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Ill-posed free boundary problems

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1. Introduction

This article is concerned with the formulation and analysis of some models for certain physical processes as free boundary problems in which the free boundary is capable of rapid variation in time and/or space. We shall describe the physical background and some formal mathematical ideas in the introduction and then give a more rigorous analysis for the Stefan problem and its specialisations in the subsequent sections.

(i) Physical Models

The Stefan problem is a famous model for heat flow in a conducting medium which undergoes a phase-change. The temperature satisfies

$$c\,\frac{\partial u}{\partial t} = \nabla\cdot(k\nabla u) \tag{1}$$

away from any phase boundaries in the medium. For simplicity we take c, k to be positive constants. The classical Stefan model then ascribes the conditions:

$$\text{constant melting temperature, } u = 0, \tag{2}$$

$$\text{and energy balance, } \left[k\,\frac{\partial u}{\partial n}\right]_S^L = -Lv_n, \tag{3}$$

at the phase boundary where $[\ ]_S^L$ denotes the jump from solid to liquid,

v_n is the velocity of the phase boundary in the direction of n and L is the latent heat. With suitable initial and fixed boundary conditions, this problem with $L > 0$ has a unique solution [16], which coincides with that of the weak formulation in which

$$\frac{\partial H}{\partial t} = k\nabla^2 u, \quad H \in \begin{cases} cu & u < 0 \\ [0, L] & u = 0 \\ cu + L & u > 0 \end{cases} \tag{4}$$

is satisfied in a distributional sense [5]. Moreover $u \gtrless 0$ in the liquid and solid regions respectively. The well-posedness of this problem is not yet completely understood but we note that as long as $L \ngtr 0$, H can easily be smoothed to make (4) a classical nonlinear parabolic equation.

Although the Stefan problem is amenable to numerical solution, [5], there are few explicit solutions against which numerical results can be compared. This is even true for one-phase Stefan problems in which $u \equiv 0$ in either the liquid or solid phase. However, there is a special class of Stefan problems for which many explicit solutions can be found, namely when the problem is one phase and in addition $c = 0$ in (1). In this case we call (1, 2, 3) a Hele-Shaw free boundary problem because liquid flow with velocity q in a Hele-Shaw cell, [29], can be described in terms of a pressure p such that

$$q = -k\nabla p, \ \nabla \cdot q = 0; \tag{5}$$

moreover at the free liquid boundary, the condition of conservation of mass is

$$k\frac{\partial p}{\partial n} = -v_n \tag{6}$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative from the liquid (cf. (3)) and also, with a suitable normalisation and some assumptions about surface tension, [24],

$$p = 0 \tag{7}$$

(cf. (2)). We note that this problem can be identified with a one-phase version of (1, 2, 3) with p being the liquid temperature; $p \gtrless 0$ means that the free boundary expands and contracts respectively. We shall see in section 3 that complex variable and variational inequality techniques can be used to great effect on the Hele-Shaw model.

The model (5, 6, 7) is also the simplest for certain electrochemical processes [15] in which $-p$ is identified as the potential and (6) is Faraday's law. Here the sign of p determines whether electroforming ($p < 0$) or electromachining ($p > 0$) is taking place. Equally, if we identify p with the pressure in a gravity-free porous medium, part of which is saturated and part dry, (6,7) are the conditions of conservation of mass and momentum at the interface between dry and saturated regions in the absence of any surface tension effects. When gravity is important, say in the y-direction, (6) becomes

$$k\left[\frac{\partial p}{\partial n}+\cos\theta\right]=-v_n$$

where θ is the angle between n and the y axis.

We have now listed some of the best known free boundary problems which are special cases of the Stefan problem. There are also many generalisations of the Stefan problem [21] but little is known about such problems, even when the field equation is still scalar. For example, the introduction of surface energy in its simplest from gives the Gibbs-Thompson relation [14]

$$u=-\gamma\varkappa$$

instead of (2) where γ is a positive constant and $\varkappa$ is the curvature of the solid interface, positive if it is convex towards the liquid. A similar modification is possible for (7) but it destroys the possibility of using complex variable methods. This is also the case for the "two-phase" version of (5–7), called the Muskat problem [18]. An even more difficult generalisation is to the case of vector field equations; these arise in the so-called alloy solidification problem [17].

There are many other free boundary problems which are unrelated to the Stefan problem. For the purpose of this article we mention two examples from two-dimensional, inviscid, incompressible hydrodynamics. The first is that of gravity waves [28] in which the field equation for the velocity potential φ is

$$\nabla^2\varphi=0 \tag{8}$$

and on the surface, say $y=\eta(x,t)$, we have:

$$\text{momentum balance,}\quad \frac{\partial\varphi}{\partial t}+\frac{1}{2}|\nabla\varphi|^2+y=0,$$

$$\text{and mass balance,}\quad \frac{\partial\varphi}{\partial y}=\frac{\partial\eta}{\partial t}+\frac{\partial\varphi}{\partial x}\frac{\partial\eta}{\partial x}.$$

In the second example of a vortex sheet $y=\zeta(x,t)$ between two liquids 1 and 2, the field equation is still (8) but now the momentum and mass balance give

$$\left[\frac{\partial\varphi}{\partial t}+\frac{1}{2}|\nabla\varphi|^2\right]_1^2=0$$

and

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_1}{\partial x}\frac{\partial \zeta}{\partial x} = \frac{\partial \varphi_2}{\partial y} - \frac{\partial \varphi_2}{\partial x}\frac{\partial \zeta}{\partial x}$$

respectively.

(ii) Formal stability analysis

Special solutions of all the problems listed above have been studied with stability analyses of varying degrees of complexity [19]. The only such analysis we will describe here is the linear stability for the Hele-Shaw problem, for which, with $k = 1$,

$$p = V(Vt - x), \; x < Vt,$$

is an exact solution with free boundary $x = Vt$.

We now seek a formal asymptotic expansion for small ε in which the free boundary is

$$x \sim Vt + \varepsilon \sin nye^{\sigma t}$$

where n is real and positive and σ is to be determined. We correspondingly write the pressure perturbation, which decays away from the boundary, as

$$p \sim V(Vt - x) + \varepsilon C \sin nye^{nx + (\sigma + nV)t} + \ldots$$

where C is a constant. (6) and (7) then give

$$Cn = -\sigma \text{ and } -V + C = 0$$

respectively. Hence the linear stability growth rate is

$$\sigma = -nV. \tag{9}$$

For $V \gtrless 0$ we have a situation reminiscent of the contrast between forward and backward heat equations

$$\frac{\partial u}{\partial t} = \pm \frac{\partial^2 u}{\partial y^2} \tag{10$\pm$}$$

where a solution $y = \sin nye^{\sigma t}$ gives $\sigma = \mp n^2$. When $V < 0$, i.e. when $p < 0$ and the fluid region contracts, and in the backward heat equation, short wavelength disturbances grow rapidly in time. This analogy can even be carried over to the question of blow-up for smooth initial data; it is known that solutions of (5–7) can blow up in finite time [20] as can (10−), say with initial data $u(x, 0) = e^{-x^2}$ (in the case of minus sign).

Results such as (9) apply to most of the models considered in (i) above, with, in all cases of instability, the shortest wavelengths growing the fastest. When $L > 0$, the Stefan problem is linearly stable as long as $u \gtrless 0$ in the liquid and solid regions respectively, but it can become unstable if either superheating or supercooling occurs, depending on the sizes of the local heat fluxes [17]. The introduction of surface energy replaces (9) by

$$\sigma = -nV - \gamma n^3$$

which means that the solution is only unstable to long wavelength perturbations when $V < 0$, and that there exists a most unstable wavelength with $n = -V/3\gamma$. Such a result has been used in [14] to predict dendrite spacing in solidification problems.

Corresponding to these results, electroforming is linearly unstable and electromachining is stable; porous medium flows without gravity are stable only if the fluid region is expanding with time or if, in the Muskat problem, the more viscous fluid displaces the less viscous one. Porous medium flows under gravity are unstable if interface with the dry region has a downward pointing normal and does not move downwards too rapidly, or if the normal points upwards and the interface moves rapidly downwards. With a suitable normalisation vortex sheets give

$$\sigma = n > 0 \tag{11}$$

irrespective of the direction of the tangential velocities in the fluids 1 and 2. However for gravity waves $\sigma = \pm in$ and this is the only of our examples which permits oscillatory behaviour to persist.

These heuristic linear stability analyses suggest that in certain cases, most of the problems listead in (i) are ill-posed or can at least exhibit solutions which can change rapidly in time. It is tempting to identify such rapid changes with phenomena such as dendritic growth in solidification or fingering in porous medium flows. Before such an identification can be made, some estimates must be made of nonlinear effects and the idea that these can stabilise linearly unstable processes and vice versa is very important [10]. It turns out that by using multiple scale methods, even the very complicated model of a solidifying alloy can be shown to exhibit such nonlinear stabilisation [32]. However, our purpose here is not to consider the possibility of new mechanisms rendering linearly unstable problems well-posed. Rather we shall examine what can be found from studying even badly behaved solutions to the primitive models listed above. Our physical motivation for this is that, notwithstanding the ill-posedness, such solutions can occasionally provide illuminating descriptions of experimentally observed phenomena. Let us cite two examples. In the Saffman-Taylor solution [24] of the Hele-Shaw problem, travelling wave solutions of (5–7) are sought in an infinite strip, with $p < 0$. It is found that a continuum of such solutions exist for a given wave speed, all of which are unstable to small disturbances as in (9). However, just one of this continuum gives good agreement with what is observed in practice [24, 30].

Similarly in three-dimensional aerofoil theory, the forces on the aerofoil can be computed [31] uing assumptions about the flow pattern at the trailing edge and also the approximation that the trailing vortex sheet is flat despite the instability (11); this gives excellent results for the lift despite the fact that in practice the vortex sheet curls up rapidly.

The layout for the remainder of the article will be to first consider ill-posed Stefan problems in one space dimension and then ill-posed Hele-Shaw problems in two space dimensions. Finally we shall use the latter results together with some comparison ideas to discuss the ill-posed Stefan problem in higher dimensions.

2. One-dimensional Stefan problem with superheating

In practice regions of supercooling may arise if a very pure liquid is cooled carefully. Superheating can arise if a substance undergoes body heating in which case (1) is replaced by an equation of the form $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F$. For example if a clear block of ice is heated by light, parts of it may become superheated before any melting occurs [33]. More importantly there is the possibility of superheating during the process of electrical welding. If the classical model for phase changes is to apply, the material must become superheated [13] (see fig. 1a). It is possible that no melting occurs until the surface of the material reaches the melting point, by which time a substantial amount is already superheated [33] (see fig. 1,b). Superheating is expected for only very pure materials; in practice, a "mushy region", in which the

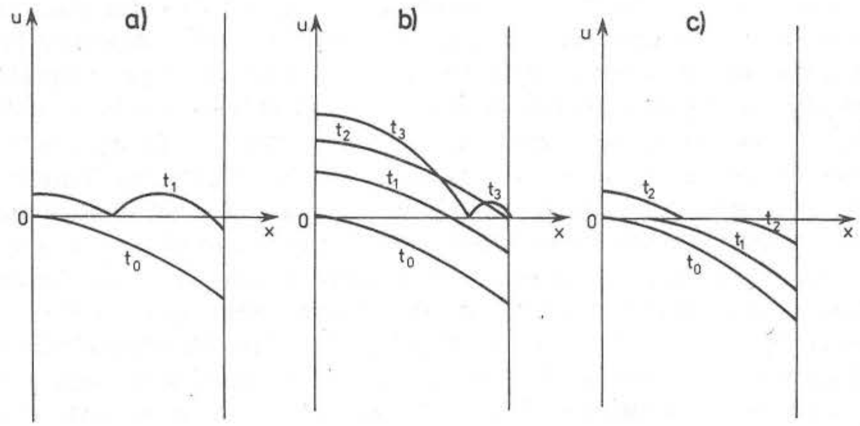


Fig. 1. Three possibilities for body heating in a symmetric, one-dimensional region (a) Melting starts at the centre. (b) Melting starts at the outside edge. (c) A mushy region forms.

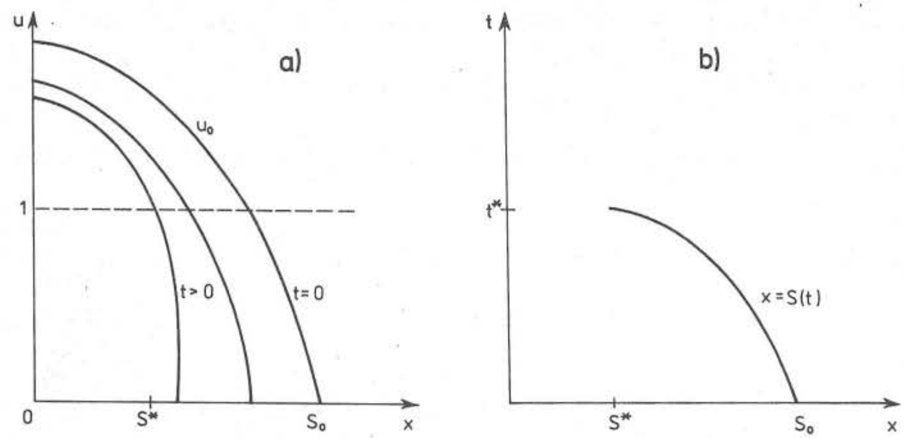


Fig. 4. The case of blow-up at $t = t^*$ when $s = s^*$. (a) Temperature profiles for different times. (b) The trajectory of the free boundary.

solution fails to exist. For both the Neumann solution and the solution to (12–15) to exist for all time, the average initial temperatures, C and $\frac{1}{s_0}\int_0^{s_0} u_0\,dx$ respectively, must be less than one.

Sufficient conditions for the occurrence of either (a) or (b) can also be found [7]. Taking u_0 to be smooth with $u_0 < 1$ for $0 \leqslant x < s_0$ then (a) occurs, while if u_0 is smooth with just one point x_0 such that $u_0(x_0) = 1$ and $\int_0^{s_0} (u_0 - 1)\,dx \leqslant 1$ then (a) or (b) occurs.

It is also possible to broaden the sufficient conditions for blow-up. Defining $H(t) = \int_0^{s(t)} h(x)\big(u(x,t) - 1\big)\,dx$ for a given h, H is an increasing function if $h'' \geqslant 0$ for $0 < x < s_0$ and $h'(0) \geqslant 0$. Hence, for cases (a) and (b), H takes its greatest values for $t = \infty$ and $t = t^*$ respectively; these are non-positive if h is non-negative. It follows that if there is some h defined for $0 \leqslant x \leqslant s_0$ such that:

$$h(0) \geqslant 0,\ h'(0) \geqslant 0,\ h''(x) \geqslant 0 \text{ for } 0 < x < s_0 \quad \text{and} \int_0^{s_0} (u_0 - 1)\,h\,dx > 0 \tag{17}$$

then blow-up, (c), occurs.

It is possible to apply an integral transformation to (12–15) and to work with the variable $U = \int_x^{s(t)} (\xi - x)\left(1 - u(\xi, t)\right) d\xi$, [7]. u and U are related $\frac{\partial^2 U}{\partial x^2} = 1 - u$ and $\frac{\partial U}{\partial t} = u$ so that U satisfies the oxygen diffusion problem [2]

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} - 1 \quad 0 < x < s, \tag{18}$$

$$U = \frac{\partial U}{\partial x} = 0 \qquad x = s, \tag{19}$$

$$\frac{\partial U}{\partial x} = -\int_0^{s_0} (1 - u_0)\, dx \qquad x = 0, \tag{20}$$

$$U = U_0(x) \equiv \int_x^{s_0} (\xi - x)\left(1 - u_0(\xi)\right) d\xi \qquad t = 0. \tag{21}$$

Note that where $\omega(x) = s^{-1}(x)$ is defined $\left((s_\infty, s_0]\right.$ for (a), $(0, s_0]$ for (b), $\left.(s^*, s_0]\right.$ for (c))

$$U = \int_t^{\omega(x)} u(x, \tau)\, d\tau. \tag{22}$$

It transpires that the condition for blow-up (17) is equivalent to either $\frac{\partial U}{\partial x}(0, t)$ being positive (the simplest blow-up condition) or the existence of some x_1 where $U_0(x_1)$ is negative. The latter clearly implies blow-up since U is decreasing and for both (a) and (b) U is ultimately non-negative.

Throughout these results for the one-dimensional problem the temperature $u = 1$ has significance. If u_0 is smooth and everywhere less than one there is no blow-up but if u_0 is somewhere greater than one, blow-up can occur.

3. Two-dimensional Hele-Shaw cell

Taking pressure to be $-u$ then with a suitable scaling the Hele-Shaw problem can be written as

$$\nabla^2 u = 0 \text{ in } \Omega(t) \text{ except at specified singularities,} \tag{23}$$

$$u = 0, \; v_n = \frac{\partial u}{\partial n} \text{ on } \partial\Omega(t), \qquad (24), (25)$$

temperature is identically that of melting, may be formed. Mushy regions are predicted by numerical computations using the enthalpy method [1] (see fig. 1c).

This motivates us to consider (1–3) in one dimension with superheated solid occupying $0 < x < s(t)$ and liquid at the melting temperature in $x > s$ [27]. We discuss the one phase problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < s(t), \tag{12}$$

$$\frac{\partial u}{\partial x} = 0 \qquad x = 0, \tag{13}$$

$$\frac{ds}{dt} = \frac{\partial u}{dx}, \quad u = 0 \quad x = s(t), \tag{14}$$

$$u = u_0(x) \quad 0 \leqslant x \leqslant s(0) = s_0 \tag{15}$$

where u_0 is smooth and non-negative and $u_0(s_0) = 0$. For simplicity we have scaled so that $k = c = L = 1$.

Three possibilities can occur [27, 6]:

(a) the solution exists for all time and $s \to s_\infty > 0$, $u \to 0$ as $t \to \infty$ (fig. 2);

(b) there is some finite, positive t^* such that the solution exists for $0 < t \leqslant t^*$ and as $t \to t^*$, $s \to 0$ and $\frac{ds}{dt} \to -\infty$ (fig. 3);

(c) there is some finite, positive t^* at which the solution blows up, i.e. the solution only exists for $0 < t \leqslant t^*$ but as $t \to t^*$ $\frac{ds}{dt} \to -\infty$ and $s \to s^* > 0$ (fig. 4).

With the present scaling the heat released in lowering the temperature of a sample of solid from $u = 1$ to $u = 0$ is precisely that required to melt it. Now if u_0 is such that the total heat content (taking liquid with $u = 0$ to have zero heat), $\int_0^{s_0} (u_0 - 1)\,dx$, is positive, then it remains positive. For (a) to occur the final amount $= -\int_0^{s_\infty} dx$ is negative while (b) gives zero at $t = t^*$. So if there is more heat present than can be taken up, in melting, blow-up must occur.

Of course if the critical case (b) is to take place $\int_0^{s_0} (u_0 - 1)\,dx = 0$ so an arbitrarily small perturbation to such initial data can produce instead (a) or (c).

The significance of the temperature $u = 1$ can also be noticed in the Neumann solution for a semi-infinite region, where now $u_t = u_{xx}$ for $x > s$, $u = 0$ and $u_x = \dot{s}$ on $x = s$, $s(0) = 0$, and $u = C = \text{const.}$ at $t = 0$. Here $u = C\{1 - \operatorname{erfc}(x/2t^{\frac{1}{2}})/\operatorname{erfc}(a)\}$ for $x > s = 2at^{\frac{1}{2}}$ where a is the solution to

$$a \exp(a^2/4) \int_{a/2}^{\infty} \exp(-\sigma^2)\, d\sigma = C. \qquad (16)$$

(16) has a solution for $C < 1$; as $C \to 1, a \to \infty$ so the boundary speed also tends to infinity, and if $C \geqslant 1$ then (16) has no solution and the Neumann

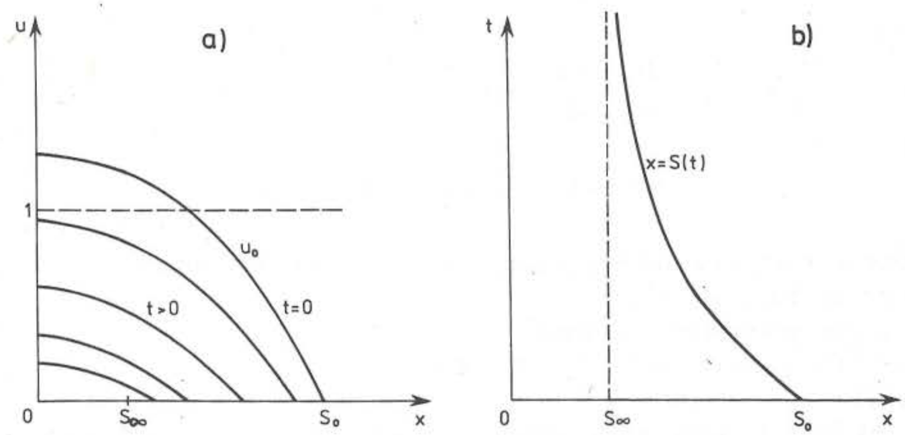


Fig. 2. A one-dimensional problem with partial melting. (a) Temperature profiles for different times. (b) The trajectory of the free boundary.

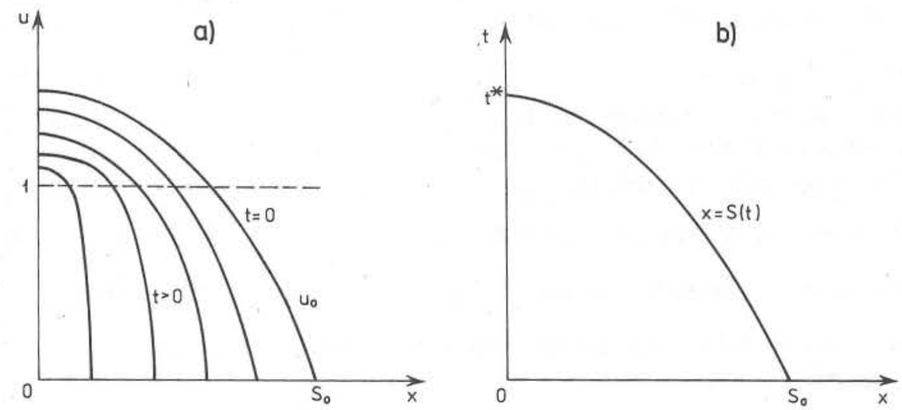


Fig. 3. The critical case of complete melting at $t = t^*$. (a) Temperature profiles for different times. (b) The trajectory of the free boundary.

where $\Omega(t)$ is the region occupied by liquid and n is the outward normal on the free surface $\partial\Omega(t)$; the specified singularities can include point sources or sinks, or sources and sinks distributed along curves. As in the previous section, we are mainly interested in problems with retreating boundaries so u generally will be positive (pressure is negative); i.e. sinks are applied.

For two-dimensional problems complex variables can be used to find a number of exact solutions which indicate the ill-posedness of these problems with shrinking regions.

For example, with a problem in a parallel-sided channel there are travelling wave solutions for which a finger of air protrudes into the retreating liquid; there is a continuum of such solutions so that for every value of λ, $0 < \lambda \leqslant 1$, there is a solution for which the finger width is λ times that of the channel [24]. Moreover, there are truly time-dependent solutions which tend to the travelling waves as $t \to \infty$ [23]. For all of the time dependent cases the boundary is like $x = t + e^t \sin y$ for large negative time. Not only do all the free boundaries corresponding to different finger widths have the same approximate shape, but so do a wider class of solutions which blow up in a finite time. For instance, if $w = u + i\psi$ is the complex potential then there is a solution in which $z = x + iy = w - a(t) + \exp[-a(t) - w]$, where $2a + \exp(-2a) = -2t$ and $a > 0$; the boundary is $x \sim t + e^t \sin y$ for large negative t and forms a cusp at $t = -\frac{1}{2}$.

Some other exact solutions involve suction from a single point lying within a finite region of fluid. A particularly simple example has a sink, strength Q, at the origin with the boundary $\partial\Omega(t)$ given by $z = a\zeta + b\zeta^n$, $|\zeta| = 1$; here $n \geqslant 2$ and a and b are positive real functions of t with $a \geqslant nb$. a and b vary such that $a^n b$ is constant while $a^2 + nb^2$ decreases at a rate Q/π. Eventually at a time t^*, $a = nb$ and $\partial\Omega$ then has $n-1$ inward pointing cusps. The solution can not be continued, using this simple model, to later times so we again say that t^* is a blow-up time. Again the ill-posedness of the problem is additionally shown from our ability, by choosing n large and b small enough, to take initial boundaries arbitrarily close to a circle (and with curvature arbitrarily close to that of the circle) but for which the blow-up time is arbitrarily small. Of course if $b = 0$ at $t = 0$ then $\partial\Omega$ is a circle which shrinks to a point as t approaches $\pi a(0)^2/Q$.

Instead of the direct approach we can use the Schwarz function of $\partial\Omega$. This is a function $g(z)$, analytic in a neighbourhood of $\partial\Omega$, such that $\partial\Omega$ is given by

$$\bar{z} = g(z). \tag{26}$$

[3, 22]. $g(z)$ has a number of singularities lying in Ω. Some of these vary

in time so that the rates of change are the singularities of $2\left[\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\right]$; these are known. Since the internal singularities of g are known at any time t, $\partial\Omega(t)$ can be found from the relationship (26) [12, 22].

In place of the function g we may alternatively use a function $U(x,y)$ which is related to g by $g=\bar{z}-2\left[\frac{\partial U}{\partial x}-i\frac{\partial U}{\partial y}\right]$ and is chosen so that it is zero on $\partial\Omega$. Now at $t=0$ $U=U_0$ is the solution to:

$$\nabla^2 U_0 = 1 \text{ in } \Omega_0, \text{ except at singularities;} \qquad (27)$$

$$U_0 = \frac{\partial U_0}{\partial n} = 0 \text{ on } \partial\Omega_0. \qquad (28, 29)$$

To find $\partial\Omega$ at a time t the free boundary problem

$$\nabla^2 U = 1 \text{ in } \Omega, \text{ except at singularities,} \qquad (30)$$

$$U = \frac{\partial U}{\partial n} = 0 \text{ on } \partial\Omega; \qquad (31, 32)$$

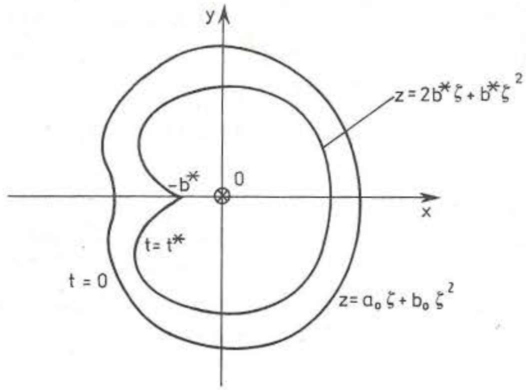
must be solved. The singularities of U are those of U_0 minus the time integrals of the singularities of u.

The ill-posedness of the problem with a retreating boundary is manifested through the necessity of solving the Cauchy problem (27–29). Small changes in the initial boundary can make a large change in the singularities of U towards which the boundary is moving.

One advantage in the use of the variable U is that information can be gained about three, as well as two, dimensional problems. We shall look at two examples, one in two and one in three dimensions.

I. Suction from a limaçon

Starting with a limaçon, $\partial\Omega_0$ is $z=r\exp(i\theta)=a_0\zeta+b_0\zeta^2$, $|\zeta|=1$ where $a_0>0$, $b_0>0$ and $a_0>2b_0$, and imposing a sink of constant strength Q at $z=0$, $\partial\Omega$ is still a limacon, $z=a\zeta+b\zeta^2$, where $a^2 b=a_0^2 b_0$ and $a^2+2b^2=$ $=a_0^2+2b_0-Qt$, provided $a\geqslant 2b$. This can be found from considering the singularities of U_0 which are $\frac{1}{2}\{-(a_0^2+2b_0^2)\ln r+a_0^2 b_0 r^{-1}\cos\theta\}$ and the integrated singularity of u, $\frac{1}{2\pi}Qt\ln r$, so that singularities of U are $\frac{1}{2}\left\{-\left(a_0^2+2b_0^2-\frac{Qt}{\pi}\right)\ln r+a_0^2 b_0 r^{-1}\cos\theta\right\}$. The solution holds up to the time t^* where $a=2b$ when $\partial\Omega$ is a cardioid with a cusp at $z=-b(t^*)$ (see



Fif. 5. Suction at $z = 0$ from the limacon $z = a\zeta + b\zeta^2$, $|\zeta| < 1$.

fig. 5); there is no solution for later times so we say that t^* is the blow-up time.

For any other example in which U has singularities $\frac{1}{2}\{-A\ln r + Br^{-1}\cos\theta\}$ it again follows that $\partial\Omega$ is a cardioid, $z = a\zeta + b\zeta^2$ where $a^2 + 2b^2 = A$ and $a^2 b = B$. If A is too small compared with B, then there are no a, b giving a simple closed curve Γ; blow-up must occur before this time.

II. Off-centre suction from a sphere

Attempting to remove an amount of fluid α through a point A and an amount β through the origin 0 from the unit sphere would produce a final boundary given by first removing the amount β and then the amount α. But this is the effect of applying a point sink at A to an initial region which is a sphere of radius $R = (1 - 3\beta/4\pi)^{1/3}$. Clearly it is impossible to remove fluid through A if a, the distance of A from 0, is greater than R. We deduce that there must be blow-up if $\beta \geqslant \frac{4\pi}{3}(1 - a^3)$.

Again we can consider the singularities associated with the problem. The singularity of U_0 is $1/3\,|\underline{x}|$ while the integrated singularities of u are $\beta/4\pi\,|\underline{x}| + \alpha/4\pi\,|\underline{x} - \underline{a}| = \frac{1}{3}(1 - R^3)|\underline{x}| - \alpha/4\pi\,|\underline{x} - \underline{a}|$ and blow-up must occur if $\alpha > 0$ and $R \leqslant |\underline{a}|$. Any other suction problem giving the same two final singularities must also blow-up if $\alpha > 0$ and $R \leqslant |\underline{a}|$.

The two variables u and U are related by

$$u = -\frac{\partial U}{\partial t}. \tag{33}$$

Indeed in the region that is crossed by the free boundary, $\Omega(t)\backslash\Omega(t^*)$, U is given by a Baiocchi transformation,

$$U(x, t) = \int_t^{\omega(x)} u(x, \tau)\, d\tau, \tag{34}$$

where the free boundary $\partial\Omega(t)$ is given by $t = \omega(x)$.

For expanding regions a similar variable to U can be used to solve the problem by means of variational inequalities [4, 5]. If we take a problem in which fluid enters the region Ω through a closed curve Γ, where Ω lies outside Γ (see fig. 6), then we can define V by

$$V = -\int_\omega^t u\, d\tau, \quad x \text{ in } \Omega\backslash\Omega_0,$$

$$V = -\int_0^t u(x, \tau)\, d\tau, \quad x \text{ in } \Omega_0.$$

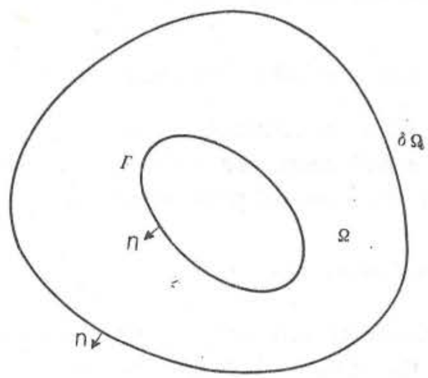


Fig. 6. Injection into the annular region Ω through the fixed boundary Γ.

If the flow rate through Γ is specified, say $\dfrac{\partial u}{\partial n} = q(x, t)$ for x on Γ, then V solves the following problem:

$$\nabla^2 V = 1 \text{ in } \Omega\backslash\Omega_0,$$

$$\nabla^2 V = 0 \text{ in } \Omega_0,$$

$$\frac{\partial V}{\partial n} = Q(x, t) \equiv \int_0^t q(x, \tau)\, d\tau \text{ on } \Gamma,$$

$$V = \frac{\partial V}{\partial n} = 0 \text{ on the free boundary } \partial\Omega.$$

This problem may be reformulated as a variational inequality:

$$V(\nabla^2 V+\chi)=0,\ V\geqslant 0,\ \nabla^2 V+\chi\geqslant 0 \text{ for } \underset{\sim}{x} \text{ outside } \Gamma,$$

where $\chi=0$ in Ω_0 and -1 elsewhere.

If Ω lies inside a fixed boundary Γ and surrounds a bubble (see fig. 7) then a problem with a retreating boundary may also be solved by using a variational inequality.

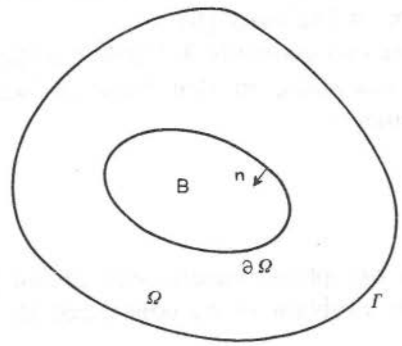


Fig. 7. Suction from the annular region Ω, which surrounds a bubble B, through the outer boundary Γ.

Taking the initial free boundary $\partial\Omega_0$ to be such that U_0, as given by (27–29), has no singularities in Ω_0, and imposing, say, pressure along Γ, $u=-q(\underset{\sim}{x},t)$ on Γ, then U satisfies (31, 33) with

$$\nabla^2 U=1 \text{ in } \Omega,$$

and

$$U=Q\equiv U_0-\int_0^t q\,d\tau \text{ on } F. \tag{35}$$

Supposing that is possible, by suitable choice of $q(\underset{\sim}{x},\tau)\geqslant 0$ for $\tau>t$, to remove all the fluid lying in $\Omega(t)$, then from (33) $U\geqslant 0$; in particular we require that $Q\geqslant 0$. It follows that U solves the variational inequality

$$U(\nabla^2 U-1)=0,\ U\geqslant 0,\ \nabla^2 U\geqslant -1, \text{ for } x \text{ in } B\cup\partial\Omega\cup\Omega \quad \text{(where } B \text{ is the bubble)} \tag{36, 37, 38}$$

together with (35).

For Hele-Shaw problems with retreating boundaries a variety of different types of cusp may occur. If $\partial\Omega$ has a cusp at a point z_0 then there is a singularity of the Schwarz function g az z_0. If g admits a power series expansion with $g=A(z-z_0)+B(z-z_0)^\alpha+o(z-z_0)^\alpha$, about z_0, where $\alpha>1$

taking, without loss of generality, $z_0 = 0$ and the cusp to point in the direction of the x-axis, $y = \pm O(x^\alpha)$ as $x \to 0$ on $\partial\Omega$.

It has been shown that the problem (36–38) can exhibit free boundaries with cusps of power 5/2, 9/2, 13/2, ... but not with cusps of power 3/2, 7/2, 11/2, ... near which U must be negative [11, 25]; a specific example of a problem which can be put in the form (36–38) which exhibit a power 5/2 cusp is given in [26]. This means that if cusps of power 3/2, 7/2, ... appear in the Hele-Shaw problem the solution blows up. Conversely it is possible that if cusps of power 5/2, 9/2, ... are formed the solution can continue to exist with disappearance of the cusp [8].

However it is believed that generally 3/2 power cusps form, as with nearly all the known explicit examples, so that blow-up occurs in almost every instance that a cusp appears.

4. Comparison results

We now turn to the one-phase, superheated Stefan problem to examine criteria for blow-up. The problem to be considered is:

$$\frac{\partial u}{\partial t} = \nabla^2 u \qquad \underset{\sim}{x} \text{ in } \Omega(t), \tag{39}$$

$$u = 0, \; v_n = \frac{\partial u}{\partial n} \qquad \underset{\sim}{x} \text{ on } \partial\Omega(t), \tag{40, 41}$$

$$u = u_0 \geqslant 0 \qquad \underset{\sim}{x} \text{ in } \Omega(0) = \Omega_0. \tag{42}$$

We firstly remark that again we have three possibilities:

(a) the solution exists for all time and $u \to 0$ as $t \to \infty$;

(b) there is some time t^* at which $\mu(\Omega) =$ measure of $\Omega = 0$;

(c) there is "blow-up" at some finite time t^* with $\mu(\Omega(t^*)) > 0$, i.e. the solution does not exists for $t > t^*$ although melting is incomplete at $t = t^*$.

Following the one-dimensional case we again find some conditions for for blow-up. On defining $H(t) = \int_{\Omega(t)} h(\underset{\sim}{x})(u-1)\, d\underset{\sim}{x}$ for some given function h. H is increasing if $\nabla^2 h \geqslant 0$ and ultimately $H \geqslant 0$ for both (a) and (b) if $h \geqslant 0$. Hence (c) must occur if the following condition is satisfied:

$$\text{there exists some } h(\underset{\sim}{x}) \text{ with } h \geqslant 0 \text{ and } \nabla^2 h \geqslant 0 \text{ in } \Omega_0 \text{ such that } \int_{\Omega_0} h(u_0 - 1)\, d\underset{\sim}{x} > 0. \tag{43}$$

We may note that if (b) is to occur than $\int_{\Omega_0} h(u_0 - 1)\, d\underset{\sim}{x} = 0$ for all harmonic functions h. (If $\int_{\Omega_0} (u_0 - 1)\, d\underset{\sim}{x} = 0$ but there is some harmonic such that $\int_{\partial\Omega_0} h(u_0 - 1)\, d\underset{\sim}{x} \neq 0$ then blow-up occurs).

A similar criterion can be applied to the Hele-Shaw problem with point sinks. For positive h with $\nabla^2 h \geqslant 0$,

$$0 \leqslant H(t) \equiv \int_{\Omega} h \, d\underset{\sim}{x} \leqslant H(0) - \sum_{1}^{N} Q_n(t) \, h(\underset{\sim}{X}_n)$$

where there are point sinks of strength $q_n(t)$ at the points $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_N$ and $Q_n(t) = \int_0^t q_n(\tau) \, d\tau$, $n = 1, \ldots, N$. It follows that if $\sum_1^N Q_n(t_1) \, h(\underset{\sim}{X}_n) > H(0)$ for some time t_1 with $\sum_1^N Q_n(t_1) \leqslant \mu(\Omega_0)$ then blow-up must occur at a time $t^* < t_1$. As above, if there is to be complete removal of fluid at $t = t^*$ then $H(t^*) = 0$ and $H(0) = \sum_1^N Q_n(t^*) \, h(\underset{\sim}{X}_n)$ for all harmonic h.

It is easily seen that for the condition (43) for blow-up to hold then u_0 must be somewhere greater than one, as was required for blow-up in the one-dimensional problem. The conditions for blow-up can be improved by comparing the Stefan problem with a suitable Hele-Shaw problem.

Before applying known results about specific Hele-Shaw problems to some specific Stefan examples we shall discuss briefly a problem related to Hele-Shaw flow.

Supposing that the flow between two parallel plates is caused by the plates moving directly towards or apart from each other, then the pressure $-u$, satisfies a free boundary problem of the form

$$\nabla^2 u + f(t) = 0 \qquad \underset{\sim}{x} \text{ in } \Omega(t)$$

together with (24) and (25). f is positive for separating plates and a contracting region Ω.

Taking U to satisfy $\nabla^2 U = 1$ in Ω except at singularities, with (31) and (32), then the new variable $w = u + fU$ satisfies a Hele-Shaw problem with driving singularities just f times those of U. Hence the time derivatives of the singularities of U are $-f$ times the singularities of U and:

$$\text{the singularities of } U \text{ at } t \text{ are } F(t) \times \text{singularities of } U_0 \quad \text{where } F(t) = \exp\left\{-\int_0^t f(\tau) \, d\tau\right\} \tag{44}$$

and U_0 is the solution to (27–29). The free boundary can then be found by solving (30–32) where the known singularities of U are given by (44).

As an example we may take $\partial\Omega_0$ to be a square and $f \equiv 1$. The singularities of U_0 take the form of jumps in normal derivatives across the diagonals of the square. The singularities of U also lie along the diagonals but decay exponentially in time; the vertices of $\partial\Omega$ remain stationary as $\partial\Omega$ retreats towards the diagonals. From time reversability for the Hele-Shaw

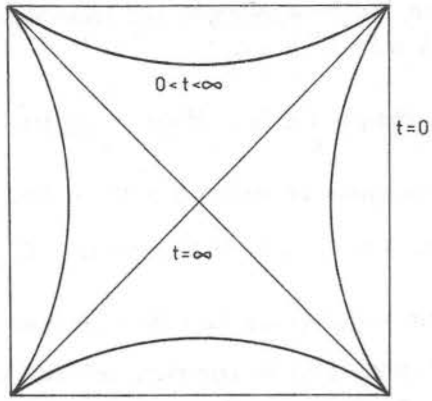


Fig. 8. Separating plates with Ω initially a square.

problem the region Ω at any given time is that formed by applying suitable line sources along the diagonals and starting with an initially empty region (see fig. 8).

Starting with a limaçon, $\partial\Omega_0$ is $z = a_0\zeta + b_0\zeta^2$, $|\zeta| = 1$, the singularities at time t are $\frac{1}{2}F(t)\{-(a_0^2+2b_0^2)\ln r + a_0^2 b_0 r^{-1}\cos\theta\} = \frac{1}{2}\{-(a^2+2b^2)\ln r + a^2 b r^{-1}\cos\theta\}$ so that $\partial\Omega(t)$ is also a limaçon, $z = a\zeta + b\zeta^2$, where $a^2+2b^2 = F(t)(a_0^2+2b_0^2)$ and $a^2 b = F(t)a_0^2 b_0^2$. The solution blows up at a time t^* when $a = 2b$: i.e. t^* satisfies $F(t^*) = 27a_0^4 b_0^2/2(a_0^2+2b_0^2)^3$.

Returning to the Stefan problem we also use a new variable U. We firstly have to solve:

$$\nabla^2 U_0 = 1 - u_0 \text{ in } \Omega_0 \text{ except at its singularities.}$$

$$U_0 = \frac{\partial U_0}{\partial n} = 0 \text{ on } \partial\Omega_0.$$

Then we take U to satisfy:

$$\frac{\partial U}{\partial t} = \nabla^2 U - 1,\ \underset{\sim}{x} \text{ in } \Omega(t) \text{ except at specified singularities which are those of } U_0, \tag{45}$$

$$U = \frac{\partial U}{\partial n} = 0, \quad \underset{\sim}{x} \text{ on } \partial\Omega(t), \tag{46, 47}$$

$$U = U_0 \text{ at } t = 0,\ \underset{\sim}{x} \text{ in } \Omega_0. \tag{48}$$

Again u and U are related by (33) throughout Ω and also by (34) in the region crossed by $\partial\Omega$.

If U is always positive, which is the case for problems with behaviour (b) and possibly for (a) but not (c), then (45–48) is an oxygen diffusion

Suppose now that (b) is to occur. Then $\omega(\underset{\sim}{x})$ is defined and finite for all $\underset{\sim}{x}$ in Ω_0 so U is also finite.

If instead (a) is to occur then as $t \to \infty$, Ω contracts towards some region Ω_∞ and $U \to U_\infty$ where

$\nabla^2 U_\infty = 1$ in Ω_∞ except at its singularities which are those of U_0, (49)

$$U_\infty = \frac{\partial U_\infty}{\partial n} = 0 \text{ on } \partial\Omega_\infty. \tag{50, 51}$$

This is seen to be the same as problem (30–32) used in the solution of the Hele-Shaw problem.

Under special circumstances the properties of U can be related to the condition for blow-up (43). If U_0 has no singularities then (a) cannot occur. Moreover if for some positive h with $\nabla^2 h \geqslant 0$, $0 < \int\limits_{\Omega_0} (u_0 - 1)\, h\, d\underset{\sim}{x} = -\int\limits_{\Omega_0} h \nabla^2 U_0\, d\underset{\sim}{x} = -\int\limits_{\Omega_0} U_0 \nabla^2 h\, d\underset{\sim}{x}$ then U_0 is somewhere negative. Conversely if U_0 is negative for some $\underset{\sim}{x}_0$ in Ω_0 we can take some positive h with $\nabla^2 h > 0$ in a neighbourhood of $\underset{\sim}{x}_0$ where $U_0 < 0$ and $\nabla^2 h = 0$ elsewhere; it then follows that $\int\limits_{\Omega_0} (u_0 - 1)\, h\, d\underset{\sim}{x} > 0$. Of course, as with one dimension, if U has no singularities then from $\frac{\partial U}{\partial t} \leqslant 0$ and $U_0 < 0$ somewhere, neither (a) nor (b) are possible so there is blow-up [9].

So far it has been required that $u > 1$ to give blow-up; now the properties of U will be used to show blow-up even for $u < 1$. It must be first noted that if U_0 is infinite somewhere in Ω_0 then (*b*) cannot take place so the problem satisfies (a) or (c). Moreover if U_0 has singularities such that (49–51) fails to have a solution then no limiting free boundary is possible and (c) must occur. Specific examples of the Hele-Shaw problem can be compared with examples of the Stefan problem to check if a feasible solution $U_\infty, \partial\Omega_\infty$ to (49–51) exists.

III. A problem with $\partial\Omega_0$ a limaçon

As earlier we shall take $\partial\Omega_0$ to be given by $z = a_0 \zeta + b_0 \zeta^2$, $|\zeta| = 1$, where $0 < b_0 < a_0/2$.

We define R and ψ by $\zeta = R \exp(i\psi)$ and choose some smooth positive function $G(R)$ satisfying

$$G(1) = G'(1) = G(0) = G'(0) = G''(0) = G''(1) - 1 = 0,$$

and for a given positive number ε,

$$-\varepsilon \leqslant \{(a_0^2+4b_0^2 \pm 4a_0 b_0)(G''+G'/R) \mp 4a_0 b_0 G/R^2\}/(a_0^2+4R^2 b_0^2 \pm \pm 4a_0 b_0 R) \leqslant 1$$

for $0 < R < 1$ with both signs.

Then we take initial data u_0 with

$$u_0 = c\,[1-\{(a_0^2+4b_0^2)(G''+G'/R)+4a_0 b_0 (G''+ + G'/R - G/R^2)\cos\psi\}/(a_0^2+4b_0^2 R^2+4a_0 b_0 R\cos\psi)$$

so

$$0 \leqslant u_0 \leqslant c\,(1+\varepsilon)$$

and

$$\nabla^2 U_0 = (a_0^2+4b_0^2 R^2+4a_0 b_0 R\cos\psi)(1-u_0).$$

It follows that U_0 has the singularities $\frac{1}{2}(1-c)\{-(a_0^2+2b_0^2)\ln r + + a_0^2 b_0 r^{-1}\cos\theta\}$.

Taking $c<1$, U_0 is infinite at the origin so (*b*) does not occur. But if (a) occurs $\partial\Omega_\infty$ would be a limacon $z=\alpha\zeta+b\zeta^2$ with $a^2+2b^2=(1-c)(a_0^2+2b_0^2)$ and $a^2 b=(1-c)a_0^2 b_0$. For $c>c^* \equiv 1-27a_0^4 b_0^2/2\,(a_0^2+2b_0^2)^3$ no such limacon is possible and blow-up must occur.

By choosing c and ε so that $\varepsilon>0, c>c^*$ and $c\,(1+\varepsilon)<1$ the initial data is smooth, $u_0<1$ throughout Ω_0, but (c) nevertheless occurs.

IV. $\partial\Omega_0$ is the unit sphere in $\mathbf{R}^3$ with an asymmetric initial temperature

We take initial data of the form $u_0=u_1(|\underset{\sim}{x}|)+u_2(|\underset{\sim}{x}-\underset{\sim}{a}|)$ where $0<|\underset{\sim}{a}|<1$. u_1 is chosen to be smooth and decreasing, also $u_1(1)=0$ and $u_1=\gamma$ for $r\leqslant r_0$ for some r_0, γ with $0<\gamma<1, 0<r_0<1$. $u_2(r)$ is also chosen to be smooth, positive for $r<1-|\underset{\sim}{a}|$; $u_2(r)=0$ for $r\geqslant 1-a$ and $\max u_2<1-\gamma$. U has singularities $R^3/3\,|\underset{\sim}{x}|-\alpha/|\underset{\sim}{x}-\underset{\sim}{a}|$ where $\alpha>0$ and $1-\gamma<R^3<1-r_0^3\gamma$.

Again the fact that U is unbounded precludes (b), and (a) is impossible if the initial data is chosen so that $R<|\underset{\sim}{a}|$ (c.f. example II). Again there must be blow-up although u_0 is smooth and everywhere less than one.

For these two examples there is blow-up despite the fact that $u_0<1$. This contrasts with the one-dimensional problem where blow-up can only occur if $u_0>1$ somewhere. It seems that blow-up normally occurs in a quite different way in two or three dimensions to one dimension, probably by the formation of some type of cusp rather like the Hele-Shaw problem. Any stabilising surface effects, such as the surface energy of section one, would prohibit the high curvatures that appear in the boundary near cusps.

Finally it can be noted that in example IV (and also in III if the limaçon is replaced by an initial boundary given by $z=a_0\zeta+b_0\zeta^n$ for arbitrarily

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Niepoprawnie postawione zagadnienia ze swobodną granicą

Praca dotyczy sformułowań i analizy modeli pewnych procesów fizycznych opisywanych zagadnieniami ze swobodną granicą, w których swobodna granica może wykazywać gwałtowne zmiany w czasie i w przestrzeni. Dyskutowane jest podłoże fizyczne zagadnień oraz przeprowadzona zostaje analiza pewnych szczególnych zagadnień typu Stefana.

Некорректные проблемы со свободной границей

Работа касается формулировок и анализа математических моделей некоторых физических процессов описыванных задачами со свободной границей, в которых свободная граница может очень быстро изменятся во времени и в пространстве. Рассуждается физическое обоснование проблем и проводится анализ некоторых специальных задач типа Стефана.

large n) then the initial data can be chosen arbitrarily close to the trivial problem $u_0 = 0$ in Ω_0 = the unit ball with blow-up still obtained.

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