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On hysteresis in phase transitions

by

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Introduction

Hysteresis effects can appear in phase transitions; they correspond to non-equilibrium situations in which memory has to be taken into account. A typical case is that in which a monotone relation between two variables u and w (see fig. 1) is replaced by a hysteresis loop (see fig. 2).

Here the couple $(u(t), w(t))$ goes along the lower part of the loop if u is increasing and along the upper part if u is decreasing. Actually, the path of (u, w) can be more complicated than so; if u is in the critical range $]u_1, u_2[$ and u inverts its movement, then (u, w) moves into the interior of the loop. Though it can be difficult to give a mathematical description of this behaviour, one can assume that the function $u(\cdot)$ determines the function $w(\cdot)$, once the suitable information on the initial state have been provided.

A useful representation of hysteresis effects is given by the classical Preisach model, also known as the "independent domain model" [1, 7, 8, 14, 16, 23, 25, 26, 27, 30, 31, 32, 34, 35, 39, ..., 43, 51]. According to this model, a fairly general class of hysteresis functionals is obtained by "composing" a possibly infinite family of simpler hysteresis functionals of the type sketched in fig. 4.

The corresponding case without hysteresis is the jump relation of fig. 3. For instance, this can represent the temperature-phase dependence for the solid-liquid transition (u is temperature; $w = -1$, i.e. ice, if $u < 0$; $w = 1$, i.e. water, if $u > 0$); in this case, the hysteresis relation accounts for supercooling and superheating effects [49].

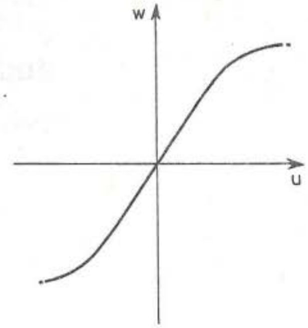

Fig. 1

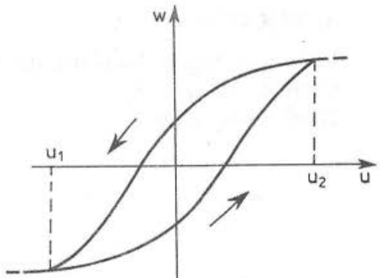

Fig. 2

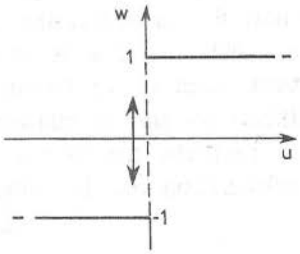

Fig. 3

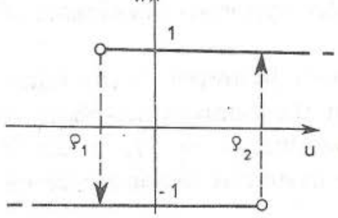

Fig. 4

Other examples of hysteresis phenomena are ferromagnetism (for which the Preisach model was originally introduced) [1, 3, 6, ..., 9, 28, 29, 34, 35, 44, 51, 55, ..., 58] and plasticity [2, 4, 12, 18, 33, 53]; hysteresis appears also in the study of filtration through porous media [14, 27, 30, 31, 32, 39, ..., 43], in biology [15, 50], in chemistry [15, 50] and in other applicative fields.

Mathematicians have not yet devoted much concern to hysteresis, despite of its evident applicative interest. It seems that the only systematic study has been conducted by Krasnosel'skii and Pokrovskii [16, ..., 26, 37, 38]. The present author has studied hysteresis effects in connection with partial differential equations [46, ..., 58], taking into account ferromagnetism in particular.

In this paper the Preisach model is considered jointly with Maxwell's equations for a ferromagnetic slab. In section 1 we give a mathematical construction of the "hysteresis functional" associated with the Preisach model. Then in section 2 a variational formulation of Maxwell's equations for a distributed univariate ferromagnetic system is introduced and the existence of a solution is proved by means of implicit time-discretization, a priori estimates and limit procedure. In section 3 numerical aspects are shortly discussed. We refer to [46, 51] for details. We conclude with a collection of references of applicative and mathematical works on hysteresis.

1. The Preisach model for ferromagnetism

We set $\mathbf{P} \equiv \{(p_1, p_2) \in \mathbf{R}^2 | p_1 < p_2\}$; we shall denote by p a generic couple $(p_1, p_2) \in \mathbf{P}$. For any $p \in \mathbf{P}$ we set

$$s_p(\eta) \equiv \begin{cases} \{-1\} & \text{if } \eta = p_1 \\ \{-1,1\} & \text{if } p_1 < \eta < p_2 \\ \{1\} & \text{if } \eta \geqslant p_2. \end{cases} \tag{1.1}$$

We fix $T > 0$ and set

$$D_p \equiv \{(H, M^0) \in C^0([0, T]) \times \{-1,1\} | M^0 \in s_p(H(0))\};$$

this is the set of the compatible evolutions of H and initial values of M for the "relay" characterized by the thresholds p_1 and p_2. For any $(H, M^0) \in D_p$ we define $M = f_p(H, M^0): [0, T] \to \{-1, 1\}$ as follows (see fig. 4)

$$\begin{cases} M(0) = M^0 \\ \forall t \in]0, T[, \text{ if } H(t) \leqslant p_1 \; (H(t) \geqslant p_2, \textit{respect.}) \\ \textit{then } M(t) = -1 \; (M(t) = 1, \textit{respect.}); \\ M \textit{ jumps from } -1 \textit{ to } 1 \; (\textit{from } 1 \textit{ to } -1, \textit{respect.}) \\ \textit{at time } t \textit{ only if } H(t) = p_2 \; (p_1, \textit{respect.}); \\ \textit{these are the only discontinuities of } M. \end{cases} \tag{1.2}$$

We note that at any instant t, $M(t) = [f_p(H, M^0)](t)$ does not depend on $H|_{]t,T]}$ (property of "*causality*"), that the dependence of M on H is "rate--independent" and that

$$\begin{cases} \forall\, [t', t''] \subset [0, T], \text{ if } H \textit{ is non-decreasing} \\ (\textit{or non-increasing}) \textit{ in } [t', t''], \textit{ so is also } M \end{cases} \tag{1.3}$$

(property od "*piecewise monotonicity*"). Notice that $f_p(., M^0)$ is not monotone, in the sense that

$$\begin{cases} (H_i, M^0) \in D,\ M_i = f_p(H_i, M^0)\ (i = 1, 2) \\ \textit{does not imply } \int_0^T (M_1 - M_2)\cdot(H_1 - H_2)\, dt \geq 0. \end{cases} \tag{1.4}$$

Now let μ be a positive, finite, complete measure over $\mathbf{P}$. By S we shall denote the family of μ-measurable functions $\mathbf{P} \to \{-1, 1\}$; for these we shall use notations of the type of $\{M_p^0\}$. We also set

$$D \equiv \{(H, \{M_p^0\}) \in C^0([0, T]) \times S \,|\, M_p^0 \in s_p(H(0))\ \mu\text{-a.e. in } \mathbf{P}\};$$

this is the set of all compatible evolutions of H and initial states M_p^0 of the relays. We introduce the *Preisach hysteresis functional* $\mathbf{F}$ associated with μ:

$$\forall\, (H, \{M_p^0\}) \in D,\ \forall t \in [0, T], \quad [\mathbf{F}(H, \{M_p^0\})](t) \equiv \int_{\mathbf{P}} [f_p(H, M_p^0)]\, d\mu_p. \tag{1.5}$$

Notice that also this functional is *casual, rate-independent* and *piecewise monotone*, since so are the f_p's; also $\mathbf{F}$ is not monotone (in the sense of (1.4)).

At any instant t the magnetic state is characterized by $H(t)$ and by the *internal variables* $\{M_p(t)\} = \{[f_p(H, M_p^0)](t)\}$; in general the latter contain more information then the macroscopic variable $M(t) = \int_{\mathbf{P}} M_p(t)\, d\mu_p$.

PROPOSITION. If

$$\begin{cases} \mu \textit{ has no masses concentrated either in points or along} \\ \quad \textit{segments parallel the axes,} \end{cases} \tag{1.6}$$

then

$$\forall\, (H, \{M_p^0\}) \in D,\ \mathbf{F}(H, \{M_p^0\}) \in C^0([0, T]) \tag{1.7}$$

$$\begin{cases} \forall \textit{ sequence } \{(H_n, \{M_p^0\})\}_{n\in\mathbf{N}} \subset D, \text{ if } H_n \to H \textit{ uniformly in } [0, T] \\ \textit{then } \mathbf{F}(H_n, \{M_p^0\}) \to \mathbf{F}(H, \{M_p^0\}) \textit{ uniformly in } [0, T]. \end{cases} \tag{1.8}$$

For the proof we refer to section 1 of [51].

The geometric representation of the internal state $\{M_p(t)\}$ in the "Preisach plane" $\mathbf{P}$ allows to recognize further properties for $\mathbf{F}$ (see [7, 8, 51]). Indeed the subset of $\mathbf{P}$ where $M_p = -1$ and that where $M_p = 1$ are separated by an antimonotone graph, if so they are at $t = 0$;

this graph contains the whole information of the internal state. Moreover there are simple rules for modifying this graph according to the evolution of H; this yields useful computational procedures.

A major question is the identification of the measure μ.

A more detailed analysis of the mathematical properties of the Preisach model can be found in [51]. Generalizations to the vectorial case have been proposed by Damlamian and the present author in [5]; of course the key point is the generalization of the "relay" functional (1.2).

2. Study of Maxwell's equations in a ferromagnetic body

We shall deal with the elctromagnetic evolution of an isolated ferromagnetic slab represented by a segment $[a, b] \subset \mathbf{R}$. We consider Maxwell's equations neglecting displacement-current and assume Ohm's law; using suitable measure units, we have

$$\sigma \frac{\partial B}{\partial t} - \frac{\partial^2 H}{\partial x^2} = f \text{ in } Q \equiv]a, b[\times]0, T[, \tag{2.1}$$

where σ denotes the conductivity and f is a datum. We shall give a variational formulation.

We assume that a.e. in Ω the initial field $H^0(x)$ and the initial values of the internal variables $\{M_p^0(x)\}$ are given and fulfill the compatibility condition

$$M_p^0(x) \in s_p\left(H^0(x)\right) \ \mu\text{-a.e. in } \mathbf{P}, \text{ a.e. in }]a, b[.$$

This yields the initial field $B^0(x) \equiv \mu_0 H^0(x) + 4\pi \int_{\mathbf{P}} M_p^0(x)\, d\mu_p$ a.e. in Ω. We also assume that

$$B^0 \in W^{1,2}(]a, b[)', \ f \in L^2(Q) \tag{2.2}$$

and that $\mathbf{F}$ is a Preisach hysteresis functional as introduced in section 1. We introduce a variational problem:

(P) Find $H \in L^2(0, T; W^{1,2}(]a, b[))$ such that

$$H(x, \cdot) \in C^0([0, T]), H(x, 0) = H^0(x) \text{ a.e. in }]a, b[\tag{2.3}$$

and such that, setting

$$M(x, t) \equiv \left[\mathbf{F}\left(H(x, \cdot), \{M_p^0(x)\}\right)\right](t)\ \forall t \in [0, T], \text{ a.e. in }]a, b[. \tag{2.4}$$

$$B \equiv \mu_0 H + 4\pi M \text{ a.e. in } Q, \tag{2.5}$$

then $B \in W^{1,2}(0, T; W^{1,2}(]a, b[)')$ and

$$\sigma \frac{d}{dt} \int_a^b B \cdot v dx + \int_a^b \frac{\partial H}{\partial x} \cdot \frac{\partial v}{\partial x} dx = \int_a^b f \cdot v dx$$

$$\forall v \in W^{1,2}(]a, b[), \ a.e. \ in \ ]0, T[\qquad (2.6)$$

$$B(\cdot, 0) = B^0 \ in \ W^{1,2}(]a, b[)'. \qquad (2.7)$$

Note that (2.6) yields $B \in W^{1,2}(0, T; W^{1,2}(]a, b[)')$; this gives a meaning to (2.7). The equation (2.6) corresponds to (2.1) and to the boundary condition $\frac{\partial H}{\partial x} = 0$ for $x = a, b$.

THEOREM. Assume that (1.6) and (2.2) hold. Then for any $m \in \mathbf{N}$ the approximate problem corresponding to an implicit time discretization with time-step $k \equiv \frac{T}{m}$ $(m \in \mathbf{N})$ has one and only one solution, denoted by $H_m(x, t)$. Moreover there exists an H such that, possibly taking a subsequence,

$$H_m \to H \ weakly \ star \ in \ W^{1,2}(0, T; L^2(]a, b[)) \cap \cap L^\infty(0, T; W^{1,2}(]a, b[)). \qquad (2.8)$$

Finally such an H is a solution of problem (**P**).

Sketch of the proof (see section 2 of [51]) for details. (i) *Approximation.* (**P**$_m$) Find $H_m^n \in W^{1,2}(]a, b[)$ for $n = 1, \ldots, m$, such that, setting

$$H_m(x, t) \equiv linear \ interpolate \ of \ \{H_m^n(x)\}_{n=0,\ldots,m} \ (H_m^0 \equiv H^0),$$

$$a.e. \ in \ ]a, b[\qquad (2.9)$$

$$M_m^n(x) \equiv [\mathbf{F}(H_m(x, \cdot), \{M_p^0(x)\})](nk) \ a.e. \ in \ ]a, b[\qquad (1.10)$$

$$B_m^n(x) \equiv \mu_0 H_m^n(x) + 4\pi M_m^n(x) \ a.e. \ in \ ]a, b[, \qquad (2.11)$$

then

$$\frac{\sigma}{k} \int_a^b (B_m^n - B_m^{n-1}) \cdot v dx + \int_a^b \frac{\partial H_m^n}{\partial x} \cdot \frac{\partial v}{\partial x} dx$$

$$= \int_a^b f_m^n \cdot v dx \quad \forall v \in W^{1,2}(]a, b[), \ n = 1, \ldots, m, \qquad (2.12)$$

where $f_m^n(x) \equiv \frac{1}{k} \int_{(n-1)k}^{nk} f(x, t) \, dt \quad a.e. \ in \ ]a, b[.$

For any m (**P**$_m$) can be solved step by step in time. Let us fix a generic $n \in \{1, \ldots, m\}$ and assume that $H_m^1, \ldots, H_m^{n-1}$ are known; then by (2.10) M_m^n depends just on H_m^n, due to the causality of **F**; i.e. (say)

$$M_m^n(x) = \Phi_m^n(H_m^n(x), x), \ a.e. \ in \ ]a, b[. \qquad (2.13)$$

Moreover, by the "piecewise monotonicity" of $\mathbf{F}$, $\Phi_m^n(\cdot, x)$ is monotone (notice that $H_m(\cdot, x)$ is linear in $[(n-1)k, nk]$, hence it is monotone either non-decreasing or non-increasing, a.e. in $]a, b[$).

(ii) *A priori estimates* — Let us take $v = H_m^n - H_m^{n-1}$ in (2.12) and sum in n. By the monotonicity of $\Phi_m^n(\cdot, x)$ we have

$$\frac{1}{k}\int_a^b (B_m^n - B_m^{n-1})\cdot(H_m^n - H_m^{n-1})\,dx \geqslant \frac{\mu_0}{k}\,\|H_m^n - H_m^{n-1}\|^2_{L^2(]a,b[)}; \qquad (2.14)$$

then by standard calculations we get

$$\|H_m\|_{W^{1,2}(0,T;L^2(]a,b[))\cap L^\infty(0,T;W^{1,2}(]a,b[))} \leqslant \textit{Constant independent of } m. \qquad (2.15)$$

Moreover of course

$$\|M_m\|_{L^\infty(Q)} \leqslant M_{sat.} \equiv \mu(\mathbf{P}). \qquad (2.16)$$

(iii) *Limit procedure* — By the previous estimates there exist H, M such that, possibly taking subsequences,

$$H_m \to H \textit{ weakly star in } W^{1,2}(0, T; L^2(]a, b[)) \cap L^\infty(0, T; W^{1,2}(]a, b[)) \qquad (2.17)$$

$$M_m \to M \textit{ weakly star in } L^\infty(Q). \qquad (2.18)$$

Taking $m \to \infty$ in (2.12) we get (2.6). Since the inclusion of $W^{1,2}(0, T; L^2(]a, b[)) \cap L^\infty(0, T; W^{1,2}(]a, b[))$ into $L^2(]a, b[; C^0([0, T]))$ is compact, possibly extracting a further subsequence we have

$$H_m(x, \cdot) \to H(x, \cdot) \textit{ uniformly in } [0, T], \textit{ a.e. in }]a, b[; \qquad (2.19)$$

then by (1.8) we obtain

$$\mathbf{F}(H_m(x, \cdot), \{M_p^0(x)\}) \to \mathbf{F}(H(x, \cdot), \{M_p^0(x)\}) \quad \textit{uniformly in } [0, T], \textit{ a.e. in }]a, b[. \qquad (2.20)$$

and this yields (2.4). ■

REMARK. The uniqueness of the solution of problem (**P**) is an open question. We note that it does not seem straightforward, as **F** is not monotone (cf. (1.14)).

3. Numerical Results

In the proof of the previous existence result, we considered just a time-discretization; however for any m ($\mathbf{P_m}$) corresponds to a family of m elliptic problems which can be numerically solved by standard space-

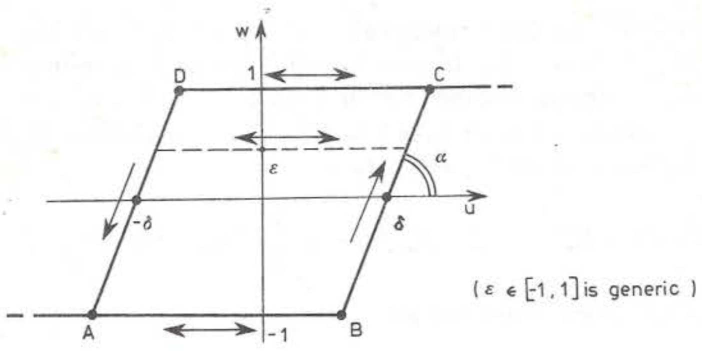

Fig. 5

-discretization procedures. In [46] Verdi and the present author considered especially simple hysteresis functionals of the type sketched in fig. 5. Along $\overline{BC}$ the couple (u, v) can only move upwards, along $\overline{AD}$ it can only move downwards; in the interior of the loop it can move horizontally in both directions. This corresponds to a measure μ with total mass 1 and support along a segment lying on a straight-line of the form $p_2 - p_1 = 2\delta$ (δ: *const* > 0). Note that in this particular case there is no internal memory: the ferromagnetic state is completely described by the couple $(H(x,t), M(x,t))$.

The problem obtained by discretization in space and time can be easily solved by the non-linear Gauss-Seidel method, of which one can prove the convergence. Several numerical tests have lead to the following conclusions (see [46]): the approximate solutions converge and costs and errors do not depend on the slope α of the hysteresis loop; however costs increase as the "amount of hysteresis" δ increases, whereas errors are insensitive to this parameter.

References

[1] Biorci G., Pescetti D. Analytical theory of the behaviour of ferromagnetic materials. *Il Nuovo Cimento*, **7** (1958), 6, 829–839.

[2] Bouc R. Modèle mathématique d'hystérésis. *Acustica*, **24** (1971), 16–25.

[3] Bouc R. Solution périodique de l'équation de la ⟨ferrorésonance⟩ avec hystérésis. *C.R. Acad. Sc. Paris*, **263** (1966), Série A, 497–499.

[4] Bouc R. Modèle mathématique d'hystérésis et application aux systèmes à un degré de liberté. Thèse, Marseille (1969).

[5] Damlamian A., Visintin A. Une généralization vectorielle du modèle de Preisach pour l'hystérésis. *C.R. Acad. Sc. Paris*, **297** (24 oct. 1983), Série I, 437–440.

[6] Del Vecchio R. The inclusion of hysteresis processes in a special class of electromagnetic finite element calculations. *I.E.E.E. Trans. on Magn.*, **MAG-18** (1982) 1, 275–284.

[7] ENDERBY J. A. The domain model of hysteresis. Part 1: Independent domains; Part 2: Interacting domains. *Trans. Faraday Soc.* **51** (1955), 835–848; **52** (1956), 102–120.

[8] EVERETT D. H. A general approach to hysteresis. Part 1 (with W. J. WHITTON); Part 2 (with F. H. SMITH): Development of the domain theory; Part 3: A formal treatment of the independent domain model of hysteresis; Part 4: An alternative formulation of the domain model. *Trans. Faraday Soc.* **48** (1952), 749–757; **50** (1954), 187–197; **50** (1954), 1077–1096; **51** (1955), 1551–1557.

[9] FABRIZIO M., GIORGI C. L'isteresi per un sistema elettromagnetico. *Fisica Matematica Suppl. B.U.M.I.*, **1** (1981), 33–45.

[10] GLASHOFF K., SPREKELS J. An application of Glicksberg's theorem to set-valued integral equations arising in the theory of thermostats. *S.I.A.M. J. Math. An.*, **3** (1981), 477–486.

[11] GLASHOFF K., SPREKELS J. The regulation of temperature by thermostats and set-valued integral equations. *J. Int. Equ.* **4** (1982), 95–112.

[12] GUELIN P. Remarques sur l'hystérésis mécanique. *J. de Méc.*, **19** (1980) 2, 217–247.

[13] HORNUNG U. Mathematical models for hysteresis effects. *Zeitschr. Angew. Math. Mech.*, **63** (1983), 324–325.

[14] HORNUNG U. The mathematics of hysteresis. *Bull. Australian Math. Soc.*

[15] JÄGER W. A diffusion-reaction system modelling spatial patterns. Equadiff Bratislava, 1981 Teubner Texte zur Math. 47 (1982).

[16] KRASNOSEL'SKIĬ M. A. Equations with non-linearities of hysteresis type (Russian) VII Int. Konf. Nichtlin. Schwing., Berlin 1975; Abh. Akad. Wiss. DDR, 3 (1977), 437–458. (English abstract in Zentr. fur Math. 406-93032).

[(7] KRASNOSEL'SKIĬ M. A., POKROVSKIĬ A. V. Operators representing non-linearities of hysteresis type (Russian). In "Theory of operators in functional spaces", ed. G. P. AKILOV, Nauka (1977).

[18] KRASNOSEL'SKIĬ M. A. et al. Hysterant operator. *Soviet Math. Dokl.*, **11** (1970) 1, 29–33.

[19] LIFŠIC E. A., KRASNOSEL'SKIĬ M. A., ZABREĬKO P. P. An elastic-plastic element oscillator. *Soviet Math. Dokl.*, **11** (1970) 1, 73–75.

[20] KRASNOSEL'SKIĬ M. A., POKROVSKIĬ A. V. Vibrostability of solutions of differential equations. *Soviet Math. Dokl.*, **11** (1970) 6, 1510–1512.

[21] KRASNOSEL'SKIĬ M. A., POKROVSKIĬ A. V. Systems of hysterons. *Soviet Math. Dokl.*, **12** (1971) 5, 1388–1391.

[22] KRASNOSEL'SKIĬ M. A., KOZJAKIN V. S., POKROVSKIĬ A. V. Vibrationally stable hysterons. *Soviet Math. Dokl.*, **13** (1972) 5, 1305–1309.

[23] KRASNOSEL'SKIĬ M. A., POKVROSKIĬ A. V. Periodic oscillations in systems with relay nonlinearities. *Soviet Math. Dokl.*, **15** (1974) 3, 873–877.

[24] KRASNOSEL'SKIĬ M. A., POKROVSKIĬ A. V. Regular solutions of equations with discontinuous nonlinearities. *Soviet Math. Dokl.*, **17** (1976) 1, 128–132.

[25] KRASNOSEL'SKIĬ M. A. POKROVSKIĬ A. V. Modelling transducers with hysteresis by means of continuous systems of relays. *Soviet Math. Dokl.*, **17** (1976) 2, 447–451.

[26] KRASNOSEL'SKIĬ M. A., POKROVSKIĬ A. V. Systems with hysteresis. (Russian) Nauka, Moscow (1983).

[27] LEES S. J., WATSON K. K. The use of a dependent domain model of hysteresis in numerical soil water studies. *Water Resour. Res.* **11** (1975), 943–948.

[28] MADELUNG E. Über Magnetisierung durch schnellverlaufende Ströme und die Wirkungsweise des Rutherford. Marconischen Magnetdetektors. *Ann. der Phys.*, **17** (1905), 861–890.

[29] MAYERGOYZ I. Theoretical investigation of the nonlinear skin effect. *Archiv für Elektrotechnik*, **64** (1981), 153–162.

[30] MUALEM Y. Modified approach to capillary hysteresis based on a similarity hypothesis. *Water Resour. Res.* **9** (1973), 1324–1331.

[31] MUALEM Y. A conceptual model of hysteresis. *Water Resour. Res.* **10** (1974), 514–520.

[32] MUALEM Y. Theory of universal hysteretical properties of unsaturated porous media. Proc. Fort Collins Fluid Intern. Hydrol. Symp. 1977, in Water Resour. Res. Public., 1979, XIV.

[33] MULLER I., VILLAGGIO P. A model for an elastic-plastic body. *Arch. Rat. Mech. An.*, **65** (1977), 25–46.

[34] NEEL L. Téorie des lois d'aimantation de Lord Rayleigh. *Cahiers de Physique*, **12** (1942), 1–20.

[35] NELL L. Théorie des lois d'aimantation de Lord Rayleigh. *Cahiers de Physique*, **13** (1943), 18–30.

[36] OWEN D. R., WILLIAMS W. O. On the concept of rate-independence. *Quat. Appl. Math.*, **26** (1968) 3, 321–329.

[37] POKROVSKIĬ A. V. On the theory of hysteresis nonlinearities. *Soviet Math. Dokl.*, **14** (1973) 3, 896–900.

[38] POKROVSKIĬ A. V. Nonlocal extendibility of solutions of vibrostable differential equations. *Soviet Math. Dokl.*, **14** (1973) 1, 267–271.

[39] POULOVASSILIS A. The effect of the entrapped air on the hysteresis curves of a porous body and on its hydraulic conductivity. *Soil Science*, **109** (1970), 154–162.

[40] POULOVASSILIS A., CHILDS E. C. The hysteresis of pure water: the non-independence of domains. *Soil Science* **112** (1970), 301–312.

[41] POULOVASSILIS A., EL-GHAMRY W. M. The dependent domain theory applied to scanning curves of any order in hysteretic soil water relationships. *Soil Science* **126** (1978), 1–8.

[42] POULOVASSILIS A., TZIMAS E. The hysteresis in the relationship between hydraulic conductivity and suction. *Soil Science* **117** (1974), 250–256.

[43] POULOVASSILIS A., TZIMAS E. The hysteresis in the relationship between hydraulic conductivity and soil water content. *Soil Science* **120** (1975), 327–331.

[44] PREISACH E. Über die magnetische Nachwirkung. *Z. für Physik*, **94** (1935), 277–302.

[45] TOLMACHEV S. T. Calculating quasi-static fields in media with hysteresis. *Izv. Akad. Nauk SSSR. Energetika i Transport.*, **18** (1980) 1, 105–118.

[46] VERDI C., VISINTIN A. Numerical approximation of hysteresis problems (to appear in *I.M.A. J. Num. Anal.*).

[47] VISINTIN A. Hystérésis dans les systèmes distribués. *C. R. Acad. Sc.* Paris, **293** (14 déc. 1981), 625–628.

[48] VISINTIN A. A model for hysteresis of distributed systems. *Ann. Mat. Pura Appl.*, **131** (1982), 203–231.

[49] VISINTIN A. A phase transition problem with delay. *Control and Cybernetics*, **11** (1982), 5–18.

[50] VISINTIN A. Evolution problems with hysteresis in the source term. (to appear in *S.I.A.M. J. Math. Anal.*).

[51] VISINTIN A. On the Preisach model for hysteresis. *Nonlinear Analysis*, **9** (1984), 977–996.

[52] VISINTIN A. On variable hysteresis operators. *Boll. Un. Mat. Ital.* **3-B** (1984), 837–857.

[53] VISINTIN A. Rheological models and hysteresis effects. (to appear in *Rend. Sem. Mat. Fis. Univ. Padova*).

[54] VISINTIN A. Continuity properties of a class of hysteresis functionals. *Atti Sem. Mat. Fis. Modena* **32** (1983), 232-247.

[55] VISINTIN A. On the evolution of ferromagnetic media. *Math. Modelling.* **5** (1984), 43–61.

[56] VISINTIN A. Partial differential equations with hysteresis functionals Proceedings of the 6th Intern. Confer. on Computing Methods in Applies Sciences and Engineering, Versailles (1983).

[57] VISINTIN A. On Landau-Lifshitz' equations for ferromagnetism *Jap. J. Appl. Math.* **2** (1985), 69–84.

[58] VISINTIN A. On some microscopic models of ferromagnetism. (to appear in *Rend. Sem. Mat. Fis.* Torino.).

Efekt histerezy w przemianach fazowych

Efekty histerezy mogą towarzyszyć przemianom fazowym, w których pamięć gra istotną rolę w stanach nierównowagi. Artykuł dotyczy klasycznego modelu Preisacha znanego też jako tzw. model obszarów niezależnych. Pokazane zostaje zastosowanie modelu Preisacha do opisu ewolucji ferromagnetycznych przemian fazowych w układach izolowanych.

Эффект гистерезиса в фазовых переходах

Эффекты гистерезиса могут соответствовать фазовым переходам, в которых память является существенным элементом в неравновесных состояниях. В работе обсуждается классическая модель Прейсаха (или модель независимых областей). Эту модель применяется к описанию эволюции ферро-магнетных фазовых переходов в изолированных системах.