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An introduction to Max-minimal sets

by

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A minimal set is a subcollection of entities such that it is internally connected stronger than with its environment. In the preceding papers (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14]) the strength of connection between two nonempty and disjoint subsets was assumed to be the sum of strengths for elementary connections, i.e. those between pairs of entities. This definition, although reasonable for a great number of problems appearing when we seek a rational partition of a network (ref., e.g., [3, 4, 7, 8]), may be inadequate for another purposes. An analogous mathematical structure is considered in this paper. The only distinction lies in a different definition of "similarity" between two nonempty and disjoint sets of entities. It occurs that this new mathematical structure has similar properties as those for the (classic) minimal sets.

1. Introductory notes

Many theoretical and practical problems in science and technology may be formulated as to obtain a reasonable partition of some nonempty and finite set of entities. To each pair of distinct elements taken from the set some weight (being a non-negative real number) is assigned. The weight represents a similarity between a pair of entities. In another words the value of this weight is greater when these entities are more similar in some sense and/or more close to each other. The way for determining the value of reciprocal similarity between two entities usually depends upon the intrinsic features of the real (source) problem under consideration and it is here beyond the scope of interest. The interpretation of weights as similarities implies, however, some general conditions for a "good" decomposition. Namely, first, each pair of entities with a relatively large value of reciprocal similarity should belong to the same class. Second, any two entities with a relatively small mutual similarity have to be in distinct classes. These conditions are usually fulfilled by the application of the minimal sets' techniques for real-life problems.

The very nature of the so-called minimally interconnected subnetworks, or, shortly, minimal sets, can be briefly described as follows. They are, roughly speaking, some collections of entities such that they are internally connected stronger than with their environment. In other words, the aggregate similarity between any proper part of minimal set and its remainder should be greater than the analogous parameter defined for this proper part and the complement of this minimal set to the collection consisting of the entities under consideration.

The problem arises how to evaluate the aggregate reciprocal similarity between sets of entities when the elementary similarities (i.e. those between pairs of entities) are known. Evidently, it depends upon the very nature of the real (source) problem under consideration. In the previous papers (see, e.g., [2, 5, 6, 9]) the aggregate reciprocal similarity $f(A, B)$ between disjoint and nonempty sets A and B was simply defined as the sum of $w(x, y)$'s (i.e. the elementary similarities between the xth and yth entities) taken over $x \in A$ and $y \in B$, i.e.

$$f(A, B)=\sum_{x \in A} \sum_{y \in B} w(x, y) \tag{1}$$

Throughout the paper minimal sets determined for the aggregate reciprocal similarity given by (1) are called classic minimal sets.

Several papers (see, e.g. [2, 5, 6, 9, 14]) were devoted to the theory of classic minimal sets and to the algorithms for finding them. The technique of classic minimal sets proves to be a convenient tool for solving a variety of networking-type problems as, e.g., structuring of a telephone interexchange network [7], partitioning a computer and/or teleprocessing network [3, 4, 11], designing a printed board in electrical networks [8], etc. Moreover, it can be applied to solve other problems arising in practice as, e.g., structuring of a group of enterprises [1] and structuring of a data base [10]. In the recent paper [14] the polynomial-type algorithm for finding the classic minimal sets is proposed.

To show the very nature of the aggregate similarity defined by (1), and thus of the classic minimal sets, we consider here three nonempty and pairwise disjoint sets, say A, B, and C. For the sake of simplicity, we assume $w=w(x, y)=w(x, t)=$ $=$const., $w>0$ for any $x \in A$, $y \in B$ and $t \in C$. Therefore, by using (1), we obtain $\max\{f(A, B), f(A, C)\}=|A| \max\{|B|, |C|\}\, w<|A|\,(|B|+|C|)\, w=f(A, B \cup C)$, i.e. that $B \cup C$ is "more similar" to A than B (or C). It shows that in the case of (1) the value of aggregate similarity depends not only on elementary similarities but also upon the number of entities composing the sets under consideration. Hence, if $w(.,.)$'s are expressed in terms of cooperation intensity, commodity of interest and/or information flow between, e.g., enterprises, then the aggregate similarity given by f is meant as the strength of reciprocal connections. Therefore, f has here a reasonable interpretation, and the classic minimal sets technique can be applied to extract some groups of enterprises which cooperate more intensively. It was confirmed by the above mentioned practical usefulness of the classic minimal sets technique, particularly to the problems of structuring a network [3, 4, 7, 8, 11].

A different situation occurs when we need to perform a data analysis or to classify a set of entities due to given similarities between them. In other words, when $w(.,.)$'s have the sense of, e.g., resemblance or likeness between two objects. In this case the aggregate reciprocal similarity should not depend on the number of entities composing appropriate sets. According to the previous remarks f does not satisfy this condition. Continuing the example given above, this case occurs when $w(.,.)$'s are expressed in terms of, e.g., assortment similarity, branch similarity and/or production and technological similarity between enterprises. Therefore, another definition of aggregate reciprocal similarity between two sets is required. Some convenient form of this parameter is introduced below.

It is convenient to begin with some remarks about similarity and dissimilarity. The concept of them has very intuitive roots. In fact, we often use the terms similarity and/or dissimilarity basing on our experience or even feelings but without taking into account any rational and systematic inspection of characteristics concerning entities under comparison. It can be also said that in common sense the statement "more similar" is equivalent to "less dissimilar" and, furthermore, that each entity cannot be more similar (or less dissimilar) to another one than to itself. But this intuitive understanding is insufficient for working out clustering problems met in data analysis, taxonomy, classification, pattern recognition, etc., because there appears the question about ranking similarities and/or dissimilarities. Therefore, the concept is formalized, e.g., by taking a mapping $v: D \to U$, where D is a nonempty set consisting of ordered pairs of entities, and U indicates a nonempty set (usually ordered by some relation useful for ranking) of values. If there exists no necessity to compare entities with themselves (this case occurs in the theory discussed in the paper), then $D = D_1 = X \times X - \{\{x, \{x, x\}\}: x \in X\}$, where X is the set of entities under consideration, and $\{x, \{x, y\}\}$ is the well-known notation for an ordered pair; otherwise $D = D_2 = X \times X$. U is usually a set of inexact, fuzzy or ordinary real (sharp) numbers. Moreover, beyond some specific purposes, the condition of symmetry for v is adopted, i.e. $v(x, y) = v(y, x)$ is assumed for any pair of entities taken from X (see, e.g., the definition of w in Section 2).

For brevity, let us now restrict our considerations to a symmetric mapping $v: A \to U$, with the image being some subset of real numbers. It is obvious that v can represent silmilarity (v=sim) as well as dissimilarity (v=dis), because it does not violate any assumption verbally introduced above. The distinction between similarity and dissimilarity lies in interpretation of v only, which immediately implies the way of ranking. Namely, x and y are more close to each other than to z if and only if $\text{sim}(x, y) > \sup\{\text{sim}(x, z), \text{sim}(y, z)\}$ and/or $\text{dis}(x, y) < \inf\{\text{dis}(x, z), \text{dis}(y, z)\}$ (see, e.g., [12]).

The above discussion gives us a simple and intuitively obvious way for such evaluation of aggregate reciprocal similarity that this parameter does not depend on the cardinalities of sets under consideration. Namely, the distance (distance is some specific form of dissimilarity, see, e.g., [13], Ch. 2) between the sets A and B is usually defined as $\inf\{\text{dis}(x, y): x \in A, y \in B\}$. Analogously, we can take sup $\{w(x, y): x \in A, y \in B\}$ as the aggregate reciprocal similarity between nonempty and disjoint sets A and B, for w=sim, or

$$m(A, B)=\max\{w(x, y): x \in A, y \in B\}, \tag{2}$$

if we consider a finite set of entities, because than A and B are also finite. Returning to the example used before to show the inadequacy of f for classification, we easily obtain $m(A, B)=m(A, C)=m(A, B \cup C)=w$, i.e. m does not depend on the cardinalities, indeed.

2. Preliminaries

We consider a finite and nonempty set X, $|X|>1$, and a function

$$w: \{\{x, y\}: x, y \in X, x \neq y\} \rightarrow R^{+} \cup \{0\},$$

where R^{+} is the set of positive real numbers, and $\{x_1, x_2, \ldots, x_n\}$, as usually, denotes an unordered n-tuple, i.e. a set consisting of n elements. In other words, w is a non-negative valued function defined on unordered pairs of distinct elements from X. The latter implies symmetry of w with respect to its arguments. Furthermore, we define

$$m: \{\{A, B\}: \emptyset \neq A, B \subset X, A \cap B=\emptyset\} \rightarrow R^{+} \cup \{0\},$$

evaluated due to (2). The definitions of w and m yield symmetry of m with respect to its arguments.

It can be easily verified that the condition $\emptyset \neq J \subset I$ implies

$$\max\{y_i: i \in J\} \leqslant \max\{y_i: i \in I\}, \tag{3}$$

$$\min\{y_i: i \in J\} \geqslant \min\{y_i: i \in I\}, \tag{4}$$

for a finite set I. In particular, the definition of m and (3) yield that the inclusions $A \subset B \subset X$ and $C \subset D \subset X$ lead to

$$m(A, C) \leqslant m(B, D), \tag{5}$$

for any nonempty sets A, C and disjoint B and D. Moreover, obviously,

$$\max\{m(A, B), m(A, C)\}=m(A, B \cup C), \tag{6}$$

for any nonempty sets A, B and C such that $A \cap (B \cup C)=\emptyset$. The last two formulae are frequently applied in proofs throughout the paper.

In the subsequent sections we investigate the most important properties of Max-minimal set defined as follows.

Definition 1. If the following inequality

$$m(R, S-R)>m(R, X-S) \tag{7}$$

holds for a nonempty set $S \subset X$, $S \neq X$, and for each its nonempty subset $R \neq S$, then S is called the Max-minimal set.

For instance, each set of points (vertices) of a graph encircled in dashed lines in Fig. 1 constitutes a Max-minimal set.

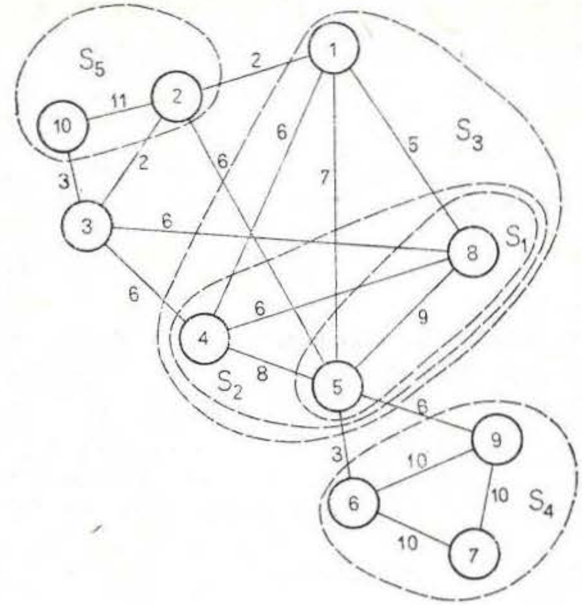

Fig. 1. Vertices of the graph represent entities, and edges correspond to the non-zero similarities (their values are attached as the weights to edges). Max-minimal sets are encircled in dashed lines.

3. Basic properties of Max-minimal sets

Applying de Morgan's rules to Definition 1 we easily obtain

COROLLARY 1. *Any* $\{x\}$, $x \in X$, *is a Max-minimal set.*

Moreover, if $R\,(R \subset X)$ and $X-S$ are nonempty and disjoint, then $m\,(R, X-S) \geqslant 0$, due to the definition of m. Thus, (7) implies

COROLLARY 2. *If S is a Max-minimal set, then $m\,(R, S-R) > 0$ for each its nonempty subset $R \neq S$.*

Let S be a Max-minimal set, and R its nonempty subset, $R \neq S$. From Definition 1 it follows that $m\,(R, S-R) > m\,(S-R, X-S)$, because $\emptyset \neq S-R \subset S$ and $S-R \neq S$. Since $X-R \supset S-R$, then, by (5), we obtain $m\,(R, X-R) \geqslant m\,(R, S-R)$. Combining it with (7) and using (6), we attain $m\,(R, X-R) > \max\,\{m\,(S-R, X-S),\, m\,(R, X-S)\} = m\,(S, X-S)$, i.e.

$$m\,(R, X-R) > m\,(S, X-S) \tag{8}$$

On the other hand, let us only assume that (8) holds. Using (5) and (6), we easily get $m(R, X-R) = \max\{m(R, X-S), m(R, S-R)\} > m(S, X-S) = \max\{m(R, X-S), m(R-S, X-S)\} \geq m(R, X-S)$. Supposing $m(R, X-S) \geq m(R, S-R)$ we obtain, therefore, $m(R, X-S) > m(R, X-S)$, i.e. a contradiction. In other words, we proved

Lemma 1. *S is Max-minimal if and only if* (8) *holds for any nonempty $R \subset S$, $R \neq S$.*

For instance, in Fig. 1 $S_1 = \{5, 8\}$ is Max-minimal. Moreover, we have $m(\{5\}, X-\{5\}) = m(\{8\}, X-\{8\}) = 9 > m(S_1, X-S_1) = 8$.

Thus, Lemma 1 gives another but equivalent definition of Max-minimal sets. Moreover, using the law of contraposition to Lemma 1, we get

Corollary 3. *If $m(S, X-S) \geq m(R, X-R)$ for some nonempty subset R of S, $R \neq S$, then S is not Max-minimal.*

Let S and Q be two distinct Max-minimal sets such that $S \cap Q = T \neq \emptyset$, $S \not\subset Q$ and $Q \not\subset S$. We denote $R = S-T$, $P = Q-T$ and $H = X-(S \cup Q)$. Evidently, $S \neq R \subset S$ and $Q \neq R \subset Q$. Using Definition 1, we obtain $m(R, T) > m(T, X-S) = m(T, H \cup P)$. From (5) it follows that $m(T, H \cup P) \geq m(P, T)$, i.e. $m(R, T) > m(P, T)$. The analogous way leads to $m(P, T) > m(R, T)$, which contradicts the previous inequality. In other words, we proved the main result of this section, i.e.

Lemma 2. *Two Max-minimal sets are either disjoint or one of them is included in the other.*

Returning to the situation depicted in Fig. 1, S_1, S_2, S_3, S_4 and S_5 are Max-minimal, and we have $S_1 \subset S_2 \subset S_3$. Moreover, S_3, S_4 and S_5 are pairwise disjoint.

4. Unions of Max-minimal sets and their proper parts

First, we introduce some convenient notation. Let T be a nonempty set of indices, and $\{H_t : t \in T\}$ — a collection of subsets taken from X. Hereinafter $\bigcup_{t \in T} H_t$ is abbreviated by H_T. Moreover, $S_T = \{S_t : t \in T\}$ denotes a collection of pairwise disjoint Max-minimal sets, and $R_T = \{R_t : t \in T\}$ — a collection of sets such that $R_t \subset S_t$, $R_t \neq S_t$, for each $t \in T$. $K(R_T)$ is here a set of indices assigned to nonempty R_t's, i.e. $K(R_T) = \{j : j \in T,\ R_t \neq \emptyset\}$.

Now, we are in a position to formulate and prove some features concerning unions of Max-minimal sets and their proper parts. In fact, the most significant result is stated in Theorem 1. Propositions 1, 2 and 3, although interesting, seem to be of less practical importance and play here rather an auxiliary role. They constitute an intermediate step in proving Theorem 1, whose introduction makes possible to construct the proof of Theorem 1 in a short and clear form without considering a large number of specific subcases. Finally, Theorem 2 concerns the application of the Max-minimal sets technique to solving real-life problems and the proper interpretation of results given by this technique.

PROPOSITION 1. Let I be a nonempty and finite set of indices, S_I, R_I, $K=K(R_I)$ be defined as above, and $Q \subset X$ be an arbitrary set such that $S_I \cap Q = \emptyset$ and $S_I \cup Q \neq X$. If at least one of the inequalities $|Q|\,|K|>0$ and $|K| \geqslant 2$ holds, then $P=Q \cup R_I$ is not Max-minimal. Moreover, the condition $Q \neq \emptyset$ implies the inequality $m(P, X-P) \geqslant$ $\geqslant m(Q, X-Q)$.

P r o o f. We choose any increasing sequence $(K(i): i \in K)$ of subsets taken from K defined by $K(i)=\{u: u \in K,\ u \leqslant i\}$. The proof proceeds by induction over $|K(i)|$.

For convenience we write here $P_{|K(i)|}$ and $S_{|K(i)|}$ instead of $Q \cup R_{K(i)}$ and $S_{K(i)}$, respectively. Moreover, we denote $R_{(i)}=P_{|K(i)|}-P_{|K(i)|-1}$ and $S_{(i)}=S_{|K(i)|}-S_{|K(i)|-1}$, for $i \in K$, where $P_0=Q$ and $S_0=\emptyset$.

Let Q be nonempty.

1. $|K(i)|=1$. According to (6), we have $m(P_1, X-P_1)-m(Q, X-Q)=a-b$, where $a=\max\{m(R_{(1)}, X-P_1),\ m(Q, X-P_1)\}$ and $b=\max\{m(Q, X-P_1),$ $m(Q, R_{(1)})\}$.

CASE 1. $a=m(R_{(1)}, X-P_1)$ and $b=m(Q, X-P_1)$. Therefore, $m(R_{(1)}, X-P_1) \geqslant$ $\geqslant m(Q, X-P_1) \geqslant m(Q, R_{(1)})$. Hence, $a-b$ is non-negative.

CASE 2. $a=m(R_{(1)}, X-P_1)$ and $b=m(Q, R_{(1)}) \neq m(Q, X-P_1)$. Since $X-P_1 \supset$ $\supset S_{(1)}-R_{(1)}$, then, by (5), we get

$$m(R_{(1)}, X-P_1) \geqslant m(R_{(1)}, S_{(1)}-R_{(1)}) \tag{9}$$

Using (5) again, we have

$$m(R_{(1)}, X-S_{(1)}) \geqslant m(Q, R_{(1)}), \tag{10}$$

because $S_I \cap Q=\emptyset$. Therefore, $a-b \geqslant m(R_{(1)}, S_{(1)}-R_{(1)})-m(R_{(1)}, X-S_{(1)})$ i.e. $a-b$ is positive, since $S_{(1)}$ is a Max-minimal set.

CASE 3. $a=b=m(Q, X-P_1) \neq m(R_{(1)}, X-P_1)$. Hence, $a-b=0$.

CASE 4. $a=m(Q, X-P_1) \neq m(R_{(1)}, X-P_1)$ and $b=m(Q, R_{(1)}) \neq m(Q, X-P_1)$. That means

$$m(R_{(1)}, X-P_1)<m(Q, X-P_1)<m(Q, R_{(1)}) \tag{11}$$

Combining (11) with (9) and (10), we get

$$m(R_{(1)}, S_{(1)}-R_{(1)})<m(R_{(1)}, X-S_{(1)}) \tag{12}$$

Thus, by applying (7) to the right-hand side of (12), we obtain $m(R_{(1)}, S_{(1)}-R_{(1)})<$ $<m(R_{(1)}, S_{(1)}-R_{(1)})$, i.e. a contradiction, which proves that Case 4 does not occur. That completes the first part of the proof, i.e. we obtain that $m(P_1, X-P_1) \geqslant$ $m(Q, X-Q)$, and, by Corollary 3, that P_1 is not Max-minimal.

2. Let us now assume that the inequality

$$m(P_i, X-P_i) \geqslant m(Q, X-Q) \tag{13}$$

is fulfilled for each i, $1 \leqslant i < |K|$. Similarly as in the first part of the proof (here P_i plays the role of Q) we obtain $m(P_{i+1}, X-P_{i+1}) = m(R_{(i+1)} \cup P_i, X-P_{i+1}) \geqslant \geqslant m(P_i, X-P_i)$, which, combined with (13), gives us $m(P_{i+1}, X-P_{i+1}) \geqslant m(Q, X+-Q)$. The last inequality implies, due to Corollary 3, that P_{i+1} is also not Max-minimal, which finishes the second step of induction.

Therefore, we have $m(Q, X-Q) \leqslant m(Q \cup R_K, X-(Q \cup R_K)) = m(P, X-P)$, since $R_K = R_I$, i.e. the proof is accomplished for the case $|Q|\,|K|>0$.

If $Q=\emptyset$, then, by assumptions, we have $|K| \geqslant 2$. From Lemma 2 it follows that P_2 is not Max-minimal. Furthermore, we can take $Q'=P_1$ and $K'=K-K_1$, where K_j denotes a set $K(t)$ chosen so thath $j=|K(t)|$ (construction of $K(t)$'s yields the uniqueness of K_j). Hence, $|Q'|\,|K'|>0$, and thus the result obtained for the case $Q \neq \emptyset$ remains in force, i.e. $P=P_1 \cup P_{K'} = R_K = R_I$ is not Max-minimal, which completes the whole proof. ■

For example, in Fig. 1 we have $m(\{2, 4, 5\}, X-\{2, 4, 5\}) = 11 \geqslant m(\{2\}, X--\{2\}) = 11$, and $m(\{2, 4, 5, 10\}, X-\{2, 4, 5, 10\}) = 9 \geqslant m(\{2, 10\}, X-\{2, 10\}) = 6$, where $R_3 = \{4, 5\} \subset S_3$. First, we take $Q=\{2\}$, and second — $Q=S_5$.

PROPOSITION 2. Let I, S_I, R_I and K have the same meaning as in Proposition 1. If K is nonempty, then

$$m(R_I, X-R_I) > \max\{m(S_i, X-S_i): i \in K\} \tag{14}$$

Proof. The proof proceeds by induction. If $|K|=1$, then (14) directly follows from Lemma 1. Now, we assume that

$$m(R_{K(i)}, X-R_{X(i)}) > \max\{m(S_j, X-S_j): j \in K(i)\} \tag{15}$$

holds for each i such that $|K(i)| < |K|$, where $K(i)$ (as well as $R_{(i)}$, $S_{(i)}$ and K_i below) is defined as in the proof of Proposition 1. By Proposition 1, we get

$$m(R, X-R) \geqslant m(R_{K(i)}, X-R_{K(i)}) \tag{16}$$

and

$$m(R, X-R) \geqslant m(R_{(i+1)}, X-R_{(i+1)}), \tag{17}$$

for $R = R_{K(i)} \cup R_{(i+1)}$. Applying Lemma 1 to $R_{(i+1)}$ and $S_{(i+1)}$ and using (17), we obtain

$$m(R, X-R) > m(S_{(i+1)}, X-S_{(i+1)}) \tag{18}$$

Thus, by (16), (15), (18) and the definition of $S_{(i+1)}$ (see the proof of Proposition 1), we reach $m(R, X-R) > \max\{m(S_j, X-S_j): j \in K_{i+1}\}$, which accomplishes the second step of induction, and thus the whole proof. ■

Returning to the example shown in Fig. 1, we, e.g., have $m(\{1, 2, 6\}, X-\{1, 2, 6\}) = 11 > \max\{m(S_i, X-S_i): i=3, 4, 5\} = 7$, where $R_3=\{1\}$, $R_4=\{6\}$ and $R_5=\{2\}$.

Propositions 1 and 2 are useful for proving Proposition 3 below, and, further, Proposition 2 is directly applied in the proof of Theorem 1.

PROPOSITION 3. Let I and S_I be as in Proposition 1, $S_I \neq X$. If S_J is not Max-minimal for any $J \subset I$ such that $|J|>1$, then

$$m(S_I, X-S_I) \geqslant \min\{m(S_i, X=S_i) : i \in I\} \tag{19}$$

P r o o f. Since S_J, $|J|>1$, is not Max-minimal, then from Lemma 1 it follows that there exists a nonempty set $H(J) \subset S_J$, $S_J \neq H(J)$, which satisfies

$$m(S_J, X-S_J) \geqslant m(H(J), X-H(J)) \tag{20}$$

We can write $H(J)=R_{T(J)} \cup S_L$, where $\emptyset \neq T(J) \cup L \subset J$, $L \neq J$, $T(J) \cap L=\emptyset$ and, additionally, $R_t \neq \emptyset$, for any $t \in T(J)$.

First, we consider the case $T(J) \neq \emptyset$ and $L=\emptyset$. Hence, denoting $T=T(J)$, we obtain

$$m(R_T, X-R_T) > \max\{m(S_i, X-S_i) : i \in T\}, \tag{21}$$

due to Proposition 2. Since $H(J)=R_{T(J)}$, then

$$m(S_J, X-S_J) \geqslant \min\{m(S_i, X-S_i) : i \in I\}, \tag{22}$$

according to (20), (21) and the obvious inequality $\max\{a_i : i \in T\} \geqslant \min\{a_i : i \in T\}$, and, finally, by (4).

Now, we assume that L and $T(J)$ are nonempty. Thus, denoting $H_0=H(J)$ and $L_1=L$, we get

$$m(H_r, X-H_r) \geqslant m(S_{L_{r+1}}, X-S_{L_{r+1}}) \tag{23}$$

for $r=0$, by Proposition 1.

If $|L|=1$, then (22) is implied by (20), (23) and (4). Otherwise, i.e. for $|L|>1$, S_L is not Max-minimal, due to the assumptions, and we can consider it in the same way as S_J.

In general, for any $L_r \subset I$, $|L_r|>1$, $r>1$, there exists a nonempty set $H_r=R_{T(L_r)} \cup$ $\cup S_{L_{r+1}}$ satisfying the inequality

$$m(S_{L_r}, X-S_{L_r}) \geqslant m(H_r, X-H_r), \tag{24}$$

by the assumptions and Lemma 1. Moreover, we have $\emptyset \neq T(L_r) \cup L_{r+1} \subset L_r$, $T(L_r) \cap L_{r+1}=\emptyset$,

$$|L_{r+1}|<|L_r| \tag{25}$$

and $R_t \neq \emptyset$, for each $t \in T(L_r)$. Now, there are possible five subcases.

SUBCASE 1. $L_{r+1}=\emptyset$. Therefore, $T=T(L_r)$ is nonempty, and we can proceed as for the case $T(J) \neq \emptyset$ and $L=\emptyset$. Thus, using the same arguments as mentioned above, we obtain

$$m(S_{L_r}, X-S_{L_r}) \geqslant \min\{m(S_i, X-S_i) : i \in I\} \tag{26}$$

SUBCASE 2. $|L_{r+1}|=1$ and $T(L_r)\neq\emptyset$. Hence, we reach the analogous situation as for $T(J)$ and L being nonempty, i.e. we get (26) again.

SUBCASE 3. $|L_{r+1}|>1$ and $T(L_r)\neq\emptyset$. Now, we define $q=r+1$ and repeat the construction described for L_r, (and H_r), replacing r by q. Since $q=r+1$, then

$$m(S_{L_r}, X-S_{L_r})\geqslant m(S_{L_{r+1}}, X-S_{L_{r+1}}), \tag{27}$$

due to (22) and (23).

SUBCASE 4. $|L_{r+1}|=1$ and $T(L_r)=\emptyset$. Hence, $H_r=S_{L_{r+1}}$, i.e. (27) holds, by (24). Thus, using (4), we easily get (26).

SUBCASE 5. $|L_{r+1}|>1$ and $T(L_r)=\emptyset$. Similarly, as in Subcase 4 we obtain (27).

It is evident that the described above construction for H_r can be applied successively until we arrive at Subcase either 1 or 2 or 4. In other words, the construction defines a sequence $L=(L_r\colon r=0, 1, 2, \ldots)$, where $L_0=J$. If Subcase 1 occurs for some $r=q$, then we interrupt the construction for L_q. Otherwise, the cardinality of successive L_r's decreases (according to (25)) and, finally we reach Subcase either 2 or 4. Therefore, L is finite. Moreover, by (27) and (26), we get (22) again.

In the case $T(J)=\emptyset$ and $L\neq\emptyset$ we proceed as in Subcases 2 and 4. It leads to (22) again. Substituting $J=I$ in (22), we obtain (19), which completes the whole proof.

For example, it can be easily verified that in Fig. 1 $S_3=\{1, 4, 5, 8\}$ and $S_5=\{2, 10\}$ are Max-minimal, but $S_{\{3,5\}}=S_3\cup S_5$ is not Max-minimal. For this case we have $m(S_{\{3,5\}}, X-S_{\{3,5\}})=7\geqslant\min\{m(S_3, X-S_3), m(S_5, X-S_5)\}=\min\{6, 7\}=6$.

The last proposition plays a crucial role in the proof of

THEOREM 1. *Let I, J and S_I be as in Proposition 3, and let S_J be not Max-minimal for each J such that $1<|J|<|I|$. Then S_I is Max-minimal if and only if the following condition*

$$m(S_I, X-S_I)<\min\{m(S_i, X-S_i)\colon i\in I\} \tag{28}$$

is satisfied.

Proof. We begin with the following inequality

$$\max\{a_i\colon i\in I\}<\min\{\max\{a_i, b_i\}\colon i\in I\}, \tag{29}$$

where a_i, b_i are real numbers, $i\in I$. From (29) follows existence of $r\in I$ such that $a_r<b_r$, because otherwise $\max\{a_i\colon i\in I\}<\min\{a_i\colon i\in I\}$, i.e. a contradiction. By the same reason, we get $\min\{\max\{a_i, b_i\}\colon i\in I\}=b_k$, for some $k\in I$. Furthermore, let an index $j\in I$ exist such that $a_j\geqslant b_j$. Therefore, evidently, $a_j\leqslant\max\{a_i\colon i\in I\}<b_k\leqslant a_j$, i.e. again a contradiction. Hence, (29) is equivalent to $a_i<b_i$ for each $i\in I$.

The above argumentation yields the equivalence of (28) and the condition

$$m(S_i, X-S_I)<m(S_i, S_I-S_i), \quad \text{for each } i\in I, \tag{30}$$

since $m(S_i, X-S_i)=\max\{m(S_i, X-S_I), m(S_i, S_I-S_i)\}$, due to (6).

The necessity of (30) and thus (28)) immediately results from Definition 1.

Sufficiency. Let us now assume that (28) is satisfied. We consider $H=H(I)$, $T=T(I)$ and L defined as below (20). If T is nonempty and $L=\emptyset$, then (21) holds, due to Proposition 2. Therefore, by (4), we obtain

$$m(H, X-H)>\min\{m(S_i, X-S_i): i\in I\}, \tag{31}$$

since here $H=R_T$. If $T=\emptyset$ and L is nonempty, then we have $H=S_L$, and thus (31) again, by Proposition 3. If both T and L are nonempty, then we get $m(H, X-H)\geqslant \geqslant m(S_L, X-S_L)$. The last result gives us (31) again, with the aid of Proposition 3. Combining (31) and (28), we obtain $m(H, X-H)>m(S_I, X-S_I)$ for any nonempty H such that $H\neq S_I$, $H\subset S_I$. The application of Lemma 1 completes the second part, and thus, the whole proof. ∎

Returning to the situation depicted in Fig. 1, we have that (28) does not hold for $I=\{3, 5\}$ (see the example below Proposition 3), and $S_{\{3,5\}}$ is not Max-minimal. On the other hand $S_1=\{5, 8\}$ as well as $\{4\}$ are Max-minimal (the latter due to Corollary 1), $m(S_1\cup\{4\}, X-(S_1\cup\{4\}))=7<\min\{m(S_1, X-S_1), m(\{4\}, X+-\{4\})\}=8$ and $S_2=S_1\cup\{4\}$ is also Max-minimal.

Theorem 1 is the main result of this section and the whole paper. It gives a criterion which makes possible to construct Max-minimal sets on a basis of those determined previously. Therefore, its application substantially reduces the computational effort required for determining Max-minimal sets when we do it by using Definition 1 only. Namely, in the latter case, i.e. by definition, we need to examine exactly $2^{|S|}-2$ inequalities for making sure that S is Max-minimal while in the former, i.e. by using Theorem 1, it is sufficient to check at most $|S|$ inequalities only.

Sometimes, the statement of a real-life (source) problem implies some additional constraints which should be fulfilled by the partition of a given set X. For instance, it can be required that the cardinality of any class is lower bounded. Moreover, it may happen that no Max-minimal set satisfies this constraint. To overcome this we can apply

THEOREM 2. *Let I, S_I, R_I and K be as in Proposition 1, $S_I\neq X$. If $K\neq\emptyset$, then the relation*

$$m(S_I-R_I, X-(S_I-R_I))\geqslant m(S_I, X-S_I) \tag{32}$$

holds. Moreover, if $K=I$, then the weak inequality in (32) can be replaced by the strong one.

Proof. Since $X-(S_I-R_I)=(X-S_I)\cup R_K$, then

$$m(S_I-R_I, X-(S_I-R_I))=\max\{m(S_I-R_I, X-S_I), m(S_I-R_I, R_K)\}, \tag{33}$$

due to (6). Using (6) again, we get

$$m(S_I, X-S_I)=\max\{m(S_I-R_I, X-S_I), m(R_K, X-S_I)\} \tag{34}$$

According to Definition 1 and (5), we have $m(S_i-R_i, R_i)>m(R_i, X-S_i)\geqslant \geqslant m(R_i, X-S_I)$ since $X-S_i\supset X-S_I$ for any $i\in K$. Analogously, we get $m(S_i+-R_i, R_i)>m(S_i-R_i, X-S_I)$, $i\in K$. That yields

$$m(S_K-R_K, R_K)>\max\{m(S_K-R_K, X-S_I), m(R_K, X-S_I)\} \tag{35}$$

Therefore, by (35), (33) and (34), we obtain that the assertion holds for $K=I$.

Let us now assume that $K\neq I$. Applying (5), we reach

$$\max\{m(S_I-R_I, R_K), m(S_I-R_I, X-S_I)\}\geqslant m(S_I-R_I, R_K)\geqslant m(S_K-R_K, R_K), \tag{36}$$

which, combined with (35) and (33), gives us

$$m(S_I-R_I, X-(S_I-R_I))\geqslant m(R_K, X-S_I) \tag{37}$$

Thus, by (33), (34) and (36), we easily obtain (32), which completes the proof.

Theorem 2 says, roughly speaking, that $S_I-R_I=S_I-R_K$, i.e. the union of S_i's (pairwise disjoint Max-minimal sets) and their proper parts is more similar to the environment than S_I. Therefore, it seems to be better to divide X into S_I and $X-S_I$, rather than into S_I-R_K and $X-(S_I-R_K)$, which is the answer to the question stated above Theorem 2. For instance, in Fig. 1 $S_3=\{1, 4, 5, 8\}$ and $S_5=\{2, 10\}$ are Max-minimal. By taking, e.g., $R_3=\{1, 4\}$ and $R_5=\{2\}$, we get $m(S_{\{3,5\}}, X-S_{\{3,5\}})=7\leqslant m(S_{\{3,5\}}-R_{\{3,5\}}, X-(S_{\{3,5\}}-R_{\{3,5\}}))=11$.

5. Max-minimal sets and a hierarchical clustering technique

In Introductory Notes an interpretation of Max-minimal sets is given, and in Section 2 their definition is formulated. Sections 3 and 4 are devoted to stating and proving some important mathematical properties of Max-minimal sets. The results derived in the preceding sections make it possible to consider Max-minimal sets from another, pure mathematical point of view, without taking into account their interpretation. By Corollary 1 and Theorem 1, we immediately obtain

COROLLARY 4. *S in a Max-minimal set if and only if one of the following conditions holds*

1. $S=\{x\}$, $x\in X$,

2. *$S=S_I$, where S_I is as defined in Section 4, S_J is not Max-minimal for any $J\subset I$, $1<|J|<|I|$, and the inequality (28) is satisfied.*

Let a set X and a function w be given, as described in Section 2. By $S(X, w)$ we denote the family consisting of all Max-minimal sets for fixed X and w. Hence, Corollary 4 and Lemma 2 imply

COROLLARY 5. *$S(X, w)$ is partially ordered with respect to the relation of inclusion.*

Therefore, a technique for constructing Max-minimal sets can be considered as a hierarchical clustering technique. In other words, a computational procedure for seeking Max-minimal sets can be constructed, e.g., as follows. Namely, we begin with the set $U_0=\{\{x\}, x\in X\}$, which consists of Max-minimal sets, due to Corollary 1. Then, we take $U\leftarrow U_0$. Now, we successively examine whether any pair, triple, quadruple, etc., of elements from U constitutes a Max-minimal set (by using Corollary 4). If so, we delete those elements from U and continue for the remaining part of U, etc., until either it becomes empty or it does not contain any Max-minimal set. In general, we take $U\leftarrow U_i$, where U_i consists of Max-minimal sets found for U_{i-1} and of the elements from U_{i-1} which did not belong to any Max-minimal set, $i=1, 2, \ldots$, etc. The procedure finishes when either $|U|=2$ (see the definition of m) or there is no Max-minimal set in U.

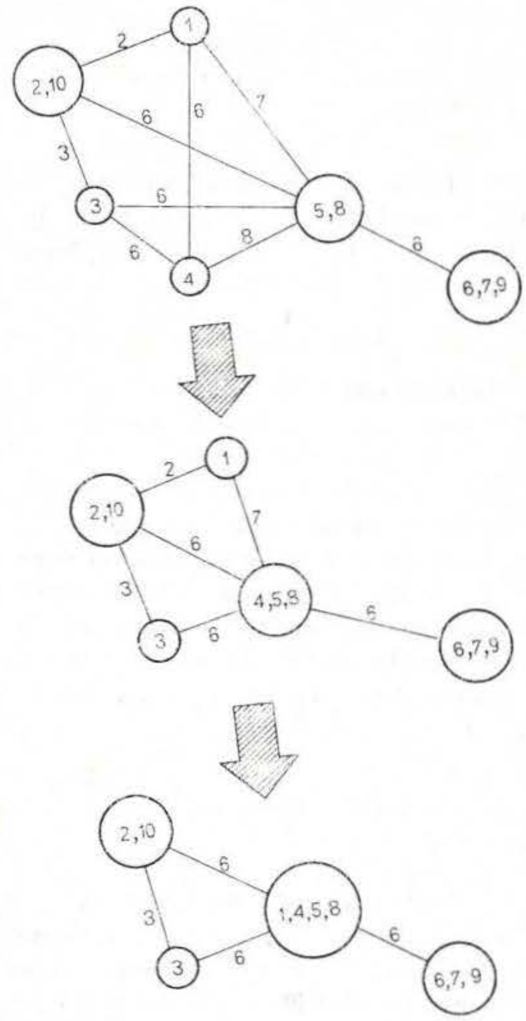

Fig. 2. Consecutive steps in searching Max-minimal sets for the case of X and $W(.,.)$'s shown in Fig. 1

As an example, let us consider the situation depicted in Fig. 1 (see also Fig. 2). We have $U_0=\{\{k\}: k=1, 2, ..., 10\}$. Then, for $U=U_0$, we find, e.g., $S_1^{(1)}=\{2, 10\}$, update $U\leftarrow U-\{\{2\}, \{10\}\}$, find $S_2^{(1)}=\{5, 8\}$, update $U\leftarrow U-\{\{5\}, \{8\}\}$, find $S_3^{(1)}=$ $=\{6, 7, 9\}$, update $U\leftarrow U-\{\{6\}, \{7\}, \{9\}\}$ and note that no union of subsets belonging to the current $U=\{\{1\}, \{3\}, \{4\}\}$ is Max-minimal. Hence, we obtain $U_1=$ $=\{S_i^{(1)}: i=1, 2, 3\}\cup\{\{1\}, \{3\}, \{4\}\}$, initialize $U\leftarrow U_1$, find $S_1^{(2)}=S_2^{(1)}\cup\{4\}$, update $U\leftarrow U-\{S_2^{(1)}, \{4\}\}$, and no more Max-minimal sets are found in this step, i.e. $U_2=\{S_1^{(2)}, S_1^{(1)}, S_3^{(1)}, \{1\}, \{3\}\}$. We take $U\leftarrow U_2$, find $S_1^{(3)}=S_1^{(2)}\cup\{1\}$, and obtain $U_3=\{S_1^{(3)}, S_1^{(1)}, S_3^{(1)}, \{3\}\}$. For $U=U_3$ we find no Max-minimal set, and thus the searching terminates.

Although the procedure outlined above seems to be easy for programming on a computer and is based on a simple idea, we should, however, consider its efficiency. To do it we notice that if, e.g., no Max-minimal set of cardinality greater than one occurs for a given pair X and w, then we examine exactly

$$2^{|X|}-|X|-2 \tag{37}$$

subsets of X for their Max-minimality. Hence, the numerical complexity of the procedure is not of polynomial-type. On the other hand, the number of subsets for a given nonempty, finite set X, $|X|>1$, being either disjoint or such that one of them is included in another, does not exceed

$$|X|\,[\gamma+\ln(|X|-1)+1], \tag{38}$$

where $\gamma=0.577...$ is the Euler constant (see the final part of Section 2 in [14]). Therefore, due to Lemma 2, (38) gives also an upper bound for $|S(X, w)|$. The comparison of (37) and (38) proves that the procedure is inefficient, indeed.

To increase the efficiency of the procedure we can proceed as for the classic minimal sets [14]. Namely, instead of taking into account each union of elements from the current U, we can restrict our examination to some specific, previously well-defined cases. This fruitful idea gave remarkable results in the case of the classic minimal sets. Namely, it led to an $0(|X|^5)$ algorithm for the classic minimal sets [14] while the previous algorithms [2, 5] were not of polynomial-type. Now, a new algorithm for Max-minimal sets based on this idea is in preparation and will be a subject of the next, forthcoming paper.

6. Concluding remarks

The idea of a minimal set lies in that some subset of a greater set is internally stronger connected with, or more similar than, its environment. This concept, introduced by Luccio and Sami [8], then first developed by Kacprzyk and Stańczak [2, 5, 6], and further — by Nieminen [9] and Stańczak [14], was described in Introductory Notes in more detail. Previous papers (see, e.g., [2, 5, 6, 9, 14]) were devoted to considering the strength of connections rather than "pure" similarity or likeness. It was historically biased by some specific interpretation of a minimal set, in fact originated by Luccio and Sami [8]. Namely, a (classic) minimal set was understood as some part of a network (see, e.g., [3, 4, 7, 8, 11]) or a group of enterprises [1] which cooperated more intensively with each other than with the rest of a system or an environment. Such interpretations lead to the definition analogous to our Definition 1, with f instead of m (see, e.g., Definition 1 and Lemma 1 in [5]).

Although the classic minimal sets technique is a useful tool for solving many practical problems [1, 3, 4, 7, 8, 10, 11], it gives sometimes, however, an inadequate description of reality, as outlined in Section 1. It is implied by taking f as the parameter of aggregate reciprocal similarity. Thus, the question arises whether f can be replaced by another index of aggregate reciprocal similarity. More precisely, whether there exists a function mapping $\{\{A, B\}: A \cap B = \emptyset,\ A, B \subset X\}$ into $R^+ \cup \{0\}$ such that, first, its value for $A = \{x\}$ and $B = \{y\}$ equals to $w(x, y)$, where $x, y \in X$, and, second, it has an appropriate interpretation for a great number of cases in which f is inadequate and, third, it produces such a mathematical structure with respect to the idea of minimal sets (outlined above) that it is interesting from an algorithmic point of view. These questions are, evidently, the classic problems which should be solved when we construct any reasonable partitioning method.

The answer to the first two questions are easy and they are contained in taking into account m given by (2) instead f and in Definition 1. The third problem is more complex, and the paper is devoted to solve it.

It seems that the answer to the third question posed above is positive. Moreover, it is interesting that Max-minimal sets have, in fact, almost identical properties as the classic minimal sets (compare, e.g., Definition 1, Lemma 1, Corollaries 1 and 2, Lemma 2, Propositions 1, 2 and 3, Theorem 1, Corollaries 4 and 5 and Theorem 2 with Lemma 1 in [5], Definition 1 in [5], Corollaries 4.1 [11] and 2 [6], Lemma 2 in [2], Theorem 4.1 in [11], Proposition 1 in [2], Theorem 1 in [2], Proposition 3 in [5], Theorem 4.4 in [11], Corollary 6.1 in [11] and Proposition 2 in [2], respectively). That permits to conjecture that there exist more expressions for the aggregate reciprocal similarity which produces a similar mathematic system with respect to the definition like Definition 1.

Finally, the directions of future researches in the minimal sets theory can be outlined. From the above remarks it follows that an efficient procedure for seeking Max-minimal sets should be constructed (it is now in preparation). Moreover, it is interesting whether it is true that there exist more functions than f and m which, together with the general definition of a minimal set, produce an analogous mathematical system as the mentioned operations f and m. That is important not only from the theoretical, but also from the practical point of view. Furthermore, it should be pointed out that in applications w's, i.e. elementary similarities, are taken from either measurements or approximate formulae. Therefore, in fact, a more adequate practical approach to the minimal sets should be based on inexact and/or fuzzy numbers rather than on ordinary (sharp) real numbers. That implies the next direction of research in the field of the minimal sets theory.

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Wprowadzenie do zespołów Max-minimalnych

Zespół minimalny jest takim podzbiorem mnogości obiektów, który jest związany silniej wewnętrznie niż z otoczeniem. W poprzednich pracach (patrz np. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14]) siłę powiązań między dwoma niepustymi i wzajemnie rozłącznymi podzbiorami określano jako sumę wartości powiązań elementarnych, tzn. powiązań występujących między parami obiektów. Definicja ta, jakkolwiek uzasadniona w przypadku licznych problemów powstających przy poszukiwaniu prawidłowego podziału sieci (patrz np. [3, 4, 7, 8]) może być nieadekwatna w zagadnieniach innej natury. W artykule rozpatruje się analogiczną strukturę matematyczną. Jedyna różnica polega na przyjęciu odmiennego określenia "podobieństwa" dwóch niepustych i rozłącznych zbiorów obiektów. Okazuje się, że nowa struktura i (klasyczne) zespoły minimalne posiadają podobne właściwości.

Введение в Макс-минимально связанные множества

Минимально связанное множество это множество объектов, которое внутренно связано сильнее чем с ее средой. В предыдущих статьях [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14] сила связанности двух непустых и непересекающихся подмножеств была суммой сил существующих для элементарных связанностей, т.е. для пар объектов. Это определение, хотя имеет применение во многих проблемах касающихся рациональной декомпозиции сети [3, 4, 7, 8], может быть неадекватно для других целей. Аналогичная математическая структура рассмотрена в этой статьи за исключением, что сходства для непустых и непересекающихся множеств объектов определяются по другому. Эта новая математическая структура имеет свойства похожие на свойства классических минимально связанных множеств.