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Asymptotic behaviour of the solution of the non-steady dam problem for incompressible fluids

by

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In this paper we consider the global behaviour ($t \to \infty$) for the solution of the nonlinear equation $\partial_t \gamma - \Delta u - \partial_y \gamma = 0$ in $]0, T[\times D$ with respect to some physical motivated boundary and initial conditions. For γ we assume that $0 \leqslant \gamma \leqslant 1$ and $u(1-\gamma) = 0$. This model describes the non-steady filtration of an incompressible fluid through an isotropic homogeneous medium D. The main result is the convergence of $u(t, \cdot)$ to the solution of the stationary problem.

1. Introduction

Let us consider two water-reservoirs which are separated by a dam D consisting of an isotropic, homogeneous, porous material. The levels of the reservoirs may be different and time-dependent and they are supposed to tend to fixed levels if t tends to infinity. We start with nonstationary initial conditions and we are interested in the asymptotic behaviour of the pressure distribution $u(t, z)$ of the water in the dam if t tends to infinity.

By an unknown interface, the free boundary, the dam is separated at time t into a wet part $\Omega(t)$ and a dry part $D \backslash \Omega(t)$. Let us assume that the water is incompressible. Then we know ([16], [12], [7]) that u is a positive harmonic function in the wet part and satisfied two boundary conditions on the free boundary:

$$u = 0 \text{ and } \partial_t \varphi + \partial_\nu u \cdot \sqrt{1 + (\partial_x \varphi)^2} = 0,$$

if there exists a parametrization $\varphi = \varphi(t, x)$ of the free boundary which is regular enough and if ν denotes the outward normal with respect to $\Omega(t)$ in the (x, y)-plane. We would like to have a formulation of the problem in which the free boundary does not occur. Therefore we extend u by zero to all of D and obtain ([16]):

$$\partial_t \chi - \Delta u - \partial_y \chi = 0 \text{ in }]0, T[\times D \tag{1.1}$$

where $\chi(t, \cdot)$ denotes the chracteristic function of $\Omega(t)$ and y is the vertical coordinate of $z = (x, y) \in D$. Now it is more convenient ([1], [12]) to replace χ by $\gamma \in L^\infty(D_T)$ satisfying

$$0 \leqslant \gamma \leqslant 1 \text{ and } u(1-\gamma) = 0 \text{ a.e. in } D_T. \tag{1.2}$$

We shall study (1.1), (1.2) with respect to boundary conditions on ∂D which are given by the physical situation. Before formulating it let us summarize the assumptions concerning ∂D. For the boundary ∂D we assume:

∂D is Lipschitz continuous
and Γ_0, Γ_1 are graphs of functions $\psi_0, \psi_1 \in C^2([a, b])$
such that $D = \{(x, y) | x \in]a, b[, \psi_1(x) < y < \psi_0(x)\}$,
$\psi_1(a) = \psi_0(a)$, $\psi_1(b) = \psi_0(b)$. Let $P := (x_0, y_0)$
denote the top of the dam. Then $\psi_0'(x) > 0$
for $a < x < x_0$ and $\psi_0'(x) < 0$ for $x_0 < x < b$. (1.3)

The last conditions ensures that the number of reservoirs remains constant. (1.3) implies that $e \cdot \nu < 0$ on Γ_1 where e is the vertical unit vector $(0, 1)$ and ν is the outward normal to Γ_1.

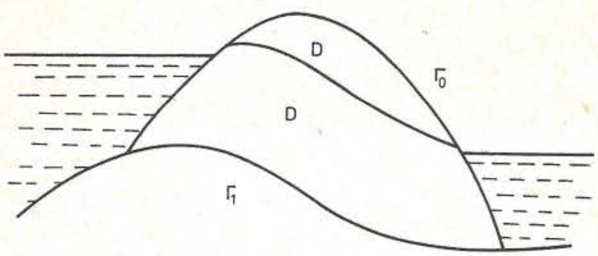

Fig. 1

On Γ_1 the dam is assumed to be impervious, i.e. ([12], [16]):

$$u_\nu + \chi \cos(\nu \cdot e) = 0 \text{ on } \Sigma_1 :=]0, T[\times \Gamma_1. \tag{1.4}$$

where ν is the outer normal to Γ_1.
We split Γ_0 into two parts

$$\Gamma_{01} := \{(x, y) \in \Gamma_0 | x \leqslant x_0\}, \; \Gamma_{02} := \Gamma_0 \backslash \Gamma_{01}$$

and describe the pressure:

$$u(t, x, y) = (y_i(t) - y)^+ \text{ on } \Sigma_{0i} :=]0, \infty[\times \Gamma_{0i}, \tag{1.5}$$

and assume

$$0 \leqslant y_i(t) < y_0 \text{ for } t \in]0, \infty[\\ y_i(t) \to Y_i \text{ if } t \to \infty \tag{1.6}$$

for $i = 1, 2$. Here $y_i(t)$ measures the water levels of the reservoirs $i = 1, 2$ at time t. The conditions (1.5) means that the pressure of the "wet part" of Γ_0 is given by the water pressure of the reservoirs and on the dry parts of Γ_0 by zero. Furthermore the levels are supposed to stay below the maximal height of the dam Y_i defines the height of the i^{th}-reservoir in the stationary situation.

Initially we prescribe the wet part of the dam that means

$$\gamma(0, \cdot) = \chi_0 \text{ a.e. in } D \tag{1.7}$$

where χ_0 is the characteristic function of some open subset $D_0 \subset D$ such that the following compatibility condition is fulfilled:

$$\chi_0|_{\Gamma_0} = \operatorname{sign}(u|_{\{0\} \times \Gamma_0}). \tag{1.8}$$

For the weak formulation of the problem we need some further technical assumptions. Let us assume that there exists an extension g of the boundary values on Γ_0 such that

$$g \in C^{0,1}(\bar{D}_\infty) \cap L^\infty(D_\infty), g \geqslant 0 \text{ in } D_\infty; \\ g(t, x, y) = (y_i(t) - y)^+ \text{ on } \Sigma_{0i}, \ i = 1, 2. \tag{1.9}$$

Since we shall look for the asymptotic behaviour for $t \to \infty$ we suppose that there exists a function G such that

$$G \in C^{0,1}(D), \\ g(t, \cdot) \to G \text{ uniformly in } D \text{ if } t \to \infty, \\ G(x, y) = (Y_i - y)^+ \text{ on } \Gamma_{0i}, \ i = 1, 2. \tag{1.10}$$

The space of test functions is

$$V = \{\omega \in H^{1,2}(D) | \omega = 0 \text{ on } \Gamma_0\}. \tag{1.11}$$

Then we shall investigate the following weak formulation of the described problem.

1.1. Non-steady Problem. The data D, g, χ_0 are supposed to satisfy (1.3), (1.8), (1.9). Then find a pair of functions $\{u, \gamma\}$ such that we have

$$u = g + L^2_{\text{loc}}(0, \infty; V), \gamma \in L^\infty(D_\infty), \partial_t \gamma \in L^2_{\text{loc}}(0, \infty; V^*); \tag{1.12}$$

$$u \geqslant 0, \ 0 \leqslant \gamma \leqslant 1, \ u(1 - \gamma) = 0 \text{ a.e. in } D_\infty; \tag{1.13}$$

$$\int_{D_\infty} (\gamma(\partial_y v - \partial_t v) + \nabla u \nabla v) \leqslant 0 \text{ for all } v \in H^1(0, \infty; H^1(D)), \tag{1.14}$$
$$v \geqslant 0 \text{ on } \Gamma_0 \cap \{g = 0\}, \ v = 0 \text{ on } \Gamma_0 \cap \{g > 0\};$$

$$\int_0^T \int_D (\gamma-\gamma_0)\, \partial_t \zeta = \int_0^T < \partial_t \gamma, \zeta >, \gamma_0 := \gamma(0, \cdot)$$

for all $\zeta \in L^2(0, T; V) \cap H^1(0, T, L^\infty(D)), \zeta(T) = 0$ for some $T > 0$. (1.15)

(1.12) contains the Dirichlet boundary conditions on Γ_0, the weak formulation (1.14) the Neumann boundary condition on Γ_1 and (1.15) is the weak formulation for the initial condition.

In this paper we intend to show, that the solution of Problem 1.1 which we get by regularization converges strongly in $L^p(D)$, $1 \leqslant p < \infty$ to the solution u_∞ of the stationary problem which is defined in

1.2 STATIONARY PROBLEM. For given data D and G satisfying (1.3), (1.10) find $\{u_\infty, \gamma_\infty\}$ such that we have

$$u_\infty \in G+V, \gamma_\infty \in L^\infty(D);$$

$$u_\infty \geqslant 0, 0 \leqslant \gamma_\infty \leqslant 1, u_\infty (1-\gamma_\infty) = 0 \text{ a.e. in } D;$$

$$\int_D (\nabla u_\infty + e\gamma_\infty) \nabla v \leqslant 0 \text{ for all } v \in H^1(D),$$

$$v \geqslant 0 \text{ on } \Gamma_0 \cap \{G = 0\},$$

$$v = 0 \text{ on } \Gamma_0 \cap \{G > 0\}.$$

1.3. REMARK. We have to assume that

Problem 1.2 has at most one solution. (1.16)

Conditions for the data of Problem 1.2 under which (1.16) is true can be found in [5], [8] and [9]. Roughly speaking we have to guarantee that each drop of water in the dam is connected with the reservoirs. For example if D has a convex bottom there exists at most one solution ([5] Theorem 9.3, [8] Remark 3, [9] Remark 3).

Let us continue with some known results in this field. For the problem 1.1, Gilardi ([12], Theorem 4.1) has proved the existence of at least one bounded solution.

In the case of a single equation where we have θ instead of γ in (1.14) and θ is Lipschitz continuous the global behaviour was studied in [14]. There it was even shown that $u(t) \to u_\infty$ strongly in $H^1(D)$ for $t \to \infty$ and the rate of convergence could be estimated. In a recent paper Friedman and DiBenedetto [11] consider a rectangular dam which separates compressible fluids. They investigate the situation of a periodical movement of the boundary values. Large time behaviour for initial boundary value problems of the form $\partial_t u = \partial_{xx}(u^m) + f(u)$ are studied in [4] and for $\partial_t u = (D(u)\, \varphi(u_x))_x$ (with degeneration in D and φ) in [10]. For investigations concerning the asymptotic behaviour of solutions of Stefan-type problems see for example [13].

2. Main results

In this section we shall formulate the main result of this paper (Theorem 2.5). But previously let us repeat an existence theorem for the stationary solution and let us describe the regularization which gives us a solution of the non-steady problem.

2.1. THEOREM (*existence for the stationary problem*). *There exists a solution of Problem* 1.2.

Proof. See Alt [3] and Brezis [8].

REMARK. In this paper we do not assume that Problem 1.2 has a solution. In §6 we shall show independently of Theorem 2.1 that there exists a solution of Problems 1.2.

Now let us describe the regularization which will give us a solution of the non-steady problem 1.1. We define

$$b_\varepsilon(t) := \begin{cases} 1 \text{ if } t \geqslant \varepsilon \\ \frac{t}{\varepsilon} \text{ if } 0 < t < \varepsilon \\ 0 \text{ if } t \leqslant 0 \end{cases} \quad . \tag{2.1}$$

For approximating a solution of Problem 1.1 we consider the following regular problem:

2.2 REGULAR PROBLEM The data $D, g, \chi_0, b_\varepsilon$ are supposed to satisfy (1.3), (1.8), (1.9), (2.1) and

$$u_0 \in g(0, \cdot) + V \cap L^\infty(D),\ u_0 > 0 \text{ in } D_0,\ u_0 = 0 \text{ in } D \backslash D_0. \tag{2.2}$$

Then find a function u_ε such that we have

$$u_\varepsilon \in g + L^\infty_{\text{loc}}(0, \infty; \text{V}),\ \partial_t b_\varepsilon(u_\varepsilon) \in L^2_{\text{loc}}(0, \infty; L^2(D)); \tag{2.3}$$

$$u_\varepsilon \geqslant 0 \text{ in } D_\infty; \tag{2.4}$$

$$\int_D \partial_t b_\varepsilon(u_\varepsilon)\, v + \int_D (\nabla u_\varepsilon + e b_\varepsilon(u_\varepsilon))\, \nabla v = 0 \text{ for all } v \in V; \tag{2.5}$$

$$b_\varepsilon(u_\varepsilon(0, \cdot)) = b_\varepsilon(u_0) \text{ on } \{0\} \times D. \tag{2.6}$$

2.3 THEOREM. *There exists for any* $\varepsilon > 0$ *one and only one solution* u_ε *of Problem* 2.2.

Proof. See Alt and Luckhaus [6] 2.2 and 2.3. ■

If ε tends to zero, the solutions u_ε of Problem 2.2 converge to a solution of Problem 1.1. This is the assertion of

2.4. THEOREM. *Suppose* (1.3), (1.6), (1.8), (1.9), (1.10), (2.1) *and* (2.2). *Then there exists a subsequence* u_ε *of solutions of the regular problem* 2.2 *such that* u_ε *converges weakly in* $L^2_{loc}(0, \infty; V)$ *to* u *and* $b_\varepsilon(u_\varepsilon)$ *converges weakly in* $L^p_{loc}(D_\infty)$ to γ, $1 \leqslant p < \infty$ *where* $\{u, \gamma\}$ *is a solution of the non-stationary problem* 1.1.

Proof: See Lemma 6.3. ■

Now we can formulate the main result of this paper.

2.5. THEOREM. *Suppose* (1.3), (1.6), (1.8), (1.9), (1.10), (1.16), (2.1) *and* (2.2). *Let* $\{u, \gamma\}$ *be the solution of Problem* 1.1 *which we get in Theorem* 2.4. *Let* $\{u_\infty, \gamma_\infty)$ *be the solution of the stationary problem* 1.2. *Then we have*

$$u(t) \to u_\infty, \; \gamma(t) \to \gamma_\infty \text{ for } t \to \infty$$

strongly in $L^p(D)$ *for all* $1 \leqslant p < \infty$.

Proof: See 6.5. ■

REMARK. If g does not depend on time the convergence $u(t) \to u_\infty, t \to \infty$ holds even weakly in $H^{1,2}(D)$.

The main idea for proving Theorem 2.5 is to construct sub- and supersolutions u^+, u^- for u, which are decreasing and increasing in t, respectively, if t tends to infinity. Actually we shall construct sub- and supersolutions $u_\varepsilon^+, u_\varepsilon^-$ for u_ε, i.e.

$$u_\varepsilon^- \leqslant u_\varepsilon \leqslant u_\varepsilon^+ \text{ in } D_\infty.$$

This can be done if we solve the regular problem 2.2 with respect to the boundary values $F^\pm(t, z) = G(z) \pm \varphi^\pm(t)$ where G is the asymptotic limit $(t \to \infty)$ of $g(t, \cdot)$ (see (1.10)) and $\varphi^\pm$ satisfy

$$\begin{gathered} F^- \leqslant g \leqslant F^+ \text{ on }]0, \infty[\times D, \\ \varphi^+ \, (\varphi^-) \text{ monotone decreasing (increasing) in } t, \\ \varphi(t) \to 0 \text{ if } t \to \infty. \end{gathered} \tag{2.7}$$

Since we have to estimate the measure of the sets $\{z \in D | 0 < G(z) < \varepsilon\}$ in terms of ε it turns out to be successful to replace G by a harmonic function with the same boundary values on Γ_0 as G. The existence of a suitable harmonic function with some additional properties will be proved in §3. The existence of sub- and supersolutions with the desired behaviour for $t \to \infty$ is the subject of §4. The most important step to control the dependence of $u_\varepsilon^\pm(t, z)$ on ε and t is to verify the estimates

$$\int_D |\nabla u_\varepsilon^\pm(t)|^2 \leqslant \text{const.}$$

uniformly in ε and t. This will be proved in §5. In §6 we intend to study the convergence of $u_\varepsilon^\pm, u_\varepsilon$ if $\varepsilon \to 0$ and $t \to \infty$. We shall show that

$$u_\varepsilon^- \to u^-, u_\varepsilon \to u, u_\varepsilon^+ \to u^+ \text{ for } \varepsilon \to 0$$

(with respect to a suitable topology) where u^-, u, u^+ are solutions of Problem 1.1 with respect to the boundary value F^-, g and F^+ and appropriate initial values and we shall show

$$u^- \leqslant u \leqslant u^+ \text{ in } D_\infty.$$

Then we consider the behaviour of $u^-(t, \cdot), u^+(t, \cdot)$ if $t \to \infty$. We shall obtain that

$$u^-(t, \cdot) \to u_\infty^-, \; u^+(t, \cdot) \to u_\infty^+ \text{ for } t \to \infty$$

(with respect to a suitable topology) where u_∞^-, u_∞^+ are solutions of the stationary problem. Since we assume that this problem is uniquely solvable (see (1.16)) we have

$$u_\infty^- = u_\infty^+ = \lim_{t \to \infty} u(t, \cdot).$$

This argument will prove Theorem 2.5.

3. Harmonic extension of the boundary values

In this section we establish some properties of a particular harmonic extension of the boundary values of G on Γ_0 into the interior of D. The details are given in the following lemma. First let us give some notations.

$$\begin{aligned} x_1 &:= \sup\{x < x_0 | G(x, y) > 0, (x, y) \in \Gamma_0\}, \\ x_2 &:= \inf\{x > x_0 | G(x, y) > 0, (x, y) \in \Gamma_0\}, \\ z_1 &:= (x_1, Y_1), z_2 := (x_2, Y_2), \end{aligned} \tag{3.1}$$

where $P = (x_0, y_0)$ is the top of the dam (see (1.3)) and y_1, y_2 are defined in (1.6). Furthermore we shall use $z = (x, y)$ and

$$S_0 := \{z \in \Gamma_0 | x < x_1\}; S_1 := \{z \in \Gamma_0 | x_1 < x < x_2\}, S_2 := \{z \in \Gamma_0 | x_2 < x\}. \tag{3.2}$$

3.1 LEMMA. *Assume* (1.3), (1.10). *Then there exists a solution* $H \in G + V$ of

$$\int_D (\nabla H + e) \nabla v = 0 \text{ for all } v \in V. \tag{3.3}$$

For H we have the following properties:

$$H \in L^\infty(D) \text{ and } H > 0 \text{ in } D. \tag{3.4}$$

There exist ϱ_0, ε_0 *such that for* $i=1,2$:

$$|\nabla H(z)| = O(|\log|z-z_i||) \text{ for all } z \in B_{\varrho_0}(z_i) \cap D; \tag{3.5}$$

$$\partial_y H \leqslant -k < 0 \text{ on } D^0_{\varrho_0} := \{z \in D | \operatorname{dist}(z, S_1) \leqslant \varrho_0\} \text{ for some } k > 0; \tag{3.6}$$

$$\operatorname{meas}\left(\{z \in D | a \leqslant H(z) \leqslant \varepsilon + a\} \cap D^1_{\varrho_0}\right) \leqslant k_{1\varepsilon} \text{ for } a \in \mathbf{R}, 0 < \varepsilon \leqslant \varepsilon_0, \tag{3.7}$$

where k_1 *is independent of* a;

$$\{z \in D | 0 \leqslant H \leqslant 2\varepsilon\} \subset D^1_{\varrho_0}; \tag{3.8}$$

$$\int_{\{a \leqslant H \leqslant \varepsilon + a\} \cap D^1_{\varrho_0}} |\nabla H|^2 = O(\varepsilon),\ 0 < \varepsilon \leqslant \varepsilon_0. \tag{3.9}$$

Proof of (3.3), (3.4): The existence of a solution H in $(G+V) \cap L^\infty(D)$ follows for example as a special case from Theorem 3 in [14]. In order to show $H > 0$ notice that $H^- = \min\{0, H\} \in V$ and test (3.3) with H^-. We obtain:

$$\int_D \nabla H \, \nabla H^- = -\int_{\Gamma_1} evH^- \leqslant 0$$

(see (1.3)). But this implies $H \geqslant 0$ in D and using the classical maximum principle for $\Delta H = 0$ in D we obtain $H > 0$ in D. ■

Proof of (3.5): We shall give the proof for $i=1$. For $i=2$ we can use the same arguments. For simplicity we can assume without loss of generality that the inner normal $-\nu(z_1)$ in $z_1 \in \Gamma_0$ points into the direction of the positive x-axis, that Γ_0 near z_1 is given locally as the graph of a function $\psi \in C^2([-\varrho_0, \varrho_0])$ and $z_1 = 0$. Now we straighten Γ_0 in a neighbourhood of z_1 and define for $(x, y) \in K^+_{\varrho_0} := \{(x, y) | 0 < x < \varrho_0, -\varrho_0 < y < \varrho_0\}$

$$\omega(x, y) := H(x + \psi(y), y).$$

Later on ϱ_0 has to be chosen small enough. The function ω satisfies the following boundary value problem:

$$L\omega := (1 + \psi'^2)\,\partial_x^2 \omega + \partial_y^2 \omega - 2\psi' \,\partial_x \partial_y \omega - \psi'' \,\partial_x \omega = 0 \text{ in } K^+_{\varrho_0},$$
$$\omega(0, y) = 0 \text{ for } 0 < y < \varrho_0,\ \omega(0, y) = -y \text{ for } -\varrho_0 < y < 0.$$

Now we would like to subtract a harmonic function η from ω, with the same boundary values as ω, on $\{x = 0\}$. For this purpose we take for $0 \leqslant \tau \leqslant \varrho_0$, $-\frac{\pi}{2} \leqslant \varphi \leqslant \frac{\pi}{2}$ (we use $z = (\tau, \varphi) = (x, y)$):

$$\eta(z) = -\frac{1}{\pi} \operatorname{Im}(iz \log(-iz)) = \frac{\tau}{\pi}\left[\left(\varphi - \frac{\pi}{2}\right) \sin\varphi - \cos\varphi \log\tau\right]. \tag{3.10}$$

η is harmonic in $K^+_{\varrho_0}$ and on $\{x = 0\}$ η has the same boundary values as ω, $\eta \geqslant 0$ in $K^+_{\varrho_0}$ if ϱ_0 is small enough since

$$\eta(z) = \frac{\tau}{\pi} \cos\varphi \left[\frac{\varphi - \frac{\pi}{2}}{\sin\left(\varphi - \frac{\pi}{2}\right)} \sin\varphi - \log\tau \right] \tag{3.11}$$

and the first term in the brackets is bounded. For the gradient of η we obtain:

$$\nabla\eta(z) = -\frac{1}{\pi}(1+\log\tau)\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\pi}\left(\varphi - \frac{\pi}{2}\right)\begin{pmatrix}0\\1\end{pmatrix} \tag{3.12}$$

and we shall prove that $\nabla\omega$ behaves like $\nabla\eta$. This can be seen as follows: For the difference $v := \omega - \eta$ we get

$$\begin{aligned} Lv &= \psi'^2\, \partial_x^2\, \eta - 2\psi'\, \partial_x\, \partial_y\, \eta - \psi''\, \partial_x\, \eta = \\ &= \frac{1}{\pi}\left[-\frac{\psi'^2}{\tau} \cos\varphi + 2\frac{\psi'}{\tau} \sin\varphi + (1+\log\tau)\,\psi'' \right] \text{ in } K^+_{\varrho_0}, \\ &\qquad \omega - \eta = 0 \text{ on } \{x = 0\}. \end{aligned} \tag{3.13}$$

Since $\psi(0) = \psi'(0) = 0$ and $\psi \in C^2([-\varrho_0, \varrho_0])$ the right side in (3.13) is in $L^p(K^+_{\varrho_0})$ for all $p \geqslant 2$. Therefore we get from the theory of elliptic operators that $\omega - \eta \in H^{2,p}(K^+_{\varrho_0})$ for all $p \geqslant 2$. To apply this theory we have to smooth out the corners of $\partial K^+_{\varrho_0}$. On the boundary of this new set, the values of $\omega - \eta$ are regular enough since ω and η are smooth functions away from 0. From the Sobolev embedding theorem we obtain in particular that $\omega - \eta \in C^1(\overline{K^+_{\varrho_0}})$. This implies

$$\begin{aligned} \nabla\omega &= \nabla\eta + O(1) = O(|\log\tau|) \text{ in } K^+_{\varrho_0}, \\ \nabla H &= A\nabla\omega = A\nabla\eta + O(1) = O(|\log\tau|) \text{ in } B_{\varrho_0}(z_1) \cap D, \end{aligned} \tag{3.14}$$

where

$$A = \begin{pmatrix} 1 & , 0 \\ -\psi'(y), & 1 \end{pmatrix}. \qquad \blacksquare$$

P r o o f o f (3.6): Now we have to take into account that we neglect the rotation of the system when we assume that the inner normal $-\nu(z_1)$ in z_1 points into the interior of D. Therefore we must prove

$$\begin{pmatrix} -b \\ a \end{pmatrix} \nabla H \leqslant -k < 0 \text{ on } B_{\varrho_0}(z_1) \cap D$$

for $a = \cos\varphi_0$, $b = \sin\varphi_0 > 0$ where φ_0 is the angle of rotation and

$$\begin{pmatrix} -b \\ a \end{pmatrix} \nabla H \leqslant -k < 0 \text{ on } \{z \in D \mid \operatorname{dist}(z, S_1) \leqslant \varrho_0\} \setminus B_{\varrho_0}(z_1).$$

The first assertion follows from (3.12), (3.14) and the second if we apply Hopf's maximum principle on compact subsets of the boundary S_1. ■

Proof of (3.7): We obtain for $a \in \mathbf{R}$

$$\operatorname{meas}\left(\{z \in D | a \leqslant H(z) \leqslant a+\varepsilon\} \cap D^1_{\varrho_0}\right) =$$
$$= \operatorname{meas}\left(\{z \in D | z = p - \lambda \nu(p), p \in \Gamma_0, 0 \leqslant \lambda \leqslant \varrho_0, a \leqslant f_p(\lambda) \leqslant a+\varepsilon\}\right)$$

where $f_p(\lambda) = H(p - \lambda \nu(p))$. Similarly to (3.6) we can prove that for all $p \in S_1$

$$f_p'(\lambda) = \partial_{-\nu} H(p - \lambda \nu(p)) > k > 0$$

is true. Therefore we can continue

$$= \operatorname{meas}\left(\{z \in D | \ldots f_p^{-1}(a) \leqslant \lambda \leqslant f_p^{-1}(a+\varepsilon)\}\right) \leqslant c\left(f_p^{-1}(a+\varepsilon) - f_p^{-1}(a)\right),$$

where c is independent of a and ε,

$$\leqslant c\,(f_p^{-1})'(\xi)\,\varepsilon, \text{ for some } a < \xi < a+\varepsilon,$$
$$= c \frac{1}{f_p'(f_p^{-1}(\xi))}\,\varepsilon \leqslant \frac{c}{k}\,\varepsilon.$$ ■

Proof of (3.8): For a fixed ϱ_0 the maximum principle applied to H in $D \setminus D^1_{\varrho_0}$ yields:

$$\varepsilon_0 := \frac{1}{4} \inf\{H(z) | z \in D \setminus D^1_{\varrho_0}\} > 0. \tag{3.15}$$

But this implies (3.8). ■

Proof of (3.9): Let us use the same assumptions concerning the geometry of D as in the proof of (3.5) and let us use the notation

$$D(a, y) := \{x | (x, y) \in D^1_{\varrho_0} | a \leqslant H(x, y) \leqslant \varepsilon + a\}$$
$$f_y(x) := H(x, y).$$

Since $f_y'(x) \geqslant k_2 > 0$ in $B_{\varrho_0}(z_1) \cap D$ (see (3.14) and (3.12)) we can define

$$b(s, y) := \begin{cases} f_y^{-1}(s) & \text{if } s > 0 \\ 0 & \text{if } s \leqslant 0. \end{cases}$$

Therefore

$$\int\limits_{\{a \leqslant H \leqslant \varepsilon + a\} \cap B_{\varrho_0}(z_1)} |\nabla H|^2 = \int\limits_{-\varrho_0}^{\varrho_0} \int\limits_{D(a,y)} |\nabla H|^2\, dx\, dy = \int\limits_{-\varrho_0}^{\varrho_0} \int\limits_{b(a,y)}^{b(a+\varepsilon,y)} |\nabla H|^2\, dx\, dy \leqslant$$
$$\leqslant c \int\limits_{-\varrho_0}^{\varrho_0} \int\limits_{b(a,y)}^{b(a+\varepsilon,y)} |\log(x^2+y^2)|^2\, dx\, dy \leqslant c \int\limits_{-\varrho_0}^{\varrho_0} \left(|\log|y|| \int\limits_{b(a,y)}^{b(a+\varepsilon,y)} 1 dx\right) dy$$ ■

Now we have

$$|b(a+\varepsilon, y)-b(a, y)| \leqslant \sup_{\zeta\in D(a,y)} (f_y^{-1})'(\zeta)\,(a+\varepsilon-\max(0, a)) \leqslant \frac{\varepsilon}{k_2}$$

and this proves (3.9) on $\{a \leqslant H \leqslant \varepsilon+a\} \cap B_{\varrho_0}(^z_1)$. On $\{a \leqslant H \leqslant \varepsilon+a\} \cap D^1_{\varrho_0} \backslash B_{\varrho_0}$ it follows from (3.7) and the boundedness of ∇H on his set. ■

4. Sub- and supersolutions

The purpose of this part is to establish the sub- and supersolution $u_\varepsilon^\pm$ with respect to u_ε. $u_\varepsilon^\pm$ are expected to be monotone decreasing and monotone increasing in t, respectively, if t tends to infinity. As boundary and initial values for the sub- and supersolution we choose

$$\begin{gathered} F^\pm(t, z) := H(z) \pm \varphi^\pm(t) \quad (t, z) \in [0, \infty[\times D, \\ u_0^+(z) := H(z) + \varphi^+(0) \quad z \in D, \end{gathered} \tag{4.1}$$

$$u_0^- \text{ solves: } \int_D \nabla u_0^- \,\nabla v = 0 \text{ for all } v \in V, u_0^- \in F^-(0, \cdot) + V,$$

such that φ satisfies

$$\begin{gathered} H(z) - \varphi^-(t) \leqslant g(t, z) \leqslant H(z) + \varphi^+(t) \text{ on }]0, \infty[\times \Gamma_0, \\ H(z) - \varphi^-(0) \leqslant u_0(z) \leqslant H(z) + \varphi^+(0) \text{ on } D. \end{gathered} \tag{4.2}$$

and

$$\begin{gathered} \|\varphi^\pm\|_{C^2([0,\infty[)} \text{ bounded; } \varphi^\pm > 0,\ (\varphi^\pm)' \leqslant 0;\ \varphi^\pm(t) \to 0 \text{ if } t \to \infty; \\ (\varphi^+)' \geqslant -k \text{ on } [0, \infty[\text{ where } k \text{ is given in (3.6);} \\ (\varphi^-)' \leqslant -K,\ K := \sup\{\partial_y H(z) | z \in D \backslash D^1_{\varrho_0}\} \text{ for } 0 < t < t_0 \\ \text{and } \varphi^-(t) \leqslant \varepsilon_0 \text{ for } t \geqslant t_0. \end{gathered} \tag{4.3}$$

This can be obtained for example if we choose $\varphi^+(0)$ and $\varphi^-(0)$ large enough. Then the sub- (super-) solutions $u_\varepsilon^\pm$ are defined as follows.

4.1 Definition of $u_\varepsilon^\pm$ $u_\varepsilon^\pm$ are supposed to fulfill the following conditions:

$$\begin{gathered} u_\varepsilon^\pm \in F^\pm + L^\infty_{\text{loc}}(0, \infty; V), \quad \partial_t b_\varepsilon(u_\varepsilon^\pm) \in L^2_{\text{loc}}(0, \infty; L^2(D)); \\ \int_D \partial_t b_\varepsilon(u_\varepsilon^\pm)\, v + \int_D (\nabla u_\varepsilon^\pm + e b_\varepsilon(u_\varepsilon^\pm)) \nabla v = 0 \text{ for all } v \in V; \\ b_\varepsilon(u_\varepsilon^\pm(0, \cdot)) = b_\varepsilon(u_0^\pm) \text{ on } D. \end{gathered} \tag{4.4}$$

Remark. For existence and uniqueness we use again [6]. Theorem 2.3 and 2.4.

For comparing u_ε^-, u_ε and u_ε^+ we need the following comparison result for sub- and supersolutions.

4.2 DEFINITION. Let u_ε be a solution of the regular problem 2.2 with respect to the boundary values g and initial values u_0. We call $\omega \in L^2(0, T; H^{1,2}(D))$ a subsolution (supersolution) for u_ε if $\omega \leqslant (\geqslant)\, g$ on Γ_0, $b_\varepsilon(0, \cdot)) \leqslant (\geqslant)\, b_\varepsilon(u_0)$ and

$$\int_D \partial_t b_\varepsilon(\omega)\, v + \int_D (\nabla\omega + e b_\varepsilon(\omega))\nabla v \leqslant 0 \;(\geqslant 0)$$

for all $v \in V$ with $v \geqslant 0$.

4.3 THEOREM. *If u^- is a subsolution and u^+ a supersolution for u such that $\partial_t(b_\varepsilon(u^-) - b_\varepsilon(u^+))$ is in $L^1(D_T)$ then $u^- \leqslant u \leqslant u^+$ a.e. on D_T.*

Proof. See [6]. Theorem 2.2.

Now we are able to prove that u_ε^-, u_ε^+ are sub- and supersolutions for u_ε.

4.4. THEOREM. *Assume* (1.3), (1.8), (1.9), (1.10), (2.1) and (2.2). *Let u_ε be the solution of the regular problem 2.2 with boundary values g and initial values u_0. Let $u_\varepsilon^\pm$ be defined as in* 4.1. *Then we obtain for a fixed sequence $\varepsilon = \varepsilon_n \to 0$*

$$u_\varepsilon^- \leqslant u_\varepsilon \leqslant u_\varepsilon^+,\; u_0^- \leqslant u_\varepsilon^- \leqslant F^-,\; 0 \leqslant u_\varepsilon^+ \leqslant F^+, \text{ a.e. in } D_\infty. \tag{4.5}$$

REMARK. In the sequel we always denote this sequence by u_ε and use the same symbol if we select subsequences.

Proof. For the initial and boundary values the corresponding inequalities are satisfied (see (4.1), (4.2), (4.3)); in particular we have $u_0^- \leqslant F^-(0, \cdot)$, since both functions are harmonic in D, the boundary values on Γ_0 are the same and on Γ_1 we have

$$\partial_\nu u_0^- = 0,\; \partial_\nu F^-(0, \cdot) > 0.$$

For u_ε and $u_\varepsilon^\pm$ we have by definition

$$\int_D \partial_t b_\varepsilon(u_\varepsilon^\pm)\, v + \int_D (\nabla u_\varepsilon^\pm + e b_\varepsilon(u_\varepsilon^\pm))\nabla v = 0$$

for all $v \in V$. Using Theorem 4.3 this yields $u_\varepsilon^- \leqslant u_\varepsilon \leqslant u_\varepsilon^+$.
For F^+ we get for $v \in V$, $v \geqslant 0$

$$\int_D \partial_t b_\varepsilon(F^+)\, v + \int_D (\nabla F^+ + e b_\varepsilon(F^+))\nabla v =$$

$$= \int_D b_\varepsilon'(F^+)(\varphi^+)'\, v + \int_D (\nabla H + e)\nabla v + \int_D (e b_\varepsilon(F^+) - e)\nabla v =$$

$$= \int_D b_\varepsilon'(F^+)(\varphi^+)'\, v + \int_a^b (b_\varepsilon(F^+)-1)\, v \Big|_{\psi_1(x)}^{\psi_0(x)} - \int_D b_\varepsilon'(F^+)\, \partial_y H v \geqslant$$

$$\geqslant \int_D b_\varepsilon'(F^+)\, v\, (\varphi^{+\prime} - \partial_y H) =$$

$$= \frac{1}{\varepsilon} \int_{\{0 \leqslant F^+ \leqslant \varepsilon\} \cap D^1_{\varrho_0}} v\, (\varphi^{+\prime} - \partial_y H) + \frac{1}{\varepsilon} \int_{\{0 \leqslant F^+ \leqslant \varepsilon\} \setminus D^1_{\varrho_0}} v\, (\varphi^{+\prime} - \partial_y H).$$

Remember that $\{0 \leqslant F^+ \leqslant \varepsilon\} \subset \{0 \leqslant H \leqslant \varepsilon\}$ by definition of F^+ in (4.1). Then (3.8) implies that the domain of integration of the second integral is empty. Therefore we can continue:

$$= \frac{1}{\varepsilon} \int_{\{0 \leqslant F^+ \leqslant \varepsilon\}} v\, (\varphi^{+\prime} - \partial_y H) \geqslant 0,$$

since on $D^1_{\varrho_0}$ we have $\partial_y H \leqslant -k < 0$ and $\varphi^{+\prime} \geqslant -k$ (see (3.6) and (4.3)). Applying the comparison theorem 4.3 we obtain

$$F^+ \geqslant u_\varepsilon^+ \text{ a.e. in } [0, \infty[\times D.$$

$u_\varepsilon^+ \geqslant 0$ is obvious. For F^- we get for $v \in V$, $v \geqslant 0$

$$\int_D \partial_t\, b_\varepsilon(F^-)\, v + \int_D (\nabla F^- + e b_\varepsilon(F^-))\, \nabla v \geqslant$$

$$\geqslant \frac{1}{\varepsilon} \int_{\{0 \leqslant F^- \leqslant \varepsilon\} \cap D^1_{\varrho_0}} v\, ((-\varphi^-)' - \partial_y H) + \frac{1}{\varepsilon} \int_{\{0 \leqslant F^- \leqslant \varepsilon\} \cap D^1_{\varrho_0}} v\, ((-\varphi^-)' - \partial_y H).$$

The first integral is nonnegative since $(\varphi^-)' \leqslant 0, v \geqslant 0$ and $\partial_y H \leqslant 0$ in $D^1_{\varrho_0}$. For the second integral we distinguish between the cases $t < t_0$ and $t \geqslant t_0$ where t_0 is defined in (4.3). If $t < t_0$ we get $(\varphi^-)' \leqslant -K$ and therefore

$$\int_{\{0 \leqslant F^- \leqslant \varepsilon\} \setminus D^1_{\varrho_0}} v\, (-(\varphi^-)' - \partial_y H) \geqslant \int_{\{0 \leqslant F^- \leqslant \varepsilon\} \setminus D^1_{\varrho_0}} v\, (K - \partial_y H) \geqslant 0.$$

If $t \geqslant t_0$ we obtain

$$\{(t,z) | 0 \leqslant F^-(t,z) \leqslant \varepsilon\} \setminus D^1_{\varrho_0} = \{0 \leqslant H(z) - \varphi^-(t) \leqslant \varepsilon\} \setminus D^1_{\varrho_0} =$$

$$= \left\{0 \leqslant \frac{H}{2} + \left(\frac{H}{2} - \varphi^-(t)\right) \leqslant \varepsilon\right\} \setminus D^1_{\varrho_0} \subset$$

$$\subset \left\{0 \leqslant \frac{H}{2} \leqslant \varepsilon\right\} \setminus D^1_{\varrho_0} = 0 \text{ (see (3.15), (4.3))},$$

since $\{0 \leqslant H \leqslant 2\varepsilon\} \subset D^1_{\varrho_0}$ for $0 < \varepsilon \leqslant \varepsilon_0$ and ε_0 is defined in (3.15). Therefore F^- is a supersolution of u_ε^-, i.e. $u_\varepsilon^- \leqslant F^-$.

It remains to show $u_0^- \leqslant u_\varepsilon^-$. Without restriction we can assume $u_0^- < 0$ in D (choose $\varphi^-(0)$ large enough). Then for all $v \in V$

$$\int_D \partial_t b_\varepsilon(u_0^-)\, v + \int_D (\nabla u_0^- + e b_\varepsilon(u_0^-))\, \nabla v = 0.$$

Since $u_0^- \leqslant u_\varepsilon^-$ on Σ_0 and $b(u_0^-) = b(u_\varepsilon^-)$ on $\{0\} \times D$ using the comparison theorem we obtain $u_0^- \leqslant u_\varepsilon^-$. ■

4.5 Lemma. *Under the conditions of Theorem 4.4 u_ε^- and u_ε^+ are monotone non decreasing and monotone non increasing in t, respectively.*

Remark. This means that for any $0 < \varepsilon \leqslant \varepsilon_0$ there exists a set $N_\varepsilon \subset \mathbf{R}^+$, meas $(N_\varepsilon) = 0$ such that for all $s, t \in \mathbf{R}_+ \backslash N_\varepsilon$, $s \leqslant t$ and almost all $x \in D$

$$u_\varepsilon^-(s, x) \leqslant u_\varepsilon^-(t, x),$$
$$u_\varepsilon^+(s, x) \geqslant u_\varepsilon^+(t, x).$$

Proof: Let us show the statement concerning u_ε^- for a fixed ε. For $u_\alpha(t, z) := u_\varepsilon^-(t+\alpha, z)$, $\alpha \geqslant 0$ we have

$$\int_D \partial_t b_\varepsilon(u_\alpha(t))\, v + \int_D (\nabla u_\alpha(t) + e b_\varepsilon(u_\alpha(t)))\, v = 0$$

for all $v \in V$ and a.a. $t \in \mathbf{R}^+$; and

$$u_\alpha(t, \cdot) = F^-(t+\alpha, \cdot) \geqslant F^-(t, \cdot) = u_\varepsilon^-(t, \cdot) \text{ on } \Gamma_0 \text{ for a.a. } t \in \mathbf{R}^+,$$
$$b_\varepsilon(u_\alpha(0, \cdot)) = b_\varepsilon(u_\varepsilon^-(\alpha, \cdot)) \geqslant b_\varepsilon(u_0^-) = b_\varepsilon^-(u_\varepsilon^-(0, \cdot)) \text{ on } D \quad \text{(see Theorem 4.4).} \tag{4.6}$$

Using the comparison theorem 4.3 we obtain

$$u_\varepsilon^-(t+\alpha, z) \geqslant u_\varepsilon^-(t, z) \text{ for all } t \in \mathbf{R}^+ \backslash N_{\varepsilon,\alpha} \text{ and a.a. } z \in D,$$

where meas $(N_{\varepsilon,\alpha}) = 0$.

Now an easy consideration shows us that this implies the statement of the lemma. Keep ε fixed and let (α_n) be a sequence of positive real numbers converging to zero such that (4.6) is fulfilled and define

$$N_\varepsilon := \bigcup_{n=1}^{\infty} N_{\varepsilon,\alpha_n} \text{ and}$$
$$M := \{s \in \mathbf{R}^+ \mid s = t + \alpha_n,\ n \in \mathbf{N},\ t \in \mathbf{R}_0^+ \backslash N_\varepsilon\}.$$

It can be easily shown that meas $(\mathbf{R}^+ \backslash M) = 0$ and therefore the statement holds for u_ε^-. The proof for u_ε^+ is similar. Notice that for the initial conditions we have

$$b_\varepsilon(u_\varepsilon^+(\alpha)) \leqslant b_\varepsilon(F^+(\alpha)) \leqslant b_\varepsilon(F^+(0)) = b_\varepsilon(u_\varepsilon^+(0)).$$ ■

5. $L^\infty(H^1)$-estimates for the sub- and supersolution

In this section we prove the $L^\infty(0,\infty; H^1(D))$—estimates for u_ε and for the sub- and supersolution $u_\varepsilon^\pm$ which are defined in 4.1. We need the following notations:

$$\partial_t^h \omega(t,z) := \frac{1}{h}(\omega(t+h,z)-\omega(t,z)),$$

$$\omega_h(t,z) := \frac{1}{h}\int_t^{t+h} \omega(s,z)\,ds,$$

$$\omega_h^-(t,z) := \frac{1}{h}\int_{t-h}^{t} \omega(s,z)\,ds,$$

$$B_\varepsilon^h(\omega,s,z) := \frac{b_\varepsilon(\omega(s+h,z))-b_\varepsilon(\omega(s,z))}{\omega(s+h,z)-\omega(s,z)}.$$

If the formulations of the following statements will be the same for $u_\varepsilon^+, u_\varepsilon^-$ we shall write ω instead of u_ε^+ and u_ε^- and F instead of F^+ and F^-, respectively.

5.1 Theorem. *Assume* (1.3), (1.8), (1.9), (1.10), (2.1) and (2.2), *let u_ε be the solution of Problem* 2.2 *and $u_\varepsilon^\pm$ be as in Definition* 4.1. *Then there exist constants $\varepsilon_0, h_0, t_1, C_0 > 0$ such that we have for all $0<\varepsilon\leqslant\varepsilon_0$, $0<|h|\leqslant h_0$ and $t\geqslant t_1$:*

$$\int_t^{t+1}\int_D B_\varepsilon^h(\omega,s,z)\,\partial_t^h\omega(s,z)^2\,dz\,ds+\int_D|\nabla\omega(t,z)|^2\,dz\leqslant C_0 \tag{5.1}$$

for $\omega=u_\varepsilon^+, u_\varepsilon^-$.

Remark. The first term in (5.1) can be estimated by $\frac{c_0}{h}$ for all $t\in\mathbf{R}^+$.

Proof: The basic idea for proving this theorem is to use $\partial_t^h\omega-\partial_t^h F$ as test function in (4.4). For $h,\tau\in\mathbf{R}^+$, $\eta\in L^2(0,\tau;V)$ get from (4.4)

$$\int_{D_\tau}\left[(\partial_t b_\varepsilon(\omega))_h\,\eta+\left(\nabla\omega_h+e\,(b_\varepsilon(\omega))_h\right)\nabla\eta\right]=0. \tag{5.2}$$

Now we take $\partial_t^h\omega-\partial_t^h F$ as test function in (5.2). For fixed t, $t\leqslant\sigma\leqslant\tau\leqslant t+1-h$ and $D':=\,]\sigma,\tau[\times D$ we obtain

$$\int_{D'}\partial_t^h b_\varepsilon(\omega)\,\partial_t^h\omega+\int_{D'}\left(\nabla\omega_h+e\,(b_\varepsilon(\omega))_h\right)\nabla\left(\partial_t^h\omega+e\partial_t^h b_\varepsilon(\omega)\right)=$$
$$=\int_{D'}\partial_t^h b_\varepsilon(\omega)\,\partial_t^h F+\int_{D'}\left(\nabla\omega_h+e\,(b_\varepsilon(\omega))_h\right)e\partial_t^h b_\varepsilon(\omega).$$

Since $\omega(t,z)$ is defined on $]0,\infty[\times D$ we can choose $t\leqslant\sigma\leqslant\tau\leqslant t+1$.

We continue

$$\int\limits_{D'} B_\varepsilon^h(\omega, s)\left(\partial_t^h \omega(s)\right)^2 dz\, ds + \frac{1}{2}\int\limits_D \left[\nabla\omega_h(t) + e\left(b_\varepsilon(\omega(t))\right)_h\right]^2 \Bigg|_\sigma^\tau dz \leqslant$$

$$\leqslant -\int\limits_{D'} \left(b_\varepsilon(\omega(s))\right)_h \partial_t^2 F_h(s)\, dz\, ds + \int\limits_D \left[\left(b_\varepsilon(\omega(t))\right)_h \partial_t^h F(t)\right]_\sigma^\tau dz +$$

$$+ \int\limits_{D'} |\nabla\omega_h(s)|\, |B_\varepsilon^h(\omega, s)|\, |\partial_t^h \omega(s)|\, dz\, ds + \frac{1}{2}\int\limits_D \left[\left(b_\varepsilon(\omega(t))\right)_h\right]^2 \Bigg|_\sigma^\tau \leqslant$$

$$\leqslant C + \frac{1}{2}\int\limits_{D'} B_\varepsilon^h(\omega, s)\, |\nabla\omega_h(s)|^2\, dz\, ds + \frac{1}{2}\int\limits_{D'} B_\varepsilon^h(\omega, s)\, |\partial_t^h \omega(s)|^2\, dz\, ds \tag{5.3}$$

where C is independent of ε, t and h. We introduce the following notations:

$$D_{\varepsilon h}(t) := \int\limits_t^{t+1} \int\limits_D B_\varepsilon^h(\omega, s)\, |\partial_t^h \omega(s)|^2\, dz\, ds,$$

$$A_{\varepsilon h}(t) := \int\limits_D \left[\nabla\omega_h(t) + e\left(b_\varepsilon(\omega(t))\right)_h\right]^2 dz,$$

$$R_{\varepsilon h}(t) := \int\limits_t^{t+1} \int\limits_D |\nabla\omega_h(s)|^2\, B_\varepsilon^h(\omega, s)\, dz\, ds.$$

$R_{\varepsilon h}$ can be estimated as follows.

$$R_{\varepsilon h}^{\pm} \leqslant 2\omega_\varepsilon^t(h) + 2M_\varepsilon^t(h) \text{ where}$$

$$\omega_\varepsilon^t(h) := \int\limits_t^{t+1} |\nabla\omega_h(s) - \nabla\omega(s)|^2\, B_\varepsilon^h(\omega, s)\, dz\, ds, \tag{5.4}$$

$$M_\varepsilon^t(h) := \int\limits_t^{t+1} \int\limits_D |\nabla\omega(s)|^2\, B_\varepsilon^h(\omega, s)\, dz\, ds.$$

Combining (5.3) and (5.4) we get for $t \leqslant \sigma \leqslant \tau \leqslant t+1$

$$\frac{1}{2} A_{\varepsilon h}(\tau) \leqslant C + \frac{1}{2} A_{\varepsilon h}(\sigma) + \omega_\varepsilon^t(h) + M_\varepsilon^t(h),$$

$$D_{\varepsilon h}(t) + \frac{1}{2} A_{\varepsilon h}(t+1) - \frac{1}{2} A_{\varepsilon h}(t) \leqslant C + \frac{1}{2} D_{\varepsilon h}(t) + \omega_\varepsilon^t(h) + M_\varepsilon^t(h) \tag{5.5}$$

where C is independent of ε, t and h.

Hence for fixed ε and t and for $h \to 0$ we obtain

$$\omega_\varepsilon^t(h) \to 0,$$

$$M_\varepsilon^t(h) \to \int\limits_t^{t+1} \int\limits_D |\nabla\omega(s)|^2\, b_\varepsilon'(\omega(s))\, dz\, ds. \tag{5.6}$$

The statements are true since $\nabla\omega_h \to \nabla\omega$ in $L^2(D_{t,t+1})$, $|B_\varepsilon^h(\omega, s)| \leqslant \frac{1}{\varepsilon}$ and since $t \to \omega(t, z)$ is monotone, ε fixed. Now we notice that for any $\varepsilon, h, t \in \mathbf{R}^+$ there exists $t^*_{\varepsilon h} \in]t, t+1[$

$$A_{\varepsilon h}(t^*_{\varepsilon h}) \leqslant \int_t^{t+1} A_{\varepsilon h}(s)\, ds. \tag{5.7}$$

If we choose $\sigma = t^*_{\varepsilon h}$ in the first inequality of (5.5) we obtain for $t \leqslant t^*_{\varepsilon h} < t+1$, $t^*_{\varepsilon h} \leqslant \tau \leqslant t+1$:

$$\frac{1}{2} A_{\varepsilon h}(\tau) \leqslant C + \frac{1}{2} A_{\varepsilon h}(t^*_{\varepsilon h}) + \frac{1}{2} D_{\varepsilon h}(t) + \omega^t_\varepsilon(h) + M^t_\varepsilon(h) \leqslant$$

$$\leqslant C + \frac{1}{2} A_{\varepsilon h}(t^*_{\varepsilon h}) + \frac{1}{2} A_{\varepsilon h}(t) - \frac{1}{2} A_{\varepsilon h}(t+1) + 2\left(\omega^t_\varepsilon(h) + M^t_\varepsilon(h)\right).$$

For $\tau = t+1$ this means using (5.5) again

$$A_{\varepsilon h}(t+1) \leqslant C + \int_t^{t+1} A_{\varepsilon h}(s)\, ds + A_{\varepsilon h}(t) - A_{\varepsilon h}(t+1) + 4\left(\omega^t_\varepsilon(h) + M^t_\varepsilon(h)\right). \tag{5.8}$$

Now let us apply

5.2 Lemma. *Under the assumptions of Theorem 5.1 there are constants $t_1, C_1 > 0$ such that we have for all $t \geqslant t_1$ and $0 < \varepsilon \leqslant \varepsilon_0$:*

$$\int_D b_\varepsilon(\omega(s))^2 \Big|_{s=t}^{s=t+1} + \int_t^{t+1} \int_D b'_\varepsilon(\omega(s))\, |\nabla\omega(s)|^2\, dz\, ds \leqslant C_1$$

where $\omega = u^\pm_\varepsilon$.

For the proof see 5.4. This means that $M^t_\varepsilon(h)$ converges for $h \to 0$ to some value which is bounded by C_1 independent of ε and t.
Therefore using (5.6), (5.8) and Lemma 5.2:

$$\limsup_{h\to 0} 2A_{\varepsilon h}(t+1) \leqslant \limsup_h \int_t^{t+1} A_{\varepsilon h}(s)\, ds + \limsup_h A_{\varepsilon h}(t) + C.$$

If we set

$$A_\varepsilon(t) := \int_D \left(\nabla\omega(t) + e b_\varepsilon(\omega(t))\right)^2 \tag{5.9}$$

we get for t, ε fixed and $h \to 0$

$$\int_t^{t+1} |A_{\varepsilon h}(s) - A_\varepsilon(s)|\, ds \to 0 \tag{5.10}$$

and hence

$$2 \limsup_h A_{\varepsilon h}(t+1) \leqslant \int_t^{t+1} A_\varepsilon(s)\, ds + \limsup_h A_{\varepsilon h}(t) + C.$$

Now we need the following lemma.

5.3 Lemma. *Under the assumptions of Theorem* 5.1 *there is a constant* $C_2 > 0$ *such that*

$$\int_t^{t+1} A_\varepsilon(s)\,ds \leqslant C_2$$

uniformly for all $t \in \mathbf{R}^+$ *and* ε *sufficiently small. The statement holds for* $\omega = u, u_\varepsilon^\pm$.

For the proof see 5.5. Using this result we get

$$2 \limsup_h A_{\varepsilon h}(t+1) \leqslant C + \limsup_h A_{\varepsilon h}(t)$$

where C is independent of ε and t. In order to apply Lemma 3 in [14] we need that $\limsup_h A_{\varepsilon h}(\tau)$ is bounded locally in τ uniformly in ε, for example:

$$\limsup_h A_{\varepsilon h}(\tau) \leqslant \text{const}$$

uniformly in $\tau \in [1, 2]$ and $0 < \varepsilon \leqslant \varepsilon_0$. From (5.5) we obtain for $1 \leqslant \tau \leqslant 2$

$$A_{\varepsilon h}(\tau) \leqslant C + A_{\varepsilon h}(1) + 2\omega_\varepsilon^1(h) + 2M_\varepsilon^1(h) \tag{5.11}$$

and

$$\begin{aligned} A_{\varepsilon h}(1) &\leqslant C + A_{\varepsilon h}(0_{\varepsilon h}^*) + 2\omega_\varepsilon^0(h) + 2M_\varepsilon^0(h) \quad (0_{\varepsilon h}^* \text{ is defined in (5.7)}) \\ &\leqslant C + \int_0^1 A_{\varepsilon h}(s)\,ds + 2\omega_\varepsilon^0(h) + 2M_\varepsilon^0(h). \end{aligned}$$

Applying $\limsup_{h \to 0}$ we get

$$\limsup_h A_{\varepsilon h}(1) \leqslant C + C_2 + 2C_1$$

and applying $\limsup_{h \to 0}$ in (5.11):

$$\limsup_h A_{\varepsilon h}(\tau) \leqslant C \text{ for } 1 \leqslant \tau \leqslant 2.$$

Then Lemma 3 in [14] implies:

$$\limsup_h A_{\varepsilon h}(t+1) \leqslant C$$

where C is independent of ε and t. For fixed ε we can select a subsequence $h \to 0$ such that (see (5.10))

$$(\nabla\omega + eb_\varepsilon(\omega))_h \to \nabla\omega + eb_\varepsilon(\omega) \text{ a.e. in } D_\infty$$

and for a.a. $t \in \mathbf{R}^+$

$$\big(\nabla\omega(t,\cdot) + eb_\varepsilon(\omega(t,\cdot))\big)_h \to \nabla\omega(t,\cdot) - eb_\varepsilon(\omega(t,\cdot)) \text{ a.e. in } D.$$

It follows (Lemma of Fatou) for a.a. $t \in \mathbf{R}^+$:

$$A_\varepsilon(t) \leqslant \liminf_h A_{\varepsilon h}(t) \leqslant \limsup_h A_{\varepsilon h}(t) \leqslant C.$$

This gives the estimate in the theorem concerning $\int_D |\nabla u_\varepsilon^\pm(t)|^2$. In order to prove the estimate concerning $D_{\varepsilon h}(t)$ we use again the second inequality in (5.5) and take $\limsup_h$ of it. Since $\limsup_h \omega_\varepsilon^t(h) = 0$ and $\limsup_h M_\varepsilon^t(h) \leqslant C_1$ (see Lemma 5.2) where C_1 is independent of ε and t, the proof of Theorem 5.1 is finished. ■

5.4. Proof of Lemma 5.2. Using $b_\varepsilon(\omega) - b_\varepsilon(F)$, $\omega = u_\varepsilon^\pm$, as test function in (4.4), respectively and integrating over $]t, t+1[$ we obtain:

$$\int_t^{t+1} \int_D [\partial_t b_\varepsilon(\omega)(b_\varepsilon(\omega) - b_\varepsilon(F)) + (\nabla\omega + e b_\varepsilon(\omega)) \nabla (b_\varepsilon(\omega) - b_\varepsilon(F))] = 0,$$

which implies

$$\frac{1}{2} \int_D b_\varepsilon(\omega)^2 \Big|_t^{t+1} + \int_t^{t+1} \int_D b_\varepsilon'(\omega) |\nabla\omega|^2 \leqslant$$

$$\leqslant \int_D b_\varepsilon(\omega) b_\varepsilon(F) \Big|_t^{t+1} - \int_t^{t+1} \int_D b_\varepsilon(\omega) b_\varepsilon'(F) \varphi' + \int_t^{t+1} \int_D \nabla\omega b_\varepsilon'(F) \nabla F +$$

$$+ \int_t^{t+1} \int_D b_\varepsilon(\omega) (\partial_y b_\varepsilon(\omega) - \partial_y b_\varepsilon(F)), \; \varphi = \pm\varphi^\pm. \quad (5.13)$$

The first integral on the right side is bounded uniformly in ε and t. For the second we obtain:

$$\left| \int_t^{t+1} \int_D b_\varepsilon(\omega) b_\varepsilon'(F) \varphi' \right| \leqslant \frac{c}{\varepsilon} \int_t^{t+1} \left(\int_{\{0 \leqslant H + \varphi(s) < \varepsilon\}} 1 \, dz \right) ds \leqslant$$

$$\leqslant \frac{c}{\varepsilon} \left[\int_t^{t+1} \int_{\{0 \leqslant H + \varphi(s) < \varepsilon\} \cap D_{\varrho_0}^1} 1 + \int_t^{t+1} \int_{\{0 \leqslant H + \varphi(s) < \varepsilon\} \setminus D_{\varrho_0}^1} 1 \right]. \quad (5.14)$$

From (3.7) we have for any $0 < \varepsilon \leqslant \varepsilon_0$ and any $s \in \mathbf{R}^+$

$$\operatorname{meas}(\{0 \leqslant H + \varphi(s) < \varepsilon\} \cap D_{\varrho_0}^1) \leqslant k_1 \varepsilon,$$

where k_1 is independent of s. Therefore the first integral in (5.14) is of order $O(\varepsilon)$. For estimating the second one in (5.14) let us notice that we have by the maximum principle $\frac{H}{2} > \varepsilon_0$ in $D \setminus D_{\varrho_0}^1$ (see (3.15)). Choose $t_1 > 0$ such that we have for $t \geqslant t_1$:

$$0 \leqslant \varphi^+(t) \leqslant \varepsilon_0, \; 0 \leqslant \varphi^-(t) \leqslant \varepsilon_0.$$

Then we obtain for all $s \geqslant t_1$

$$\{0 \leqslant H + \varphi(s) \leqslant \varepsilon\} \setminus D^1_{\varrho_0} = \left\{0 \leqslant \frac{H}{2} + \left(\frac{H}{2} + \varphi(s)\right) \leqslant \varepsilon\right\} \setminus D^1_{\varrho_0} \subset$$
$$\subset \left\{0 \leqslant \frac{H}{2} < \varepsilon\right\} \setminus D^1_{\varrho_0} = \{0 \leqslant H < 2\varepsilon\} \setminus D^1_{\varrho_0} = \emptyset \quad (5.15)$$

if $0 < \varepsilon \leqslant \varepsilon_0$ (see (3.8)). This implies that the second integral on the right side in (5.13) is of order $O(1)$ if $t \geqslant t_1$. Let us proceed with an estimate of the third integral on the right side in (5.13). For $\delta > 0$ we have

$$\left|\int_t^{t+1} \int_D \nabla\omega b'_\varepsilon(F) \nabla H\right| \leqslant \frac{\delta}{2} \int_t^{t+1} \int_D |\nabla\omega|^2 b'_\varepsilon(F) + \frac{1}{2\delta} \int_t^{t+1} \int_D |\nabla H|^2 b'_\varepsilon(F) =$$
$$= \frac{\delta}{2\varepsilon} \int_t^{t+1} \int_{\{0 \leqslant F(s) \leqslant \varepsilon\}} |\nabla\omega|^2 \, dz \, ds + \frac{1}{2\delta\varepsilon} \int_t^{t+1} \int_{\{0 \leqslant F(s) \leqslant \varepsilon\}} |\nabla H|^2 \, dz \, ds =$$
$$=: I_1 + I_2. \quad (5.16)$$

For I_2 we obtain

$$2\delta\varepsilon I_2 = \int_t^{t+1} \int_{\{z | -\varphi(s) \leqslant H \leqslant -\varphi(s) + \varepsilon\}} |\nabla H|^2 \, dz \, ds = O(\varepsilon) \; \text{(see (3.9))}.$$

Now we have to estimate the first integral I_1 in (5.16).
Theorem 4.4 implies

$$\left.\begin{matrix} u_\varepsilon^- \leqslant u_\varepsilon \leqslant u_\varepsilon^+ \\ u_\varepsilon^- \leqslant F^-, \, u_\varepsilon^+ \leqslant F^+ \end{matrix}\right\} \text{ a.e. in } D_\infty.$$

We have to distinguish between the cases where $\omega = u_\varepsilon^+, \omega = u_\varepsilon^-$.

$$\omega = u_\varepsilon^+ := \{0 \leqslant F^+(s) \leqslant \varepsilon\} = \{z | 0 \leqslant F^+(s, z) \leqslant \varepsilon\} \subset$$
$$\subset \{x | 0 \leqslant u_\varepsilon^+(s, z) \leqslant \varepsilon\} \; (u_\varepsilon^+ \geqslant 0!)$$
$$\omega = u_\varepsilon^- : \{0 \leqslant F^-(s) \leqslant \varepsilon\} = \{z | 0 \leqslant F^-(s, z) \leqslant \varepsilon\} \subset$$
$$\subset \{z | 0 \leqslant (u_\varepsilon^-(s, z))^+ \leqslant \varepsilon\}$$
$$\text{where } (\alpha)^+ := \max(\alpha, 0).$$

Hence

$$\frac{1}{\varepsilon} \int_t^{t+1} \int_{\{0 \leqslant F^+(s) \leqslant \varepsilon\}} |\nabla u_\varepsilon^+|^2 \leqslant \frac{1}{\varepsilon} \int_t^{t+1} \int_{\{0 \leqslant u_\varepsilon^+ \leqslant \varepsilon\}} |\nabla u_\varepsilon^+|^2 = \int_t^{t+1} \int_D b'_\varepsilon(u_\varepsilon^+) |\nabla u_\varepsilon^+|^2$$

and

$$\frac{1}{\varepsilon}\int_t^{t+1}\int_{\{0\leqslant F^-(s)\leqslant\varepsilon\}}|\nabla u_\varepsilon^-|^2\leqslant\frac{1}{\varepsilon}\int_t^{t+1}\int_{\{0\leqslant(u_\varepsilon^-(s))^+\leqslant\varepsilon\}}|\nabla u_\varepsilon^-|^2=$$

$$=\int_t^{t+1}\int_D b_\varepsilon'((u_\varepsilon^-)^+)\,|\nabla u_\varepsilon^-|^2=\int_t^{t+1}\int_D b_\varepsilon'(u_\varepsilon^-)\,|\nabla u_\varepsilon^-|^2$$

and in any case the integrals appear on the left side of (5.13). Estimating the 4th-integral on the right side in (5.13) we obtain

$$\left|\int_t^{t+1}\int_D b_\varepsilon(\omega)\,\partial_y\,b_\varepsilon(\omega)\right|\leqslant \text{const and in the same way as in (5.16):}$$

$$\left|\int_t^{t+1}\int_D b_\varepsilon(\omega)\,\partial_y\,b_\varepsilon(F)\right|=O(1)$$

uniformly in t and ε (see (3.9)). This finishes the proof of Lemma 5.2. ■

5.5. Proof of Lemma 5.3. In (2.5) and (4.4) respectively we test with $(\omega-F)$ where $\omega=u, u_\varepsilon^+, u_\varepsilon^-$ and $F=g, F^+, F^-$, respectively.

$$\int_t^{t+1}\int_D \partial_t\,b_\varepsilon(\omega)\,\omega+\int_t^{t+1}\int_D(\nabla\omega+eb_\varepsilon(\omega))^2=$$

$$=\int_t^{t+1}\int_D\partial_t\,b_\varepsilon(\omega)\,F+\int_t^{t+1}\int_D(\nabla\omega+eb_\varepsilon(\omega))\,(eb_\varepsilon(\omega)+\nabla F).$$

Define $B_\varepsilon(t):=\int_0^t(b_\varepsilon(t)-b_\varepsilon(s))\,ds\ (\Rightarrow B_\varepsilon'(t)=b_\varepsilon'(t)\,t)$ and continue:

$$\int_t^{t+1}\int_D\frac{d}{dt}B_\varepsilon(\omega)+\int_t^{t+1}A_\varepsilon(s)\,ds\ (A_\varepsilon\text{: see (5.9))}$$

$$\leqslant\int_D(b_\varepsilon(\omega)\,F)\Big|_t^{t+1}-\int_t^{t+1}\int_D b_\varepsilon(\omega)\,F'+\frac{1}{2}\int_t^{t+1}A_\varepsilon(s)\,ds+$$

$$+\frac{1}{2}\int_t^{t+1}\int_D(eb_\varepsilon(\omega)+\nabla F)^2.$$

$\int_D B_\varepsilon(\omega)\Big|_t^{t+1}$ and the terms on the right side except $\frac{1}{2}\int_t^{t+1}A_\varepsilon$ are bounded and therefore Lemma 5.3 is proved. ■

5.6. REMARK. If the boundary values g are time independent (5.1) holds even for $\omega = u_\varepsilon$. Then as test function we choose $(\partial_t^h \omega - \partial_t^h H) = \partial_t^h \omega$ in Theorem 5.1, $b_\varepsilon(\omega) - b_\varepsilon(H)$ in Lemma 5.2 and $\omega - H$ in 5.3.

6. Convergence

In the preceding sections we have constructed sub- and supersolution $u_\varepsilon^\pm$ which are monotone decreasing and increasing in t, respectively, and we have estimated the $L^\infty(H^1)$-norms of them. Now we intend to study first the convergence of $u_\varepsilon^\pm$ if $\varepsilon \to 0$ and then if $t \to \infty$.

6.1. LEMMA. *Under the assumption of Theorem 5.1 there exist* $u, u^\pm \in L^2_{\mathrm{loc}}(0, \infty; H^1(D))$ *such that we have for a suitable subsequence* $\varepsilon \to 0$

$$u_\varepsilon^\pm \to u^\pm \begin{cases} \text{weakly in } L^2_{\mathrm{loc}}(0, \infty; H^{1,2}(D)) \text{ and} \\ \text{weakly star in } L^\infty_{\mathrm{loc}}(t_1, \infty; V); \end{cases} \tag{6.1}$$

$$u_\varepsilon \to u \text{ weakly in } L^2_{\mathrm{loc}}(0, \infty; H^{1,2}(D)).$$

Furthermore

$$\int_D |\nabla u^\pm(t)|^2 \leqslant \text{const}$$

uniformly for all $t \geqslant t_1$ $u^\pm$ *are monotone decreasing and increasing in* t, *respectively, (in the sense of Lemma 4.5) and*

$$u^- \leqslant u \leqslant u^+ \quad \text{a.e. in } D_\infty. \tag{6.2}$$

Proof. For all $T \in \mathbf{R}^+$, Lemma 5.3 yields $\int_0^T \int_D |\nabla u_\varepsilon|^2 \leqslant c(T)$ uniformly in ε. Therefore using a diagonal procedure we can select a subsequence u_ε such that for any $T \in \mathbf{R}^+$ we have

$$u_\varepsilon \to u \text{ weakly in } L^2(0, T; H^{1,2}(D))$$

for $u \in L^2_{\mathrm{loc}}(0, \infty; H^{1,2}(D))$. The same arguments hold for $u_\varepsilon^\pm$. Since $\|u_\varepsilon^\pm(t, \cdot)\|_{H^1(D)} \leqslant$ const uniformly for all ε and $t \geqslant t_1$ we get the weak-star convergence in $L^\infty(t_1, T; V)$. The $L^\infty(H^1)$-estimates of $u^\pm$ follow from Theorem 5.1 and the lower semicontinuity of the weak-star convergence in $L^\infty(t_1, T; V)$ (see [17], p. 125).

Let us show that u^+ is monotone decreasing in t. From Lemma 4.5 we know that we have for all $\alpha \in C_0^\infty(]0, T[)$, $\alpha \geqslant 0$, $\varphi \in L^2(D)$, $\varphi \geqslant 0$,

$$\int_0^T \int_D \partial_t^h u_\varepsilon^+(t, z)\, \alpha(t)\, \varphi(z)\, dz\, dt \leqslant 0$$

and if $h \to 0$

$$-\int_0^T \int_D u_\varepsilon^+ (t, z)\, \alpha'(t)\, \varphi(z)\, dz\, dt \leqslant 0,$$

and if $\varepsilon \to 0$ (see (6.1))

$$-\int_0^T \int_D u^+ (t, z)\, \alpha'(t)\, \varphi(z)\, dz\, dt \leqslant 0.$$

This establishes the monotonicity of u^+ with respect to t.

Similar arguments will hold to prove that u_ε^- is monotone increasing in t. (6.2) follows since the estimates in Theorem 4.4 are conserved for weak convergence. ■

6.2. LEMMA. *Under the assumptions of Theorem 5.1 there exist* $\gamma, \gamma^\pm \in L^p_{\text{loc}}(D_\infty)$, $2 \leqslant p < \infty$, *such that we have for a suitable subsequence* $\varepsilon \to 0$:

$$\left.\begin{aligned} b_\varepsilon(u_\varepsilon) &\to \gamma \\ b_\varepsilon(u_\varepsilon^\pm) &\to \gamma^\pm \end{aligned}\right\} \begin{aligned} &\text{weakly in } L^p_{\text{loc}}(D_\infty), \\ &p > 1. \end{aligned}$$

$\gamma^\pm$ *are monotone decreasing and increasing, respectively in t (in the sense of Lemma* 4.5), *and*

$$\gamma^- \leqslant \gamma \leqslant \gamma^+ \text{ a.e. in } D_\infty.$$

Proof. This is obvious since $0 \leqslant b_\varepsilon \leqslant 1$, b_ε is monotone increasing and because of Theorem 4.4. The monotonicity can be proved in the same way as in Lemma 6.1. ■

6.3 LEMMA. (Proof of Theorem 2.4) *Under the assumptions of Theorem 5.1 we have*:

$$u \in g + V,\ u^\pm \in F^\pm + V,\ \partial_t \gamma, \partial_t \gamma^\pm \in L^2_{\text{loc}}(0, \infty; V^*), \tag{6.3}$$

$$\int_0^\infty \int_D \gamma^\pm (\partial_y - \partial_t)\, v + \int_0^\infty \int_D \nabla u^\pm\, \nabla v = 0 \quad \textit{for all } v \in \mathring{H}^1(0, \infty; V). \tag{6.4}$$

$$\int_0^\infty \int_D \gamma\, (\partial_y - \partial_t)\, v + \int_0^\infty \int_D \nabla u\, \nabla v \leqslant 0 \quad \textit{for all } v \in \mathring{H}^1(0, \infty; H^1(D)),$$

$$v \geqslant 0 \textit{ on } \Gamma_0 \cap \{g = 0\},\ v = 0 \textit{ on } \Gamma_0 \cap \{g > 0\}.$$

Furthermore

$$\gamma(0, \cdot) = \chi_0$$

$$\gamma^\pm(0, \cdot) = b_0(u_0^\pm(0)) = \begin{cases} 1 \text{ if } \text{“}+\text{”} \\ 0 \text{ if } \text{“}-\text{”} \end{cases} \tag{6.5}$$

(*in the weak sense, see* (1.15)) *where* b_0 *is the pointwise limit of* b_ε *and*

$$u \geqslant 0,\ 0 \leqslant \gamma \leqslant 1,\ u(1-\gamma) = 0 \ \text{a.e. in } D_\infty;$$
$$0 \leqslant \gamma^\pm \leqslant 1, \begin{Bmatrix} u^\pm > 0 \Rightarrow \gamma^\pm = 1 \\ u^\pm < 0 \Rightarrow \gamma^\pm = 0 \end{Bmatrix} \text{a.e. in } D_\infty. \tag{6.6}$$

Proof. (6.3) and (6.4) follow immediately from the weak convergence of $b_\varepsilon(u_\varepsilon^\pm)$ and $u_\varepsilon^\pm$, respectively. The variational inequality for u_ε we obtain in the same manner as in [8], Proof of Theorem 1.

Proof of (6.5): From (2.6) we have for all $\zeta \in L^2(0, T; V) \cap H^1(0, T; L^\infty(D))$, $\zeta(T) = 0$, $T \in \mathbf{R}^+$

$$\int_0^T \int_D \partial_t\, b_\varepsilon(u_\varepsilon)\, \zeta + \big(b_\varepsilon(u_\varepsilon) - b_\varepsilon(u_0)\big)\, \partial_t\, \zeta = 0. \tag{6.7}$$

The weak equation (2.5) for u_ε together with (6.1) implies

$$\|\partial_t\, b_\varepsilon(u_\varepsilon)\|_{L^2(0,T;V^*)} \leqslant \text{const}\,(T)$$

uniformly in ε. Therefore we can select a subsequence such that

$$\partial_t\, b_\varepsilon(u_\varepsilon) \to \partial_t\, \gamma \ \text{ in } L^2(0, T; V^*)$$

for any $T \in \mathbf{R}^+$ (diagonal procedure).

Now we go to the limit in (6.7) and obtain (6.5) for u_ε. For $u_\varepsilon^\pm$ the arguments are the same.

Proof of (6.6): $u \geqslant 0, 0 \leqslant \gamma \leqslant 1, 0 \leqslant \gamma^2 \leqslant 1$ are obvious. It remains to prove $u(1-\gamma) = 0$ a.e. in D_∞ and the corresponding assertion for $u^\pm$ and $\gamma^\pm$. By virtue of the following Lemma 6.4 we have to verify that

$$\int_A \big(b_\varepsilon(u_\varepsilon(t-\sigma, z)) - b_\varepsilon(u_\varepsilon(t, z))\big)\big(u_\varepsilon(t-\sigma, z) - u_\varepsilon(t, z)\big)\, dz\, dt$$

tends to zero if $\sigma \to 0$ uniformly for subsets $A \subset\subset D_T$ and for $0 < \varepsilon \leqslant \varepsilon_0$. But this we have already shown (see Remark after Theorem 5.1). Then Lemma 6.4 yields $\gamma \in b(u)$ which means just the statements in (6.6). For $u_\varepsilon^\pm$ we use the same argument. ■

6.4 Lemma. *Let* b_ε *be as in* (2.1), $u_\varepsilon \to u$ *weakly in* $L^2(0, T; H^1(D))$, $b_\varepsilon(u_\varepsilon) \to \beta$ *weakly in* $L^2(D_T)$ *and* $\tau_h(t, x) := (t+h, x)$, $h \in \mathbf{R}$. If

$$\int_A \big(b_\varepsilon(u_\varepsilon \circ \tau) - b_\varepsilon(u_\varepsilon)\big)(u_\varepsilon \circ \tau - u_\varepsilon) \to 0$$

for $h \to 0$ *uniformly in* ε *and for subsets* $A \subset\subset D_T$, *then we have* $\beta \in b(u)$ *where* $b := \lim b_\varepsilon$ *(pointwise)* $= \nabla\varphi$, $\varphi(s) = \max(0, s)$.

Proof ([6], Lemma 4.3). ■

6.5. Proof of Theorem 2.5. Since $\{u^-, \gamma^-\}$, $\{u^+, \gamma^+\}$ are bounded and monotone increasing and decreasing functions in t, respectively (see Lemma

6.1 and 6.2) we can define the following pointwise limits:

$$\begin{aligned} u_\infty^\pm (z) &:= \lim_{t\to\infty} u^\pm (t, z), \\ \gamma_\infty^\pm (z) &:= \lim_{t\to\infty} \gamma^\pm (t, z) \text{ for a.e. } z \in D. \end{aligned} \tag{6.8}$$

The $L^\infty (H^1)$—estimates of $u^\pm$ in Lemma 6.1 imply that the convergence

$$u^\pm (t, \cdot) \to u_\infty^\pm, \tag{6.9}$$

holds weakly in $H^{1,2}(D)$ (for subsequences) and by Lebesgue's convergence theorem strongly in $L^p(D)$ $(1 \leqslant p < \infty)$. Moreover

$$u_\infty^\pm \in H + V. \tag{6.10}$$

Now consider the weak equation for $u^\pm$ in (6.4). From (6.4) we obtain for any $t \in \mathbf{R}^+$ and for all $v \in H^1(]t, t+1[, V)$

$$\int_t^{t+1} \int_D \gamma^\pm (\partial_y - \partial_\tau) v \, dz \, d\tau + \int_t^{t+1} \int_D \nabla u^\pm \nabla v \, dz \, d\tau = 0. \tag{6.11}$$

Let $\varphi \in C_0^\infty(]0,1[)$ such that $\int_0^1 \varphi(s)\, ds \neq 0$, $\psi \in V$ and take $v(\tau, z) := \varphi(\tau - t)\psi(z)$, $t \in \mathbf{R}^+$ as test function in (6.11). Then we obtain

$$\int_t^{t+1} \int_D \varphi(\tau - t)\, \partial_y \psi(z)\, \gamma^\pm (\tau, z) - \int_t^{t+1} \int_D \varphi'(\tau - t)\, \psi(z)\, \gamma^\pm(\tau, z) + \int_t^{t+1} \varphi(\tau - t) \int_D (\nabla u^\pm (\tau, z) \nabla\psi(z))\, dz\, d\tau = 0.$$

Changing variables $\sigma = \tau - t$ we get:

$$\int_0^1 \int_D \varphi(\sigma)\, \partial_y \psi(z)\, \gamma^\pm(\sigma + t, z)\, dz\, d\sigma - \int_0^1 \int_D \varphi'(\sigma)\psi(z)\gamma^\pm(\sigma+t, z)\, dz\, d\sigma + \int_0^1 \varphi(\sigma) \int_D \nabla u^\pm(\sigma + t, z) \nabla\psi(z)\, dz\, d\sigma = 0. \tag{6.12}$$

Now we define $u_t^\pm(\sigma, z) := u^\pm(\sigma + t, z)$ and $\gamma_t^\pm := \gamma^\pm(\sigma + t, z)$. We have (see (6.8), Lemma 6.1) for suitable subsequences $t \to \infty$:

$$\begin{aligned} u_t^\pm &\to u_\infty^\pm \text{ in } L^p(]0,1[\times D) \text{ strong, } 1 \leqslant p < \infty, \\ \nabla u_t^\pm &\to \nabla u_\infty^\pm \text{ in } L^2(]0,1[\times D) \text{ weak}, \\ \gamma_t^\pm &\to \gamma_\infty^\pm \text{ in } L^2(]0,1[\times D) \text{ strong.} \end{aligned}$$

Then passing to the limit $t \to \infty$ in (6.12) it follows

$$\int_0^1 \int_D \varphi(\sigma)\, \partial_y \psi(z)\, \gamma_\infty^\pm\, dz\, d\sigma - \int_0^1 \int_D \varphi'(\sigma)\, \psi(z)\, \gamma_\infty^\pm\, dz\, d\sigma + \int_0^1 \varphi(\sigma) \int_D \nabla u_\infty^\pm(z) \nabla\psi(z)\, dz\, d\sigma = 0.$$

Since $\varphi(0)=\varphi(1)=0$ the second integral vanishes and since $\int_0^1 \varphi(\sigma)\,d\sigma \neq 0$ we obtain

$$\int_D (\nabla u_\infty^\pm(z)+e\gamma_\infty^\pm(z))\,\nabla\psi(z)\,dz=0 \tag{6.13}$$

for all $\psi\in V$.

It remains to show

$$u_\infty^\pm \geqslant 0,\ 0\leqslant \gamma_\infty^\pm \leqslant 1,\ u_\infty^\pm(1-\gamma_\infty^\pm) \text{ a.e. in } D,$$

but this follows from the pointwise convergence (see (6.6), (6.8) and (4.5), (6.1)). Then using the same arguments as in [8], Theorem 1, we have

$$\int_D (\nabla u_\infty^\pm + e\gamma_\infty^\pm)\,\nabla v \leqslant 0$$

for all $v\in H^1(D)$, $v\geqslant 0$ on $\Gamma_0\cap\{G=0\}$, $v=0$ on $\Gamma_0\cap\{G>0\}$.

Now the statement of Theorem 2.5 can be shown as follows, $u_\infty^\pm$ are solutions of the stationary problem 1.2. Because of the assumption (1.16) we have $u_\infty^+=u_\infty^-$. On account of $u^-\leqslant u\leqslant u^+$ a.e. on D_∞ (see (6.2)) and $u^\pm(t,\cdot)\to u_\infty^\pm$ in $L^p(D)$, $1\leqslant p<\infty$ (see (6.9)) we get

$$u(t,\cdot)\to u_\infty := u_\infty^\pm \text{ in } L^p(D).$$

This proves Theorem 2.5. In the case where g are time independent then $\int_D |\nabla u(t)|^2$ is bounded uniformly in t (see Remark 5.6). Now if (t_n) is any sequence with $t_n\to\infty$ then there is a subsequence such that (see Lemma 6.1)

$$u(t_n,\cdot)\to u_\infty \text{ weakly in } H^1(D).$$

This proves the Remark of Theorem 2.5 ■

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Asymptotyczne zachowanie rozwiązania niestacjonarnego zagadnienia tamy w przypadku płynów nieściśliwych

Rozważane jest zachowanie globalne (przy $t \to \infty$) rozwiązania nieliniowego równania $\partial_t \gamma - \Delta u - \partial_y \gamma = 0$ w $]0, T[\times D$ przy pewnych, fizycznie umotywowanych warunkach brzegowych i początkowych. O γ zakłada się, że $0 \leqslant \gamma \leqslant 1$ i $u(1-\gamma) = 0$. Model opisuje niestacjonarną filtrację nieściśliwego płynu w izotropowym, jednorodnym ośrodku D. Główny wynik pracy dotyczy zbieżności $u(t, \cdot)$ do rozwiązania zagadnienia stacjonarnego.

Асимптотическое поведение решения нестационарной проблемы плотины для несжимаемых жидкостей

Рассуждается глобальное поведение (при $t \to \infty$) решения нелинейного уравнения $\partial_t \gamma - \Delta u - \partial_y \gamma = 0$ в $]0, T[\times D$, при некоторых физически обоснованных краевых и начальных условиях. Предполагается, что $0 \leqslant \gamma \leqslant 1$ и $u(1-\gamma) = 0$. Модель описывает нестационарную фильтрацию несжимаемой жидкости в изотропной, однородной среде D. Важнейшим результатом работы является сходимость $u(t, \cdot)$ к решению стационарной проблемы.