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Analysis of two identification algorithms of stochastic approximation type

by

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Two stochastic approximation type algorithms applied to parameter idenuification of a single input-single output dynamic system are analysed in this paper. The corrections at time k are computed knowing, what is belived, to be the gradient direction of a given quality function. Ljung's ordinary differential equation method is applied to the algorithms analysis. The algorithms are proved to converge with probability 1. Computer simulation results are presented and the qualitative analysis of algorithms behaviour is made.

1. Introduction

This paper analyses two stochastic approximation type algorithms applied to parameter identification of a dynamic system. The identification problem involves minimizing the expectation of a quadratic function

$$W\left(\varepsilon\left(\tilde{\theta}\,(k)\right)\right)=\varepsilon^T\left(\tilde{\theta}\,(k)\right)\,\Lambda\varepsilon\left(\tilde{\theta}\,(k)\right),$$

where the output prediction error $\varepsilon\,(\cdot)$ is a function of the estimation error $\tilde{\theta}\,(k)=$ $=\theta\,(k)-\hat{\theta}$, $\hat{\theta}$ is an unknown parameter vector and Λ is a weighting matrix.

The corrections at time k are made in a "stochastic gradient" direction, denoted by $\eta\,(k)$.

The first algorithm's search direction at time k is simply $-\eta\,(k)$, which gives

$$(*)\qquad \theta\,(k+1)=\theta\,(k)-\tau\,(k)\,\eta\,(k+1), \tag{1}$$

where $\tau\,(k)>0$ is a stepsize coefficient.

The second algorithm's search direction at time k is found by filtering the directions negative to those of the "stochastic gradients" $\eta\,(i)$ $i=0, 1, 2, \ldots, k$. That gives the algorithm

$$(**)\qquad \begin{cases} \theta\,(k+1)=\theta\,(k)+\tau\,(k)\,d\,(k) \\ d\,(k+1)=d\,(k)+\alpha\cdot\tau\,(k)\left(-\eta\,(k+1)-d\,(k)\right) \end{cases} \tag{2}$$

where $\tau(k)>0$ denotes the stepsize and $p=\alpha\cdot\tau(k)\in(0,1)$ is the coefficient of filtering.

Algorithms (1) and (2) will be further called algorithm with no filtering and algorithm with filtering, or shortly: algorithm (*) and algorithm (**), respectively.

Let the sequence of stepsize coefficients $\{\tau(k)\}$ satisfy the conditions

$$\begin{aligned}
&\tau(k+1)\leqslant\tau(k), \text{ for } k=1,2,\ldots \\
&\sum_{i=1}^{\infty}\tau(i)=\infty, \\
&\sum_{i=1}^{\infty}\tau^{p}(i)<\infty, \text{ for some } p>0, \\
&\limsup_{k\to\infty}|1/\tau(k+1)-1/\tau(k)|<\infty.
\end{aligned} \tag{3}$$

We shall prove that, if $\{\tau(k)\}$ fulfils (3), then the sequence $\{\theta(k)\}$ generated by the algorithms (1) and (2) converge to $\hat{\theta}$ wpl (with probability 1).

This is a generalization of the well-known results [1] and [2]) concerning the convergence of stochastic approximation algorithms in the case when the sequence $\{\tau(k)\}$ satisfies some stronger assumptions for example $\tau(k)$ decreases in a way that yields $\sum_{i=1}^{\infty}\tau(i)=\infty$ and $\sum_{i=1}^{\infty}\tau^2(i)<\infty$, or simply $\tau(k)=1/k$, $k=1,2,\ldots$.

Convergence of algorithm (1) when the stepsize sequence satisfies (3) is discused in [6]. We are not aware of analogous results for algorithm (2).

The ordinary differential equation method is reviewed in section 2 and then applied to prove the algorithms convergence.

In many practical applications of algorithms (1) and (2) (in tracking problems, for instance, when $\hat{\theta}$ is a function of k, $\tau(k)$ is chosen to have a small constant value) so that conditions (3) cannot be fulfilled. It is known that many properties of the algorithms can be also in that case studied with the ODE method.

Algorithms (1) and (2) with constant stepsize coefficients were simulated in a few simple cases. Simulation results made the comparison of the algorithms possible.

2. Ordinary diferential equation method

We shall now present the method of Ljung [6] for studying asymptotic properties of stochastic algorithms. Instead of a given stochastic algorithm we can deal with an ordinary differential equation (ODE) associated with the algorithm. The trajectory of ODE approximates the path of the stochastic algorithm (if, of course, some assumptions on the algorithm hold).

Let us consider the general case of the stochastic approximation algorithm

$$x(k+1)=x(k)+\tau(k+1)\,Q\big(k,x(k),\varphi(k+1)\big). \tag{4}$$

Here the sequence of N-dimensional vectors $\{x(k)\}$ is intended to converge to the unknown vector x, $\tau(k)>0$ is a stepsize $Q(k, x(k), \varphi(k+1))$ is the "stochastic gradient" $\varphi(k)$ is the m-dimensional vector of observations usually input and output of the system. $\varphi(k)$ is generated by

$$\varphi(k+1)=A(x(k))\,\varphi(k)+B(x(k))\,e(k+1), \tag{5}$$

where A and B are matrices of the appropriate dimensions $e(k)$ is the white noise uncorrelated with $\varphi(k)$. Such algorithms often appear in identification or adaptive control.

Since $Q(k, x(k), \varphi(k+1))$ is a random vector (with nonzero variance) the following condition is needed for the algorithm's convergence

$$\tau(k)\to 0 \quad \text{as} \quad k\to\infty. \tag{6a}$$

In tracking problems (when x becomes a function of k) condition (6a) do not imply the convergence, so we need

$$\tau(k)\to\tau>0 \quad \text{as} \quad k\to\infty. \tag{6b}$$

Let $x(k)\to\hat{x}(k)$ wpl, and let

$$\tau(k)=a\cdot k^{-\beta} \quad 0<\beta\leqslant 1. \tag{6c}$$

Then (under some other assumptions) the process

$$\mu(k)=(x(k)-\hat{x}(k))/\sqrt{\tau(k)}$$

converges weakly to a Gaussian diffusion [4].

It is known [6] that some properties of the algorithm (4), (5) (if condition (6a) and some other assumptions on the function $Q(\cdot)$ hold) such as: convergence wpl, stationary points and asymptotic behaviour of the algorithm, can be deduced from the analysis of the deterministic differential equation associated with the algorithm

$$\frac{d}{dt}X^D(t)=f(X^D(t)), \tag{7}$$

where

$$f(x)\stackrel{\text{def}}{=}\lim_{k\to\infty} E\{Q(k, x, \varphi)\}+o(t),\ \lim_{t\to\infty} o(t)/t=0$$

The proofs of the theorems giving the precise relationships between the algorithm (4), (5) and its ODE need some assumptions on the functions $Q(\cdot)$, $A(\cdot)$, $B(\cdot)$ and the noise $e(\cdot)$ [6]. Some of them are technical but do not limit the applicability of the method.

Let $s(A)$ denote the spectrum of matrix A. Define

$$D_s=\{x\colon \lambda\in s(A(x))\Rightarrow|\lambda|<1\}.$$

Let now take $\bar{x}\in D_s$. We define random variables $\bar{\varphi}(k, \bar{x})$ and $v(k, \lambda, c)$, $\lambda<1$, c is a constant

$$\bar{\varphi}(k+1, \bar{x})=A(\bar{x})\,\bar{\varphi}(k, \bar{x})+B(\bar{x})\,e(k+1),\ \bar{\varphi}(0, \bar{x})=0 \tag{8}$$

$$v(k+1, \lambda, c)=\lambda v(k, \lambda, c)+c\,|e(k+1)|,\ v(0, \lambda, c)=0 \tag{9}$$

Let D_R denote an open subset of D_S. The following regularity conditions [6] are assumpted to be satisfied by the algorithm (4), (5) for all $x \in D_R$ ($S(\bar{x}, \rho)$ denotes ρ-neighbourhood of $\bar{x}$).

A1: $\{e(\cdot)\}$ is a sequence of independent random variables,

A2: $\{|e(k)|^p\}$ exists and is bounded for all $p>1$.

A3: $Q(k, \cdot, \cdot)$ is locally Lipschitz continuous, i.e. for any $x \in D_R$ and φ there exist $\rho(x)>0$, $v>0$ such that $|Q(k, x_1, \varphi_1)-Q(k, x_2 \varphi_2)|<K_1(x, \varphi, \rho, v)\{|x_1+ -x_2|+|\varphi_1-\varphi_2|\}$ for all $x_1 \in S(x, \rho)$ and $\varphi_i \in S(\varphi, v)$.

A4: For all $x \in D_R$, φ, $\rho>0$, $v>0$ and $w>0$ $|K_1(x, \varphi_1, \rho, v_1)-K_1(x, \varphi_2, \rho, v_2)|< <K_2(x, \varphi, \rho, v, w)\cdot\{|\varphi_1-\varphi_2|+|v_1-v_2|\}$ for $\varphi_i \in S(\varphi, w)$ and $v_i \in S(v, w)$.

A5: $A(\cdot)$ and $B(\cdot)$ are Lipschitz continuous on D_R.

A6: $f(\bar{x})=\lim\limits_{k\to\infty} E\{Q(k, \bar{x}, \varphi(k, \bar{x}))\}$ exists for all $\bar{x} \in D_R$.

A7: For $x \in D_R$ the following random variables $Q(k, x, \bar{\varphi}(k, x))$, $K_1(x, \bar{\varphi}(k, x)$, $\rho(x), v(k, \lambda, c))$ and $K_2(x, \bar{\varphi}(k, x), \rho(x), v(k, \lambda, c), v(k, \lambda, c))$ have p-moments bounded for all $p>1$ and all $\lambda<1$, $c<\infty$.

A8 : $\sum\limits_{i=1}^{\infty} \tau(i)=\infty$.

A9 : $\sum\limits_{i=1}^{\infty} \tau^p(i)<\infty$ for some $p>0$.

A10: $\tau(\cdot)$ is a decreasing sequence.

A11: $\limsup\limits_{k\to\infty} |1/\tau(k+1)-1/\tau(k)|<\infty$.

Let $\bar{D}$ denote a compact subset of D_R such that the trajectories of (7) that start in $\bar{D}$ remain in some closed subset $\bar{D}_R$ of D_R for $t>0$.

A12: In the set of all realizations of process (5) there exists a subset Ω ($P(\Omega)=1$), and random variable $C(\omega)$ such that for all $\omega \in \Omega$ $x(k) \in \bar{D}$ and $|\varphi(k)|<C(\omega)$ io (infinitely often) with k.

A13: The set of stationary points of (7), denoted by D_C, has its domain of attraction D_A such that $\bar{D} \subset D_A$.

Theorems giving the precise relationships between the algorithm (4), (5) and its ODE (equation (7)) are given in [7]. They can be shortly formulated in the following way.

THEOREM 1 (Ljung). *If algorithm* (4), (5) *satisfies* A1-A13 *then*

1° $x(k)$ *can converge only to stable stationary points of* (7).

2° *If* $x(\cdot)$ *belongs to the domain of attraction of a stable stationary point* $x^* \in D_C$ *then* $x(k)$ *converges to* x^* *wpl as* $k\to\infty$.

It is also known [6] that the trajectory of (7) approximates, in a certain sense the asymptotic path of $x(k)$ generated by the algorithm (4), (5).

Conditions A8 ÷ A11 concerning the stepsize sequence $\{\tau(k)\}$ limit the applicability of the ODE method. They are satisfied, for example, by sequence (6c). But any sequence fulfilling (6b) does not satisfy them.

In [5] asymptotic properties of $x(k)$ in the case $\tau(k)=\tau=\text{const}$ are considered. It is proved that if the ODE associated with the algorithm (4), (5) is globally asymptotically stable about point $\hat{x}$, then (under some additional assumptions) the following process

$$\mu(k)=(x(k)-\hat{x})/\sqrt{\tau}$$

converges weakly to a Gauss-Markovian process as $\tau \to 0$.

So the ODE method can also be applied in the case when $\tau(k)$ satisfies (6b).

3. Analysis of the algorithms

3.1. The problem formulation

Let a single input-single output dynamic system be given by

$$y(k)+\hat{a}_1 y(k-1)+\hat{a}_2 y(k-2)+\ldots+\hat{a}_n y(k-n)=$$
$$=\hat{b}_1 u(k-1)+\hat{b}_2 u(k-2)+\ldots+\hat{b}_1 u(k-1)+e_1(k), \qquad (10)$$

where $u(k)$ and $y(k)$ are the input and output signals, respectively, $e_1(k)$ is a stationary white noise, $\hat{\theta}=(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n, \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_1)^T$ is the vector of unknown parameters. We assume that the parameters $\hat{\theta}$ make the dynamic system (10) stable.

Having undisturbed measurements of inputs and outputs of the system till moment $k-1$, we are to find an estimate $\theta(k)$ of $\hat{\theta}$.

3.2. Prediction error identification algorithms

We shall use the Robbins-Monro Stochastic approximation procedure (see [7]) and its modification for solving the problem considered.

Let us denote

$$\psi(k)=(-y(k-1), -y(k-2), \ldots, -y(k-n), u(k-1), u(k-2), \ldots, u(k-1))^T,$$
$$\varphi(k)=(y(k), \psi^T(k))^T.$$

Knowing $\theta(k)$ and $\varphi(k+1)$ at moment k we can compute the prediction of the next output

$$\hat{y}(k+1, \theta(k))=\theta^T(k)\cdot\psi(k+1).$$

The error of that prediction at time $k+1$ is

$$\varepsilon(k+1, \theta(k))=y(k+1)-\hat{y}(k+1, \theta(k))=(\hat{\theta}-\theta(k))^T\cdot\psi(k+1)+e_1(k+1)$$

The criterion to be minimized is

$$J_W = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} W\big(\varepsilon\,(k+1, \theta\,(k))\big),$$

where

$$W\big(\varepsilon\,(k+1, \theta\,(k))\big) = \frac{1}{2} E\,\{\varepsilon^T\,(k+1, \theta\,(k)) \cdot A \cdot \varepsilon\,(k+1, \theta\,(k))\}. \tag{11}$$

At the k-th step of the algorithm $\theta\,(k)$ is changed so as to minimize the expected value of the second power of the prediction error at time $k+1$, i.e. to minimize $W\big(\varepsilon\,(k+1, \theta\,(k))\big)$. So we have the following direction, a step along which should improve the estimates

$$\eta\,(k+1) = \psi\,(k+1) \cdot \varepsilon\,(k+1, \theta\,(k)) \tag{12}$$

This direction is opposite to that of the "stochastic gradient" of (11) (i.e. to the gradient of the function that appeared in (11) under the expected value).

So from the general models (1) and (2) we get the following two algorithms for indentifying the parameters $\hat{\theta}$:

the algorithm with no filtering

$$(*) \quad \begin{cases} \theta\,(k+1) = \theta\,(k) + \tau\,(k)\,\eta\,(k+1) \\ \theta\,(0) = \theta_0 \end{cases}$$

and the algorithm with filtering

$$(**) \quad \begin{cases} \theta\,(k+1) = \theta\,(k) + \tau\,(k)\,d\,(k) \\ d\,(k+1) = d\,(k) + \alpha\tau\,(k)\,\big(\eta\,(k+1) - d\,(k)\big) \\ \theta\,(0) = \theta_0, \; d\,(0) = 0. \end{cases}$$

We shall assume that the sequence $\{\tau\,(k)\}$ satisfies (3).

3.3. Convergence of the algorithms

We shall apply the ODE method to the analysis of the algorithms convergence. We shall first reformulate both algorithms to the standard form (equality (4)), and then we shall specify the rule for generating $\varphi\,(k)$ (equality (5)).

For (13) we get

$$\begin{aligned} & x\,(k) = \theta\,(k), \\ & Q\,\big(k+1, x\,(k), \varphi\,(k+1)\big) = \psi\,(k+1)\,\big(y\,(k+1) - x^T\,(k)\,\psi\,(k+1)\big). \end{aligned} \tag{15}$$

The observation vector $\varphi\,(k)$ is generated as follows

$$\varphi\,(k+1) = A \cdot \varphi\,(k) + B \cdot \big(e_1\,(k+1), e_2\,(k+1)\big)^T, \tag{16}$$

where

$$A=\begin{bmatrix} -\hat{\theta}_1 & \hat{\theta}_2 & \hat{\theta}_3 \dots \hat{\theta}_n & 0 & \hat{\theta}_{n+2} \dots \hat{\theta}_{n+1} & 0 \\ -1 & 0 & 0 \dots 0 & 0 & 0 \quad \dots 0 & 0 \\ 0 & 1 & 0 \dots 0 & 0 & 0 \quad \dots 0 & 0 \\ 0 & 0 & 1 \dots 0 & 0 & 0 \quad \dots 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 \dots 1 & 0 & 0 \quad \dots 0 & 0 \\ 0 & 0 & 0 \dots 0 & 0 & 0 \quad \dots 0 & 0 \\ 0 & 0 & 0 \dots 0 & 0 & 1 \quad \dots 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 \dots 0 & 0 & 0 \quad \dots 1 & 0 \end{bmatrix} \qquad B=\begin{bmatrix} 1 & \hat{\theta}_{n+1} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \\ 0 & 0 \end{bmatrix}$$

$e_1(k)$ and $e_2(k)$ are stationary white noises uncorrelated with $y(i)$, $i=0, 1, 2, \dots, k$, having zero expected values and finite variances. The matrices A and B can be easily found using the $\varphi(k)$ definition and equation (10).

For algorithm (14) we have

$$x(k)=(\theta^T(k), d^T(k))^T, \tag{17}$$

$$Q(k+1, x(k), \varphi(k+1))=\begin{bmatrix} d(k) \\ \alpha\{\psi(k+1)(y(k+1)-\theta^T(k)\psi(k+1))-d(k)\} \end{bmatrix}.$$

The observation vector $\varphi(k)$ is generated in the same way as in the case of algorithm (13) (equation (16)).

We shall now prove the convergence of the algorithms

THEOREM 2. *Algorithm* (13) *identifying parameters of the dynamic system* (10) *generates a sequence* $\{\theta(k)\}$ *such that*

$$\theta(k)\to\hat{\theta} \text{ wpl as } k\to\theta$$

for all $\theta_0 \in R^{n+1}$

The theorem will be proved in the following way

— we shall prove that the algorithm considered satisfies A1 ÷ A13,

— we shall find an ODE associated with the algorithm,

— $\hat{\theta}$ will be shown to be the only stable stationary point of ODE with its domain of attraction equal to R^{n+1},

— the desired conclusion will then follow from Theorem 1.

P r o o f o f T h e o r e m 2 Let us consider algorithm (13), (16) (with notation (15)). $D_s=R^{n+1}$, since A does not depend on x. For $\bar{x}\in D_s$ we can now define the following random variables (see equations (8) and (9))

$$\bar{\varphi}(k+1, \bar{x})=A\varphi(k, \bar{x})+B(e_1(k+1), e_2(k+1))^T, \bar{\varphi}(0, \bar{x})=0, \tag{18a}$$

$$v(k+1, \lambda, c)=\lambda\cdot v(k, \lambda, c)+c\left|(e_1(k+1), e_2(k+1))^T\right|, v(0, \lambda, c)=0 \tag{18b}$$

Let $D_R=D_S$. We shall show that A1 ÷ A13 are satisfied for all $x\in D_R$.

The assumptions on $e_1(\cdot)$ and $e_2(\cdot)$ imply A1 and A2 From (15) we get

$$Q(k, x, \varphi)=\psi(y-\theta^T \psi).$$

In the appendix we prove that functions

$$K_1(x, \varphi, p, v)=(|\varphi|+v)(2+|\varphi|+v+2|\theta|+2p)$$

and

$$K_2(x, \varphi, p, v, w)=2|\varphi|+4w+2v+2|\theta|+2p+2$$

satisfy A3 and A4, respectively A and B are constant, and this implies A5

Since dynamic system (10) is stable, the following limits

$$f(\bar{\theta})=\lim_{k\to\infty} E\{\bar{\psi}(k)(\bar{y}(k)-\bar{\theta}^T\bar{\psi}(k))\} \tag{19a}$$

$$G(\bar{\theta})=\lim_{k\to\infty} E\{\bar{\psi}(k)\cdot\bar{\psi}^T(k)\} \tag{19b}$$

exist, so A6 is satisfied.

The definitions of functions $K_1(\cdot)$ and $K_2(\cdot)$ imply A7, since p-moments of random variables (18) exist. The stepsize sequence fulfils A8$\div$A11.

We shall now evaluate limits (19). Matrices A and B do not depend on θ, so $\bar{\varphi}(k, \bar{x})=\varphi(k)$ and $\bar{y}(k, \bar{x})=y(k)$. The limits are then input–output covariances of dynamic system (10). $G(\cdot)$, in particular, does not depend on θ.

From the definition of $e_1(\cdot)$ we get

$$E\{\psi(k)\cdot e_1(k)\}=0,$$

so

$$f(\theta)=G(\hat{\theta}-\theta), \tag{20a}$$

$$G(\theta)=G. \tag{20b}$$

ODE associated with algorithm (13) is then

$$\frac{d}{dt}\theta(t)=-G(\theta(t)-\hat{\theta}). \tag{21}$$

$\hat{\theta}$ is, of course, the only stable stationary point of (21). To show the stability of (21) we define the function

$$V(\Delta\theta)=\Delta\theta^T\cdot\Delta\theta, \tag{22}$$

where $\theta(t)=\theta(t)-\hat{\theta}$. Since G is positive definite and

$$\frac{d}{dt}V(\Delta\theta(t))=-2\Delta\theta^T(t)\,G\Delta\theta(t)\leqslant 0,$$

function (22) defines a Lyapunov function for equation (21). This also implies that D_R is the domain of attraction of $\hat{\theta}$, so A13 is satisfied.

To prove A12 let us take equation (13) for large $\theta(k)$

$$\theta(k+1) \simeq (I - \tau(k)\,\psi(k+1)\,\psi^T(k+1))\,\theta(k).$$

Since $\psi \cdot \psi^T$ tends to G and $\tau(k)$ tends monotonously to zero there is a compact set $\tilde{D} \subset D_R$ such that $\theta(k) \in \tilde{D}$ io wpl, $e_1(\cdot)$ and $e_2(\cdot)$ have finite moments so there exists constant C such that $|\psi(k)| < C$ and $|y(k)| < C$ io wpl, which implies $|\varphi(k)| < 2C$ io wpl. A12 is then satisfied.

We have proved that algorithm (13) fulfils A1÷A13, so the sequence $\{\theta(k)\}$ converges to $\hat{\theta}$ wpl as $k \to \infty$. This completes the proof. ■

THEOREM 3. *Algorithm* (14) *identifying parameters of the dynamic system* (10) *generates a sequence* $\{\theta(k)\}$ *such that*

$$\theta(k) \to \hat{\theta} \text{ wpl as } k \to \infty$$

for all $\theta_0 \in R^{n+1}$.

Proof of Theorem 3. The proof is analogous to the proof of Theorem 2. Let us consider algorithm (14), (16) (with notation (17)). A does not depend on x, so $D_S = R^{2n+21}$. Random variables $\bar{\varphi}(k, \bar{x})$ and $v(k, \lambda, c)$ are defined by equations (18).

Let $D_R = D_S$. We shall prove that conditions A1÷A13 are fulfilled for all $x \in D_R$.

Assumptions on $e_1(\cdot)$ and $e_2(\cdot)$ imply A1 and A2. We get from equation (17)

$$Q(k, x, \varphi) = \begin{bmatrix} d \\ \alpha(\psi(y - \theta^T \psi) - d) \end{bmatrix}.$$

A3 and A4 are satisfied by the functions

$$K_1(x, \varphi, p, v) = \alpha(|\varphi| + v)(2 + |\varphi| + v + 2|\theta| + 2p) + 1$$

$$K_2(x, \varphi, p, v, w) = \alpha(2|\varphi| + 4w + 2v + 2|\theta| + 2p + 2).$$

which are easy to find, since $K_1(\cdot)$ and $K_2(\cdot)$ for algorithm (13) are known (see the proof of Theorem 2).

A5 is fulfilled. As in the proof of Theorem 2, A6 is satisfied, that is the following limits exist

$$f(\bar{x}) = \begin{bmatrix} \lim\limits_{k\to\infty} E\{\bar{d}(k)\} \\ \lim\limits_{k\to\infty} E\{\alpha[\bar{\psi}(k)(\bar{y}(k) - \bar{\theta}^T \cdot \bar{\psi}(k)) - \bar{d}(k-1)]\} \end{bmatrix}, \tag{23a}$$

$$G(\bar{x}) = \lim_{k\to\infty} E\{\psi(k)\,\psi^T(k)\}. \tag{23b}$$

Assumptions on the noises $e_1(\cdot)$ and $e_2(\cdot)$ and definitions of functions $K_1(\cdot)$ and $K_2(\cdot)$ implies A7, stepsize sequence $\{\tau(k)\}$ fulfils A8÷A11.

Limits (23) are evaluated like in the proof of Theorem 2

$$f(\bar{x})=\begin{bmatrix} \bar{d} \\ \alpha\,(G\,(\hat{\theta}-\bar{\theta})-\bar{d}) \end{bmatrix} \tag{24a}$$

$$G(\bar{x})=G, \tag{24b}$$

which gives the ODE associated with algorithm (14)

$$\begin{cases} \dfrac{d}{dt}\,\theta(t)=d(t) \\ \dfrac{d}{dt}\,d(t)=\alpha\left(G\,(\hat{\theta}-\theta(t))-d(t)\right) \end{cases} \tag{25}$$

Equation (25) has one stationary point $\hat{x}=(\hat{\theta}^T, 0^T)^T$. We shall show it is stable. Let $\Delta x=x-\hat{x}$. The following function

$$V(\Delta x)=\Delta\theta^T\cdot\alpha G\cdot\Delta\theta+\Delta d^T\cdot\Delta d \tag{26}$$

is a Lyapunov function for equation (25). In fact, since $\alpha>0$ and G is positive definite we have

$$V(\Delta x)\geqslant 0 \quad \text{for all } \Delta x,$$

and

$$\frac{d}{dt}\,V(\Delta x(t))=2\Delta\theta^T(t)\cdot\alpha G\cdot\Delta d(t)+2\Delta d^T(t)\cdot\alpha\left(-G\Delta\theta(t)-\Delta d(t)\right)= \\ =-2\alpha\Delta d^T(t)\cdot\Delta d(t)\leqslant 0$$

So $\hat{x}$ is a stable stationary point of (25) and D_R is its domain of attraction, and this means A13 is statisfied.

We shall now prove A12. Algorithm (14) can be written for large $x(k)$ in the approximate equation form

$$x(k+1)=\begin{bmatrix} I & \tau(k)\,I \\ -\alpha\tau(k)\,\psi(k+1)\,\psi^T(k+1) & (1-\alpha\tau(k))\,I \end{bmatrix} x(k). \tag{27}$$

(we used the definition of "stochastic gradient" (12) and neglected elements with $e(k+1)$ in equation (14)).

We shall deduce that the approximate relation (27) determines a contraction "mapping" for large values of k. Let $F(k+1)$ denote its matrix

$$F(k+1)=\begin{bmatrix} I & \tau(k)\,I \\ -\alpha\tau(k)\,\psi(k+1)\cdot\psi^T(k+1) & (1-\alpha\tau(k))\,I \end{bmatrix}.$$

We shall prove that if λ is an eigenvalue of $F(k+1)$, then $|\lambda|<1$ for large k, that is the solutions of the equation

$$\det\left(F(k+1)-\lambda I\right)=0 \tag{28}$$

are inside the unit circle. Let

$$G_{k+1}=\psi(k+1)\,\psi^T(k+1).$$

Applying the well-known properties of the determinant, we get

$$\det\left(F(k+1)-\lambda I\right)=\det\begin{bmatrix}(1-\lambda)I & \tau I\\ -\alpha\tau\, G_{k+1} & (1-\alpha\tau-\lambda)I\end{bmatrix}=$$

$$=\det\begin{bmatrix}(1-\lambda)I & 0\\ -\alpha\tau\, G_{k+1} & (1-\alpha\tau-\lambda)\,I+\dfrac{\tau}{1-\lambda}\cdot\alpha\tau G_{k+1}\end{bmatrix}=$$

$$=(\alpha\tau^2)^{n+1},\ \det\,(G_{k+1}-\mu I)=0, \qquad (29)$$

where

$$\mu=\frac{1-\lambda}{\alpha\tau^2}\,(\lambda+\alpha\tau-1). \qquad (30)$$

Since G_k tends to G, all eigenvalues of G_{k+1}, that is solutions of (29), are positive for large k. From (30), we than have

$$\lambda^2+\lambda\,(2-\alpha\tau)+1-\alpha\tau+\alpha\tau^2\,\mu=0, \qquad \mu>0$$

Let $\Delta=\alpha^2\,\tau^2-4\alpha\tau^2\,\mu$. Two cases are possible:

1° $\Delta\geqslant 0$ ($\alpha\geqslant 4\mu$), which gives $\lambda_i\in(-1,1)$, $i=1,2$, since $\lambda_{1,2}=(2-\alpha\tau\pm\sqrt{\Delta})/2$.

2° $\Delta<0$ ($\alpha<4\mu$), which gives $\lambda_{1,2}=(2-\alpha\tau\pm i\sqrt{-\Delta})/2$, $\tau(k)$ tends to zero as $k\to\infty$, so for large k we have $\mu\cdot\tau(k)<1$. This implies $|\lambda_i|<1$, $i=1,2$.

We have proved that for large k solutions of (28) are inside the unit circle, i.e. the approximate equation (27) determines a contraction "mapping". Hence there exists a compact set $\bar{D}\subset D_R$ such that $x(k)\in\bar{D}$ io wpl. As in the proof of Theorem 2, we find that there exists $C<\infty$ such that $|\varphi(k)|<2C$ io wpl, which yields A12.

We have proved that algorithm (14) satisfies A1÷A13, so it converges to $\hat{\theta}$ wpl as $k\to\infty$, as desired. ■

Observe that, since the sequence $\{\tau(k)\}$ satisfies (3), the trajectories of (21) and (25) aproximate, in a certain sense, the asymptotic path of algorithms (13) and (14), respectively.

As we have already said, many properties of the algorithms (13) and (14) with constant stepsize coefficients can also be studied with the ODE method. In particular, the trajectories of (21) and (25) approximate the algorithms' asymptotic paths.

4. Simulation results

There are two different stages in the algorithms' behaviour: the first one, in which the error $\tilde{\theta}(k)$ quickly decreases, and the second one, in which $\tilde{\theta}(k)$ rondomly oscillates around zero.

Since our interest here lies in the first stage of the algorithms' behaviour, we shall compare their initial rates of convergence. To make the analysis easier, we shall use an algorithms' performance index of the form

$$J=\sum_{i=0}^{T}\|\theta(i)-\hat{\theta}\|^2, \tag{31}$$

where T is the time when the first stage finishes.

Algorithms (13) and (14) with constant stepsize coefficient were simulated in many different cases [3].

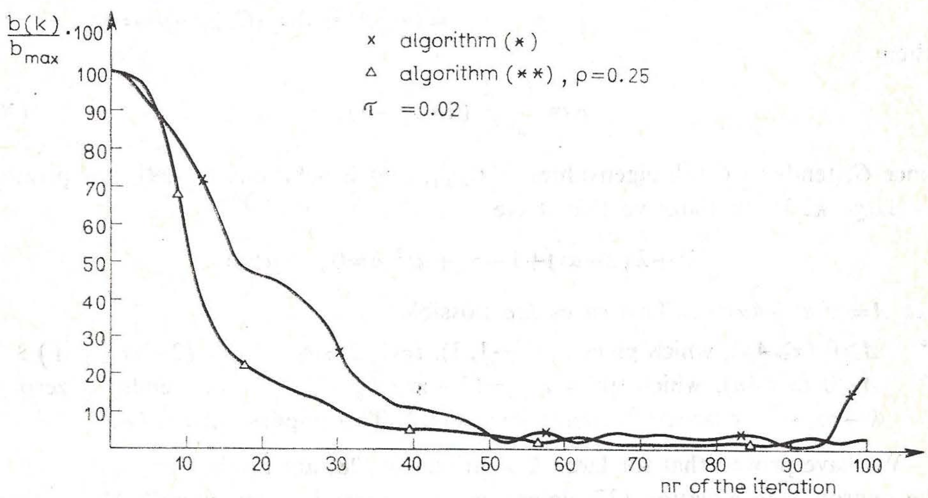

Fig. 1. Identification algorithms

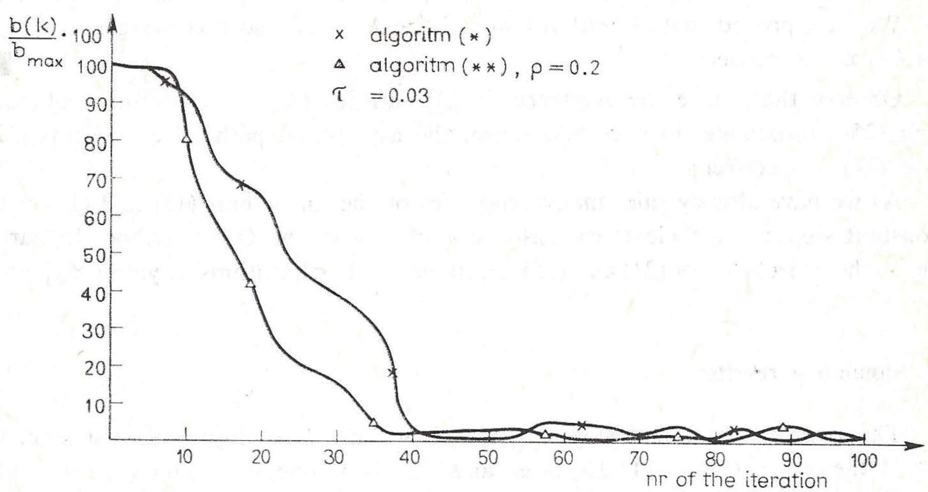

Fig. 2. Identification algorithms

Figures 1 and 2 present the results for $n=2$, $1=0$, $\hat{\theta}=(-1/10,-8/9)^T$. They show the normalized error

$$\frac{b(k)}{b_{\max}}\cdot 100,$$

where $b(k)=\|\theta(k)-\hat{\theta}\|$ and $b_{\max}=\max_k b(k)$.

Both in the case $\tau=0.02$, $p=\alpha\tau=0.25$ ($\alpha=12.5$) (presented in Fig. 1), and in the case $\tau=0.03$, $p=\alpha\tau=0.2$ ($\alpha=6.7$) (presented in Fig. 2), algorithm (**) is better. in the sense of (31), then algorithm (*).

So the filtering procedure of algorithm (14) can improve the parameter identification of the dynamic system. But there is a problem with finding a priori the value of coefficient α, which will yield improvement. There are only heuristic methods for finding such a coefficient a [3].

5. Conclusions

Algorithms (1) and (2) applied to the parameter identification of a single input — single output dynamic system, are proved to converge wpl when the sequence $\{\tau(k)\}$ satisfies conditions (3).

The ordinary differential equation (ODE) method is found to be the convenient tool for analysing the algorithms asymptotic properties. Verifying the assumptions necessary for the application of the method may need some effort, since they are technical.

The solution of the ODE associated with a given algorithm approximates its path. This makes the numerical quality analysis of algorithms possible. We can, in particular, replace (31) by

$$J=\int_0^\infty \|\Delta\theta(t)\|^2\, dt,$$

where $\Delta\theta(t)$ is the solution of the appropriate ODE. This solution can be found analitically or, if it is impossible, numerically.

Simulation results discussed in section 4 are representative for all numerical analysis of the algorithms (presented in [4]). They show that applying the filtering procedure to the algorithm with a constant stepsize can accelerate its convergence.

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Appendix

We shall prove that functions

$$K_1(x,\varphi,\rho,v)=(|\varphi|+v)\,(2+|\varphi|+v+2\,|\theta|+2\rho)$$
$$K_2(x,\varphi,\rho,v,w)=2\,|\varphi|+4w+2v+2\,|\theta|\;+2\rho+2$$

satisfy conditions A3 and A4 for algorithm (13). We have

$$Q(k,x,\varphi)=\psi\,(y-\theta^T\psi)$$

Let $\theta_i\in S(\theta,\rho)$, $\rho=\rho(x)>0$, $\varphi_i\in S(\varphi,v)$, $v>0$ for $i=1,2$. The above assump tions imply

$$|y_1-y_2|<|\varphi_1-\varphi_2|,\quad |\psi_1-\psi_2|<|\varphi_1-\varphi_2|$$
$$|\psi_i|<|\varphi_i|<|\varphi|+v,\quad |v_i|<|\varphi_i|<|\varphi|+v$$
$$|\theta_i|<|\theta|+\rho\qquad \text{for } i=1,2$$

From these inequalities and the properties of a norm we get

$$\begin{aligned}
&|Q(k,x_1,\varphi_1)-Q(k,x_2,\varphi_2)|\leqslant|\psi_1 y_1-\psi_1 y_2+\psi_1 y_2-\psi_2 y_2|+\\
&+|\psi_1\psi_1^T\theta_1-\psi_1\psi_1^T\theta_2+\psi_1\psi_1^T\theta_2-\psi_1\psi_2^T\theta_2+\psi_1\psi_2^T\theta_2-\psi_2\psi_2^T\theta_2|\leqslant\\
&\leqslant|\psi_1|\,|y_1-y_2|+|y_2|\cdot|\psi_1-\psi_2|+\\
&+|\psi_1|\cdot|\psi_1|\cdot|\theta_1-\theta_2|+|\psi_1|\cdot|\psi_1-\psi_2|\cdot|\theta_2|+|\psi_1-\psi_2|\cdot|\psi_2|\cdot|\theta_2|\leqslant\\
&\leqslant(|\varphi|+v)\,|\varphi_1-\varphi_2|+(|\varphi|+v)\,|\varphi_1-\varphi_2|+\\
&+(|\varphi|+v)^2|\theta_1-\theta_2|+2\,(|\varphi|+v)\,(|\theta|+\rho)\,|\varphi_1-\varphi_2|\leqslant\\
&\leqslant(|\varphi|+v)\,(2+|\varphi|+v+2\,|\theta|+2\rho)\cdot\{|\theta_1-\theta_2|+|\varphi_1-\varphi_2|\},
\end{aligned}$$

which completes the proof of A3.

Next, let $\varphi_i\in S(\varphi,w)$, $v_i\in S(v,w)$, $i=1,2$.

Then

$$v_i<v+w,\ |\varphi_i|<|\varphi|+w\qquad\text{for } i=1,2.$$

We have

$$
\begin{aligned}
&|K_1(\theta, \varphi_1, \rho, v_1) - K_1(\theta, \varphi_2, \rho, v_2)| \leqslant \\
&\leqslant \left|(|\varphi_1| + v_1)^2 - (|\varphi_2| + v_2)^2\right| + (2|\theta| + 2\rho + 2)\left||\varphi_1| - |\varphi_2| + v_1 - v_2\right| \leqslant \\
&\leqslant (|\varphi_1| + |\varphi_2|) \cdot |\varphi_1 - \varphi_2| + (v_1 + v_2)|v_1 - v_2| + 2|\varphi_1|\,|v_1 - v_2| + 2v_2|\varphi_1 - \varphi_2| + \\
&+ (2|\theta| + 2\rho + 2)\{|\varphi_1 - \varphi_2| + |v_1 - v_2|\} \leqslant \\
&\leqslant (2|\varphi| + 2w)|\varphi_1 - \varphi_2| + (2v + 2w)|v_1 - v_2| + (2|\varphi| + 2w)|v_1 - v_2| + \\
&+ (2v + 2w)|\varphi_1 - \varphi_2| + (2|\theta| + 2\rho + 2)\{|\varphi_1 - \varphi_2| + |v_1 - v_2|\} \leqslant \\
&\leqslant (2|\varphi| + 4w + 2v + 2|\theta| + 2\rho + 2)\{|\varphi_1 - \varphi_2| + |v_1 - v_2|\}
\end{aligned}
$$

so A4 is proved.

Analiza dwóch algorytmów aproksymacji stochastycznej zastosowanych w problemie identyfikacji

W pracy tej analizowane są dwa algorytmy typu aproksymacji stochastycznej zastosowane do identyfikacji parametrów układu dynamicznego o jednym wejściu i jednym wyjściu. K-ty krok algorytmów wykonywany jest w kierunku przeciwnym do kierunku będącego dostępną w danej chwili oceną gradientu danej funkcji celu. Do analizy algorytmów zastosowano opracowaną przez Ljunga metodę stowarzyszonego równania różniczkowego. Udowodniono zbieżność algorytmów z prawdopodobieństwem 1. Praca prezentuje wyniki symulacji oraz jakościową analizę algorytmów.

Анализ двух алгоритмов стохастической аппроксимации применених к идентификации

Исследуются свойства двух алгоритмов стохастической аппроксимации примененых к параметрической идентификации динамической системы. На κ-том итерационном шагу алгоритма вычисляется оценка градиента целевой функции. Текущая точка продвигается в направлении противиположном этой оценке. Приводится анализ альгоритмов с исспользованием метода дифференциального уравнения Люнга. Доказано что альгоритмы сходяйтся с вероятностю 1. В работе приведены результаты моделирования на ЭВМ и качественный анализ алгоритмов.