta].$$

Next, take a sequence $\{g_n\}$ of smooth functions on $[0, \infty)$ such that g_n is non-increasing on $[0, \infty)$, $g_n \leqq g$ on $[0, \infty)$ and $g_n(t) \to g(t)$ as $n \to \infty$ for a.e. $t \geqq 0$. Then, putting

$$v_n(x, t) = \left(1 - \frac{x}{\delta}\right) g_n(t) \text{ on } [0, \delta] \times [T, \infty),$$

we see that

$$v_{n,t} - v_{n,xx} = \left(1 - \frac{x}{\delta}\right) g_n'(t) \leqq 0 = \hat{u}_t - \hat{u}_{xx} \text{ a.e. on } [0, \delta] \times [T, \infty),$$

$$v_n(x, T) = \left(1 - \frac{x}{\delta}\right) g_n(T) \leqq \left(1 - \frac{x}{\delta}\right) g(T) \leqq \hat{u}(x, T) \text{ for } 0 \leqq x \leqq \delta,$$

$$v_n(0, t) = g_n(t) \leqq g(t) \leqq \hat{u}(0, t) \text{ for } T \leqq t < \infty,$$

$$v_n(\delta, t) = 0 \leqq \hat{u}(\delta, t) \text{ for } T \leqq t < \infty,$$

so that by the maximum principle for the linear heat equation we have $\hat{u} \geqq v_n$ on $[0, \delta] \times [T, \infty)$. Letting $n \to \infty$ yields (5.8).

Proof of Theorem 5.1. First assume that $\hat{u}(0, t) = g(t)$ for a.e. $t \geqq 0$. Then, it follows from (1.12) of (*d*) in Remark 1.1 that

$$\hat{l}_\infty^2 \equiv \lim_{t \to \infty} \hat{l}(t)^2 \leqq \hat{l}_\infty^2 + 2 \int_0^\infty x u_0(x)\, dx + 2 \int_0^\infty g(\tau)\, d\tau < \infty.$$

Therefore, noting (5.7), we get $l_\infty < \infty$. Next, assume that $\hat{u}(0, t) > g(t)$ for t in a set with positive linear measure. Then, by Lemma 5.2, for some $T > 0$ and $0 < \delta < \hat{l}_0$ we have

$$\frac{\hat{u}(x, t) - g(t)}{x} \geqq -\frac{1}{\delta} g(t) \text{ for } (x, t) \in [0, \delta] \times [T, \infty). \tag{5.9}$$

Note that $[T, \infty)$ can be divided into two sets $J = \{t \geqq T; \hat{u}(0, t) = g(t)\}$ and $J' = \{t \geqq T; \hat{u}(0, t) > g(t)\}$, since $\hat{u}(0, t) \geqq g(t)$ for all $t \geqq T$. If $t \in J$ and $\hat{u}_x(0+, t)$ exists, then we infer from (5.9) that

$$\hat{u}_x(0+, t) \geqq -\frac{1}{\delta} g(t).$$

Also, if $t \in J'$ and $\hat{u}_x(0+, t) \in \partial \hat{b}^t(\hat{u}(0, t))$, then we have by the monotonicity of $\partial \hat{b}^t$ with (5.1) and (ii) of Lemma 5.1

$$\hat{u}_x(0+, t) \geqq g^*(t).$$

Therefore,

$$-\hat{u}_x(0+, t) \leqq \frac{1}{\delta} g(t) + |g^*(t)| \text{ for a.e. } t \geqq T.$$

Using (1.11) of (*d*) in Remark 1.1, we obtain

$$\hat{l}_\infty \leqq \hat{l}(T)+\int_0^\infty \hat{u}(x, T)\,dx+\int_T^\infty \left\{\frac{1}{\delta} g(t)+|g^*(t)|\right\} dt < \infty,$$

so that $l_\infty < \infty$.

THEOREM 5.2. *Let* $\{b_1^t\} \in B_\infty(\beta_{1,0}, \beta_{1,1})$, $0 < l_{1,0} < \infty$, *and* $u_{1,0} \in H$ *such that* $u_{1,0} \geqq 0$ *a.e. on* $[0, \infty)$ *and* $u_{1,0} = 0$ *a.e. on* $[l_{1,0}, \infty)$. *Suppose that corresponding to these* $\{b_1^t\}$, $l_{1,0}$ *and* $u_{1,0}$ *there exist* $\{b^t\}$, g, g^*, l_0 *and* u_0, *such that all the assumptions of Theorem 5.1 are satisfied, and moreover*

$$l_{1,0} \leqq l_0,\ u_{1,0} \leqq u_0 \ \textit{a.e. on } [0, \infty)$$

and

$$b_1^t(r_1 \wedge r_2)+b^t(r_1 \vee r_2) \leqq b_1^t(r_1)+b^t(r_2) \ \textit{for any } t \in [0, \infty) \textit{ and } r_1, r_2 \in \mathbf{R}.$$

Then

$$l_{1,\infty} \equiv \lim_{t\to\infty} l_1(t) < \infty,$$

where $\{l_1, u_1\}$ *is the solution to* QV $(b_1^t; l_{1,0}, u_{1,0})$ *on* $[0, \infty)$.

Proof. By Theorems 1.1 and 4.1, QV $(b^t; l_0, u_0)$ has a unique solution $\{l, u\}$ on $[0, \infty)$ and $l_1 \leqq l$ on $[0, \infty)$. Besides, by Theorem 5.1, $\lim_{t\to\infty} l(t) < \infty$, so that $l_{1,\infty} < \infty$. ■

Next, under the assumption $l_\infty < \infty$, we investigate the asymptotic behaviour of u.

THEOREM 5.3. *Let* $\{b^t\} \in B_\infty(\beta_0, \beta_1)$ *with* $\beta_0' \in L^1(0, \infty) \cap L^2(0, \infty)$ *and* $\beta_1' \in L^1(0, \infty)$, *and suppose* $b^t \to b^\infty$ *on* $\mathbf{R}$ *as* $t \to \infty$ *in the sense of Mosco for a proper l.s.c. convex function* b^∞ *on* $\mathbf{R}$. *Also, let* $0 < l_0 < \infty$ *and* $u_0 \in H$ *with* $u_0 \geqq 0$ *a.e. on* $[0, \infty)$ *and* $u_0 = 0$ *a.e. on* $[l_0, \infty)$, *and let* $\{l, u\}$ *be the solution to* QV $(b^t; l_0, u_0)$ *on* $[0, \infty)$. *If* $l_\infty < \infty$, *then*

$$u(\cdot, t) \to 0 \ \textit{in } X \textit{ as } t \to \infty$$

and

$$0 \in \partial b^\infty(0).$$

Proof. By virtue of Lemma 3.2, $u(t)$ converges in X as $t \to \infty$ and the limit u_∞ is given by

$$u_\infty(x) = \begin{cases} c_\infty\left(1-\dfrac{x}{l_\infty}\right) & \text{for } 0 \leqq x \leqq l_\infty, \\ 0 & \text{for } l_\infty < x < \infty, \end{cases}$$

with the non-negative constant c_∞ satisfying

$$-\frac{c_\infty}{l_\infty} \in \partial b^\infty (c_\infty).$$

If $c_\infty = 0$ were shown, the proof of the theorem would be complete. Suppose for a contradiction that $c_\infty > 0$. Then, since $u(t) \to u_\infty$ in X and $l(t) \to l_\infty$ as $t \to \infty$, for each positive number ε with $\varepsilon < c_\infty$ there is $t_\varepsilon > 0$ such that

$$\varepsilon > l(t)^2 + 2\int_0^{l(t)} xu(x, t)\, dx - l(s)^2 - 2\int_0^{l(s)} xu(x, s)\, dx \left(= 2\int_s^t u(0, \tau)\, d\tau\right)$$

and

$$u(0, s) \geqq c_\infty - \varepsilon$$

for all s, t with $t_\varepsilon \leqq s \leqq t < \infty$. Hence

$$\varepsilon > 2(c_\infty - \varepsilon)(t - s)$$

for all s, t with $t_\varepsilon \leqq s \leqq t < \infty$, which is impossible. Thus $c_\infty = 0$ must be true. ∎

6. Further investigations of the asymptotic behaviour

In this section we investigate the asymptotic behaviour of u in the case where $\lim_{t\to\infty} l(t)$ may be infinite.

THEOREM 6.1. *Let $\{b^t\} \in B_\infty(\beta_0, \beta_1)$ with $\beta'_0 \in L^1(0, \infty) \cap L^2(0, \infty)$ and $\beta'_1 \in L^1(0, \infty)$, and suppose $b^t \to b^\infty$ on $\mathbf{R}$ as $t \to \infty$ in the sense of Mosco for a proper l.s.c. convex function b^∞ on $\mathbf{R}$. Let $0 < l_0 < \infty$, $u_0 \in H$ with $u_0 \geqq 0$ a.e. on $[0, \infty)$ and $u_0 = 0$ a.e. on $[l_0, \infty)$, and let $\{l, u\}$ be the solution to $\mathrm{QV}(b^t; l_0, u_0)$ on $[0, \infty)$.*
Then we have:

(i) *If $0 \notin \bigcup_{r \geqq 0} \partial b^\infty(r)$, then $u(x, t) \to \infty$ as $t \to \infty$ uniformly on each bounded interval of x.*

(ii) *If $0 \in \bigcup_{r \geqq 0} \partial b^\infty(r)$, then $\liminf_{t\to\infty} u(x, t) \geqq c_*$ uniformly on each bounded interval of x where*

$$c_* = \inf\{r \geqq 0; 0 \in \partial b^\infty(r)\} \quad (\textit{note that } 0 \in \partial b^\infty(c_*)).$$

In our proof of Theorem 6.1 we consider an auxiliary Cauchy problem for given $0 < t_0 < \infty$ and $0 < L < \infty$:

$$\begin{cases} -v'(t) \in \partial \psi^t_L(v(t)), & t_0 < t < \infty, \\ v(t_0) = 0, \end{cases} \tag{6.1}$$

where ψ_L^t is a proper l.s.c. convex function on H given by

$$\psi_L^t(z)=\begin{cases}\frac{1}{2}|z_x|_H^2+b^t(z(0)) & \text{if } z\in X,\ z(0)\in D(b^t) \text{ and } z=0 \text{ on } [L,\infty),\\ \infty & \text{otherwise.}\end{cases}$$

LEMMA 6.1. *Let* $\{b^t\}\in B_\infty(\beta_0,\beta_1)$ *with* $\beta_0'\in L^2(0,\infty)\cap L^1(0,\infty)$ *and* $\beta_1'\in L^1(0,\infty)$ *and* b^∞ *be a proper l.s.c. convex function on* $\mathbf{R}$ *such that* $b^t\to b^\infty$ *on* $\mathbf{R}$ *as* $t\to\infty$ *in the sense of Mosco. Then problem* (6.1) *has a unique solution* v *on* $[t_0,\infty)$, *and* $v(t)\to v_L$ *in* X *as* $t\to\infty$, *where*

$$v_L(x)=\begin{cases}c_L\left(1-\frac{x}{L}\right) & \text{for } 0\leqq x\leqq L,\\ 0 & \text{for } L<x<\infty,\end{cases}\tag{6.2}$$

with the constant c_L *satisfying* $-c_L/L\in\partial b^\infty(c_L)$.

This lemma is a direct consequence of Lemma 3.2. ■

Proof of Theorem 6.1. On account of Theorem 5.3, it suffices to prove the theorem in the case of $l_\infty\equiv\lim_{t\to\infty} l(t)=\infty$. In the rest of the proof, suppose that $l_\infty=\infty$. Let t_0 be any positive number and take $l(t_0)$ as L. Then, by Corollary 2 to Lemma 3.1,

$$v\leqq u \text{ on } [0,\infty)\times(t_0,\infty),\tag{6.3}$$

where v is the solution to (6.1). Hence, by Lemma 6.1,

$$v_L(x)\leqq\liminf_{t\to\infty} u(x,t) \text{ uniformly in } x\in[0,\infty).\tag{6.4}$$

Now, in addition, suppose $0\in\bigcup_{r\geqq 0}\partial b^\infty(r)$. Then we have $0\leqq c_L\leqq c_*$ by the monotonicity of ∂b^∞, where c_L is as in (6.2). Also, let c' be any cluster point of c_L as $L\to\infty$. Then $0\leqq c'\leqq c_*$ and $0\in\partial b^\infty(c')$, so that $c'=c_*$, i.e.

$$c_*=\lim_{L\to\infty} c_L.\tag{6.5}$$

We can derive (ii) of Theorem 6.1 from (6.2), (6.4) and (6.5). Next, suppose $0\notin\bigcup_{r\geqq 0}\partial b^\infty(r)$. In this case, we see easily that $c_L\to\infty$ as $L\to\infty$, so that (i) of the theorem follows from (6.3) and (6.4).

We have given the asymptotic evaluation of u from below. In the next theorem we evaluate it from above.

THEOREM 6.2. *Let* $\{b^t\}\in B_\infty(\beta_0\cdot\beta_1)$, *and suppose there are two functions* g *and* g^* *on* $[0,\infty)$ *such that*

$$g\geqq 0 \quad \text{on } [0,\infty),\ g^*\in W^{1,1}(0,\infty)$$

and

$$g^*(t) \in \partial b^t(g(t)) \text{ for all } t \geqq 0.$$

Let $0 < l_0 < \infty$, $u_0 \in H$ *such that* $u_0 \geqq 0$ *a.e. on* $[0, \infty)$ *and* $u_0 = 0$ *a.e. on* $[l_0, \infty)$, *and let* $\{l, u\}$ *be the solution to* $\mathrm{QV}(b^t; l_0, u_0)$ *on* $[0, \infty)$. *Then*

$$\limsup_{t \to \infty} u(x, t) \leqq g^\infty \text{ uniformly in } x \in [0, \infty), \tag{6.6}$$

where $g^\infty = \limsup_{t \to \infty} g(t)$.

In order to prove this theorem, we consider the following problem:

$$\begin{cases} -w'(t) \in \partial\psi^t(w(t)), & t_0 < t < \infty, \\ w(t_0) = u(t_0), \end{cases} \tag{6.7}$$

where $\{l, u\}$ is the solution to $\mathrm{QV}(b^t; l_0, u_0)$ on $[0, \infty)$, $0 < t_0 < \infty$ and

$$\psi^t(z) = \begin{cases} \frac{1}{2}|z_x|_H^2 - |g^*(t)|\, z(0) & \text{if } z \in X,\ z(0) \geqq 0 \text{ and } z = 0 \text{ on } [l(t), \infty), \\ \infty & \text{otherwise.} \end{cases} \tag{6.8}$$

LEMMA 6.2. *Under the same assumptions and notations as in Theorem 6.2, problem* (6.7) *has a unique solution* w *on* $[t_0, \infty)$ *such that* $w \geqq 0$ *on* $[0, \infty) \times (t_0, \infty)$ *and*

$$w(x, t) \to 0 \text{ as } t \to \infty \text{ uniformly in } x \in [0, \infty).$$

Proof. We set

$$b_*^t(r) = \begin{cases} -|g^*(t)|\, r & \text{if } r \geqq 0, \\ \infty & \text{if } r < 0 \end{cases}$$

for each $t \geqq 0$ and

$$b_*^\infty(r) = \begin{cases} 0 & \text{if } r \geqq 0, \\ \infty & \text{if } r < 0. \end{cases}$$

Since $g^*(t) \to 0$ as $t \to \infty$, it follows easily that $\{b_*^t\} \in B_\infty(0, g^*)$ and $b_*^t \to b_*^\infty$ on $\mathbf{R}$ as $t \to \infty$ in the sense of Mosco, so that $\psi^t \to \psi^\infty$ on H as $t \to \infty$ in the sense of Mosco (cf. Lemma 2.2), where ψ^t is as given by (6.8) and

$$\psi^\infty(z) = \begin{cases} \frac{1}{2}|z_x|_H^2 & \text{if } z \in X,\ z(0) \geqq 0 \text{ and } z = 0 \text{ on } [l_\infty, \infty), \\ \infty & \text{otherwise,} \end{cases}$$

with $l_\infty \equiv \lim_{t \to \infty} l(t)$; note here that the restriction $z = 0$ on $[l_\infty, \infty)$ is deleted if $l_\infty = \infty$. Also, by (i) of Lemma 2.4 and (i) of Theorem A.1 in the Appendix, (6.7) has a unique solution w on $[t_0, \infty)$. Since $-w'(\tau) \in \partial\psi^\tau(w(\tau))$ for a.e. $\tau \geqq t_0$, we have

$$\frac{1}{2}\frac{d}{d\tau}|w(\tau)|_H^2 = (w'(\tau), w(\tau))_H \leqq -|w_x(\cdot, \tau)|_H^2 + |g^*(\tau)|\, w(0, \tau) \tag{6.9}$$

for a.e. $\tau \geqq t_0$. We note here that

$$|w(0, \tau)| \leqq |w_x(\cdot, \tau)|_H + |w(\cdot, \tau)|_H \text{ for a.e. } \tau \geqq t_0. \tag{6.10}$$

From (6.9) and (6.10) it follows that

$$\frac{d}{d\tau}|w(\tau)|_H^2 \leqq |g^*(\tau)|\,|w(\tau)|_H^2 + |g^*(\tau)|^2 + |g^*(\tau)| \text{ for a.e. } \tau \geqq t_0.$$

Since $|g^*| \in L^1(0, \infty)$ and $|g^*|^2 \in L^1(0, \infty)$, we have $w \in L^\infty(t_0, \infty; H)$ by Gronwall's inequality. Accordingly, Theorem A.3 in the Appendix implies that $w(t) \to w_\infty$ weakly in H and $\psi^t(w(t)) \to \psi^\infty(w_\infty) = \min \psi^\infty (= 0)$ as $t \to \infty$ for some $w_\infty \in X$. It is not difficult to see that $w_\infty \equiv 0$ and $w(x, t) \to 0$ as $t \to \infty$ uniformly in $x \in [0, \infty)$. ■

Proof of Theorem 6.2. It suffices to show (6.6) in the case of $g^* < \infty$. In this case, let ε be an arbitrary positive number, and choose $t_\varepsilon > 0$ so that

$$g(t) < g^\infty + \varepsilon \text{ for all } t \geqq t_\varepsilon.$$

Also, consider problem (6.7) with $t_0 = t_\varepsilon$ and denote by w_ε its solution on $[t_\varepsilon, \infty)$. Then, by Lemma 6.2,

$$w_\varepsilon(x, t) \to 0 \text{ as } t \to \infty \text{ uniformly in } x \in [0, \infty).$$

Now we are going to show that $u \leqq w_\varepsilon + g^\infty + \varepsilon$ on $[0, \infty) \times (t_\varepsilon, \infty)$. In fact, we have

$$u_t - u_{xx} = w_{\varepsilon,t} - w_{\varepsilon,xx} = 0 \text{ a.e. on } \{(x, t); 0 < x < l(t), t_\varepsilon < t < \infty\},$$
$$u(x, t_\varepsilon) = w_\varepsilon(x, t_\varepsilon) \text{ for a.e. } x \geqq 0,$$
$$u = w_\varepsilon = 0 \text{ on } \{(x, t); l(t) \leqq x < \infty, t_\varepsilon < t < \infty\}.$$

Besides, if $t > t_\varepsilon$, $u(0, t) > w_\varepsilon(0, t) + g^\infty + \varepsilon\ (> g(t))$ and $u_x(0+, t) \in \partial b^t(u(0, t))$, then we see from the monotonicity of ∂b^t that

$$u_x(0+, t) \geqq g^*(t).$$

Also we note

$$w_{\varepsilon,x}(0+, t) \leqq -|g^*(t)| \text{ for a.e. } t \geqq t_\varepsilon.$$

Hence

$$(u_x(0+, t) - w_{\varepsilon,x}(0+t))\,(u(0, t) - w_\varepsilon(0, t) - g^\infty - \varepsilon)^+ \geqq 0 \text{ for a.e. } t \geqq t_\varepsilon,$$

so that on account of Lemma 3.1

$$u \leqq w_\varepsilon + g^\infty + \varepsilon \text{ on } [0, \infty) \times (t_\varepsilon, \infty).$$

Hence

$$\limsup_{t\to\infty} u(x,t) \leqq \lim_{t\to\infty} w_\varepsilon(x,t) + g^\infty + \varepsilon = g^\infty + \varepsilon \text{ uniformly in } x \in [0,\infty).$$

Since ε is arbitrary, we get (6.6). ■

Appendix Some abstract results on nonlinear evolution equations

Let H be an abstract Hilbert space and $\{\varphi^t\} = \{\varphi^t; 0 \leqq t < \infty\}$ be a family of proper l.s.c. convex functions on H. Consider the Cauchy problem $\mathrm{CP}(\varphi^t; u_0)$ on $[0, T], 0 < T < \infty$:

$$\mathrm{CP}(\varphi^t; u_0): \begin{cases} -u'(t) \in \partial\varphi^t(u(t)), \ 0 < t < T, \\ u(0) = u_0, \end{cases}$$

where u_0 is given in H. A function $u: [0, T] \to H$ is called a solution to $\mathrm{CP}(\varphi^t; u_0)$ on $[0, T]$, if it fulfills:

(a) $u \in C([0, T]; H) \cap W^{1,2}(\delta, T; H)$ for every $0 < \delta < T$ and $u(0) = u_0$,

(b) $t \to \varphi^t(u(t))$ is integrable on $[0, T]$ and is bounded on $[\delta, T]$ for every $0 < \delta < T$, and

(c) $-u'(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in [0, T]$.

Also, $u: [0, \infty) \to H$ is called a solution to $\mathrm{CP}(\varphi^t; u_0)$ on $[0, \infty)$, if it is a solution to $\mathrm{CP}(\varphi^t; u_0)$ on every finite interval $[0, T]$.

Let $u_{0,i} \in H$ and u_i be a solution to $\mathrm{CP}(\varphi^t; u_{0,i})$ on $[0, T]$, $i = 1, 2$. Then we have (cf. [9; §1.1])

$$|u_1(t) - u_2(t)|_H \leqq |u_1(s) - u_2(s)|_H \text{ for every } 0 \leqq s \leqq t \leqq T,$$

and therefore $u_{0,1} = u_{0,2}$ implies $u_1 = u_2$ on $[0, T]$. This shows that $\mathrm{CP}(\varphi^t; u_0)$ has at most one solution for each $u_0 \in H$.

The existence of a solution to $\mathrm{CP}(\varphi^t; u_0)$ is shown for $\{\varphi^t\}$ belonging to the following class $\Phi(\{\alpha_{0,r}\}, \{\alpha_{1,r}\})$: given two families $\{\alpha_{0,r}\} = \{\alpha_{0,r}; 0 \leqq r < \infty\} \subset W^{1,2}_{\mathrm{loc}}([0, \infty))$ and $\{\alpha_{1,r}\} = \{\alpha_{1,r}; 0 \leqq r < \infty\} \subset W^{1,1}_{\mathrm{loc}}([0, \infty))$, we denote by $\Phi(\{\alpha_{0,r}\}, \{\alpha_{1,r}\})$ the set of all $\{\varphi^t\}$ having the property

(*) for each $0 \leqq s \leqq t < \infty$ and each $z \in D(\varphi^s)$ with $|z|_H \leqq r$ there is $\tilde{z} \in D(\varphi^t)$ such that

$$|\tilde{z} - z|_H \leqq |\alpha_{0,r}(t) - \alpha_{0,r}(s)| \left(1 + |\varphi^s(z)|^{1/2}\right)$$

and

$$\varphi^t(\tilde{z}) - \varphi^s(z) \leqq |\alpha_{1,r}(t) - \alpha_{1,r}(s)| \left(1 + |\varphi^s(z)|\right).$$

THEOREM A.1 (cf. [9; §1.1, §2.8]). *Let* $\{\varphi^t\} \in \Phi(\{\alpha_{0,r}\}, \{\alpha_{1,r}\})$ *and* $u_0 \in \overline{D(\varphi^0)}$. *Then we have*:

(i) $\mathrm{CP}(\varphi^t; u_0)$ *admits one and only one solution* u *on* $[0, \infty)$ *such that*

$$\sqrt{t}\, u' \in L^2(0, T; H) \text{ for every finite } T > 0$$

and

$$t \to t\varphi^t(u(t)) \text{ is bounded on } (0, T] \text{ for every finite } T > 0.$$

In particular, if $u_0 \in D(\varphi^0)$, *then* $u' \in L^2(0, T; H)$ *and* $t \to \varphi^t(u(t))$ *is bounded on* $[0, T]$ *for every finite* $T > 0$.

(ii) *The solution* u *to* $\mathrm{CP}(\varphi^t; u_0)$ *on* $[0, \infty)$ satisfies

$$\varphi^t(u(t)) - \varphi^s(u(s)) + \frac{1}{2}\int_s^t |u'(\tau)|_H^2\, d\tau \leqq \int_s^t k_r(t)\left(1 + |\varphi^\tau(u(\tau))|\right) d\tau$$

for every $0 < s \leqq t < \infty$ *with* $\sup_{0 \leqq \tau \leqq t} |u(\tau)|_H < r$, *where*

$$k_r(\tau) = 4\,|\alpha'_{0,r}(\tau)|^2 + |\alpha'_{1,r}(\tau)| \text{ for } \tau \geqq 0 \text{ and } r \geqq 0.$$

Next, we recall a notion of the convergence of convex functions due to Mosco [12]. Given a sequence $\{\psi_n\}$ of proper l.s.c. convex functions on H and a proper l.s.c. convex function ψ on H, we say that ψ_n converges to ψ on H as $n \to \infty$ in the sense of Mosco if the following (a) and (b) are satisfied:

(a) If $z_n \to z$ weakly in H (as $n \to \infty$), then

$$\liminf_{n \to \infty} \psi_n(z_n) \geqq \psi(z).$$

(b) For each $z \in D(\psi)$ there is a sequence $\{z_n\}$ such that $z_n \to z$ in H and $\psi_n(z_n) \to \psi(z)$.

With this notion we give a convergence result of solutions to our Cauchy problems.

THEOREM A.2 (cf. [7; Theorem 1] or [9; §2.7]). *Let* $\{\varphi^t\}$ *and* $\{\varphi_n^t\}$, $n = 1, 2, \ldots$, *be in* $\Phi(\{\alpha_{0,r}\}, \{\alpha_{1,r}\})$ *such that*

$$\varphi_n^t \to \varphi^t \text{ on } H \text{ as } n \to \infty \text{ in the sense of Mosco for each } t \geqq 0.$$

Let $u_0 \in \overline{D(\varphi^0)}$ *and* $u_{0,n} \in \overline{D(\varphi_n^0)}$, $n = 1, 2, \ldots$, *such that* $u_{0,n} \to u_0$ *in* H. *Then, denoting by* u *and* u_n *the solutions to* $\mathrm{CP}(\varphi^t; u_0)$ *and* $\mathrm{CP}(\varphi_n^t; u_{0,n})$ *on* $[0, \infty)$, *respectively, we have*

$$u_n \to u \text{ in } C([0, T]; H)$$

and

$$\int_0^T \varphi_n^t(u_n(t))\,dt \to \int_0^T \varphi^t(u(t))\,dt$$

as $n \to \infty$ *for each finite* $T > 0$.

Finally we mention a result concerning the asymptotic behaviour of the solution to $\mathrm{CP}(\varphi^t; u_0)$ on $[0, \infty)$. Given a family $\{\varphi^t\}$ and a proper l.s.c. convex function φ^∞ on H, we say that $\varphi^t \to \varphi^\infty$ on H as $t \to \infty$ in the sense of Mosco, if $\varphi^{t_n} \to \varphi^\infty$ on H as $n \to \infty$ in the sense of Mosco for every sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$.

THEOREM A.3 (cf. [8; Theorem 1]). *Let* $\{\varphi^t\} \in \Phi(\{\alpha_{0,r}\}, \{\alpha_{1,r}\})$ *with* $\alpha'_{0,r} \in L^2(0, \infty)$ *and* $\alpha'_{1,r} \in L^1(0, \infty)$ *for any* $r \geqq 0$, *and suppose that* $\varphi^t \to \varphi^\infty$ *on* H *as* $t \to \infty$ *in the sense of Mosco for a proper l.s.c. convex function* φ^∞ *on* H. *Further, let* $u_0 \in \overline{D(\varphi^0)}$, *and* u *be the solution to* $\mathrm{CP}(\varphi^t; u_0)$ *on* $[0, \infty)$. *If* φ^∞ *is strictly convex on* $D(\varphi^\infty)$ *and* $\sup_{0 \leqq t < \infty} |u(t)|_H < \infty$, *then there exists* $u_\infty \in D(\varphi^\infty)$ *such that* $u(t) \to u_\infty$ *weakly in* H, $\varphi^t(u(t)) \to \varphi^\infty(u_\infty)$ *as* $t \to \infty$ *and* $\varphi^\infty(u_\infty) = \min \varphi^\infty$, *i.e.* $0 \in \partial\varphi^\infty(u_\infty)$.

REMARK. In applying [8; Theorem 1], for each $z \in D(\varphi^\infty)$ it is necessary to show the existence of a function w: $[0, \infty) \to H$ such that $w(t) \to z$ in H and $\varphi^t(w(t)) \to \varphi^\infty(z)$ as $t \to \infty$. Under the assumptions of Theorem A.3, given z in $D(\varphi^\infty)$, such a function w can be constructed as follows: First, take a sequence $\{z_n\}$ in H such that $z_n \to z$ in H and $\varphi^n(z_n) \to \varphi^\infty(z)$ as $n \to \infty$. Let r and L be non-negative numbers satisfying $|z_n|_H \leqq r$ and $|\varphi^n(z_n)| \leqq L$ for all n, respectively. Then, by assumption, for each $t \in [n, n+1)$, $n = 0, 1, \dots,$ there exists $w(t) \in D(\varphi^t)$ such that

$$|w(t) - z_n|_H \leqq |\alpha_{0,r}(t) - \alpha_{0,r}(n)|\,(1 + L^{1/2})$$

and

$$\varphi^t(w(t)) - \varphi^n(z_n) \leqq |\alpha_{1,r}(t) - \alpha_{1,r}(n)|\,(1 + L).$$

It is easy to see that this function w has the desired property.

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Jednofazowe zagadnienia Stefana z nieliniowymi warunkami brzegowymi na ustalonym brzegu

W pracy rozważa się jednowymiarowe jednofazowe zagadnienia Stefana z jednostronnymi warunkami brzegowymi na ustalonym brzegu. Dowodzi się istnienia globalnego rozwiązania takiego zagadnienia odpowiadającego jego sformułowaniu quasi-wariacyjnemu. Dyskutowane są następujące problemy:

a) monotoniczna zależność rozwiązania od warunków brzegowych i początkowych, jednoznaczność rozwiązania:

b) asymptotyczne zachowanie swobodnej granicy $x = l(t)$, warunki dostateczne na dane brzegowe, zapewniające skończoność $l(t)$ przy $t \to \infty$;

c) asymptotyczne zachowanie rozwiązania $u = u(x, t)$, zależność $\liminf u(x, t)$ i $\limsup u(x,t)$, przy $t \to \infty$ od danych brzegowych.

Однофазная проблема Стефана с нелинейными краевыми условиями на фиксированном крае

В работе рассуждается одномерную однофазную проблему Стефана с односторонними краевыми условиями на фиксированном крае. Доказывается существование глобального решения такой проблемы, ответствующего ее квази-вариационной постановке. Рассуждены следующие вопросы:

a) монотонная зависимость решения от краевых и начальных условий, однозначность решения,
b) асимптотическое поведение свободной границы $x = l(t)$, достаточные условия для краевых условий, при которых $\lim l(t)$ при $t \to \infty$ конечная,
c) асимптотическое поведение решения $u = u(x, t)$, зависимость $\liminf u(x, t)$ и $\limsup u(x, t)$ при $t \to \infty$ от краевых условий.