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Uniform null controllability of nonlinear discrete-time systems

by

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The paper deals with controllability of nonlinear discrete-time systems. The conditions of equivalence of null controllability and uniform null controllability are derived.

1. Introduction

For linear systems controllability criteria are simple and well-known. In the nonlinear case, however, this is a much more difficult problem and the existing results are far from a general solution. For continuous time nonlinear systems the applications of the theory of Lie groups seems to be most promising [1, 7, 10]. Along this line Ritt's differential groups were used to analyze controllability of discrete-time nonlinear systems [4]. A special class of bilinear discrete-time systems with a scalar control was considered by many authors [2, 3, 5, 6, 8, 9]. The criteria of controllability in $R^n - \{0\}$ and null controllability were formulated in [2, 3] and [8, 9] respectively.

In the paper we shall consider the relation between null controllability and uniform null controllability (on some finite time interval) for nonlinear discrete-time systems. First we shall define these notions and then give the conditions of their equivalence.

Consider the following discrete-time nonlinear system

$$x(t+1)=F\big(x(t), u(t)\big), \quad t=0, 1, 2, \ldots, \tag{1}$$

where the state $x(t) \in R^n$ and the control $u(t) \in U \subset R^m$. The sequence of controls $u(0), u(1), \ldots, u(r)$ is admissible if $u(t) \in U$ for each $t \in [0, r]$.

DEFINITION 1

(i) The system (1) is controllable from $D_0 \subset R^n$ to $D_1 \subset R$ if for each initial state $x(0) \in D_0$ and each final state $x_f \in D_1$ there exist a natural number s and an admissible sequence of controls $u(0), u(1), \ldots, u(s-1)$ such that $x(s)=x_f$.

(ii) The system (1) is uniformly controllable from $D_0 \subset R^n$ to $D_1 \subset R^n$. if there exists a natural number s such that for each initial state $x(0) \in D_0$ and each final state $x_f \in D_1$ there exists an admissible sequence of controls $u(0), u(1), \ldots, u(r-1)$, $r \leqslant s$, providing $x(r)=x_f$.

In case (ii) we also say that the system is controllable from D_0 to D_1 on the interval $[0, s]$. If $D_0=R^n$ and $D_1=\{0\}$ we say that the system is null controllable or controllable to zero. Clearly, uniform controllability implies controllability.

2. Main results

Now we present two theorems which characterise the relation between null controllability and uniform null controllability. The first theorem states their equivalence in the class of linear in the state homogenous systems. The second one deals with a general nonlinear case.

THEOREM 1. *Consider the following linear in the state system*

$$\begin{aligned} &x(t+1)=A(u(t))\,x(t), \qquad t=0, 1, 2, \ldots, \\ &x(t) \in R^n, \quad u(t) \in U \subset R^m, \end{aligned} \tag{2}$$

where the elements of $n \times n$ matrix $A(u)$ are any functions of u defined on U. The system (2) is null controllable iff it is uniformly null controllable.

P r o o f. Clearly it is enough to prove that null controllability implies uniform null controllability. Assume that the system (2) is controllable to zero. First we prove that for any k — dimensional subspace E^k, $0<k \leqslant n$, there exists a (at most) $(k-1)$ — dimensional subspace E^{k-1} such that the system (2) is controllable from E^k to E^{k-1} on some finite interval $[0, r]$. Let $E^k=\text{lin}\,\{z_1, z_2, \ldots, z_k\}$. Since the system is controllable to zero, there exist $r \in N$ and an admissible sequence of controls $u(0), u(1), \ldots, u(r-1)$ such that $x(r)=0$, if $x(0)=z_k$.

If we denote

$$y_i=\prod_{t=0}^{r-1} A(u(t))\, z_i, \qquad i=1, 2, \ldots, k-1$$

then the system is controllable from E^k to $E^{k-1}=\text{lin}\,\{y_1, y_2, \ldots, y_{k-1}\}$ on a finite interval $[0, r]$, since y_i is final point of trajectory starting from $x(0)=z_i$, $i=1, \ldots, k-1$. Using the above procedure n times (at most) we obtain that system (2) is uniformly null controllable from the whole state space R^n. ■

The following example proves that there is no natural number s such that each system of the form (2) is controllable to zero on the interval $[0, s]$.

EXAMPLE 1. Consider the system (2) with

$$A(u)=\begin{bmatrix} u_2 \cos\alpha, & u_1-u_2\sin\alpha \\ u_2\sin\alpha, & u_2\cos\alpha \end{bmatrix}$$

where $u_1>0$, $u_2>0$ and $\alpha=\frac{\pi}{2p}$, $p=1, 2, \ldots$.

We can rewrite $A(u)$ in the following way

$$A(u)=A_1(u)+A_2(u),$$

where

$$A_1(u)=\begin{bmatrix} u_2\cos\alpha, & -u_2\sin\alpha \\ u_2\sin\alpha, & u_2\cos\alpha \end{bmatrix}$$

and

$$A_2(u)=\begin{bmatrix} 0, & u_1 \\ 0, & 0 \end{bmatrix}.$$

$A_1(u)$ is a rotation matrix by the angle α and $A_2(u)$ changes only the first component of the state vector Let $Z\subset R^2$ denote the set of states controllable to zero on the interval $[0, r]$ and not controllable on the interval shorter then r. For $p=1$

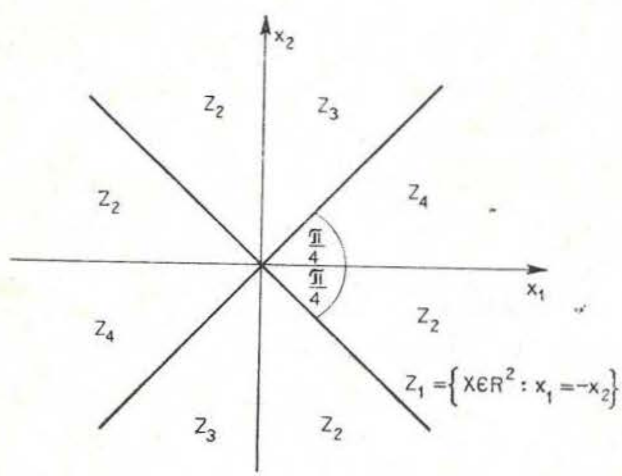

Fig. 1.

we obtain $Z_1=\{x\in R^2: x_1=0\}$ and $Z_2=R^2\setminus Z_1$. The figure 1 shows the sets Z_r for $p=2$. Similarly we can analyze each system for $p>2$. Clearly, each system is controllable to zero on the interval $[0, 2p]$. Thus the length of the interval tends to infinity if p tends to infinity.

THEOREM 2. *Consider system* (1). *Assume that F is continuous with respect to x, u for $x \in R^n$, $u \in U$, and the system is controllable from a neighbourhood of zero to zero on some finite interval* $[0, r_1]$.

Then for each compact subset K of R^n the system is controllable from K to zero iff it is uniformly controllable from K to zero.

Proof. Of course it is enough to prove that controllability from K to zero implies uniform null controllability from K to zero for any compact K. Let $x(0) \in K$ and $U_1(0)$ be a neighbourhood of zero from which the system is controllable to zero on $[0, r_1]$. We can choose a natural number $r(x(0))$ and an admissible sequence of controls $u(0), u(1), \ldots, u(r(x(0))-1)$ such that $x(r(x(0)))=0$.

Since F is continuous there exists a neighbourhood $U_2(x(0))$ satisfying the condition

$$\forall \bar{x}(0) \in U_2(x(0)): \quad \bar{x}(r(x(0))) \in U_1(0),$$

where $\bar{x}(r(x(0)))$ is the endpoint of the trajectory corresponding to the new initial state $\bar{x}(0)$ and the sequence of control $u(0), u(1), \ldots, u(r(x(0))-1)$. Thus the system is controllable from $U_2(x(0))$ to zero on the finite interval $[0, r_1+$ $+r(x(0))]$.

Let $U(x)$, $x \in K$, be a neighbourhood of x, from which the system is controllable to zero on the finite interval $[0, r_1+r(x)]$, where $r(x)$ is the length of interval on which the system is controllable from x to zero. For each $x \in K$ the set $U(x)$ is open and

$$K \subset \bigcup_{x \in K} U(x).$$

Since K is compact, we may chose a finite number of points x^i, $i=1, 2, \ldots, p$, satisfying

$$K \subset \bigcup_{i=1}^{p} U(x^i)$$

Putting $r = \max\limits_{1 \leq i \leq p} [r(x^i)+r_1]$ we conclude that from every point of K zero state may be reached on the interval not longer then $[0, r]$, which proves that the system is uniformly null controllable from K to zero. ■

The following example proves that we cannot put R^n instead of a compact set K in the theorem above.

EXAMPLE. Consider the system

$$x(t+1)=x(t)+u^2(t)-u(t), \qquad i=0, 1, \ldots,$$

$$x(t) \in R, \quad u(t) \in R.$$

It can be easily seen that the system is controllable to zero and controllable to zero from the set $\left(-\frac{1}{4}, \frac{1}{4}\right)$ on the interval $[0, 1]$. However, for any finite r there exists

$x(0)\left(\text{for example } x(0)>\frac{r}{4}\right)$, from which we cannot reach zero state on the interval $[0, r]$. Thus the system is not uniformly controllable to zero from the whole state space.

The Theorems 1 and 2 give the conditions of null controllability on some finite interval $[0, r]$. But we still do not know how to estimate the length of this interval. In the simple case of a bilinear system with two-dimensional state and unlimited control it was proved in [9] that null controllability is equivalent to null controllability on the interval [0, 3].

For a special class of bilinear systems

$$x(t+1)=\big(A+u(t)B\big)\,x(t)$$
$$u\in R, \quad \operatorname{rank} B=1$$

the conditions of null controllability on the interval $[0, n]$ were formulated in [6], but they are stronger then necessary and sufficient conditions of null controllability presented in [8].

3. Conclusions

Null controllability of discrete-time nonlinear systems has been considered. The conditions of null controllability on some finite interval have been given. The determination of the length of the interval on which the system is null controllable in a general nonlinear case requires further research.

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Jednostajna sterowalność do zera nieliniowych układów dyskretnych

W pracy rozważono zagadnienie sterowalności w klasie nieliniowych układów dyskretnych. Sformułowane zostały warunki równoważności sterowania do zera i jednostajnej sterowalności do zera.

Равномерная управляемость нелинейных дискретных систем

В работе рассуждается управляемость нелинейных дискретных систем. Представлено условия эквивалентности управляемости к нулью и равномерной управляемости к нулью.