Control
and Cybernetics
VOL. 15 (1986) No. 2

Some general structure implied by the idea of minimal sets

by

WIESŁAW STAŃCZAK
Polish Academy of Sciences
Institute of Computer Science
Warszawa, Poland

In the previous papers of the author and collaborators see, e.g., references some specific cases of the theory and technique of so-called minimal sets were considered. Now, it is shown that the fundamental idea of minimal sets concerns a greater number of partitioning-type problems. Basing on this idea, generalized and g-minimal sets are defined and their properties are derived. Several examples are given.

1. Preface

The idea of classical minimal sets was introduced by Luccio and Sami (1969), and developed by Kacprzyk and Stańczak (1976, 1978b). Some general aspects of this partitioning method and its biases were discussed by Owsiński (1981). Stańczak (1984) proposed a polynomial-type algorithm for finding classical minimal sets.

The idea of classical minimal sets was originated from the network partitioning problems by Luccio and Sami (1969), and thus it took into account aggregated strengths of connections between groups of entities rather than their "pure" similarity or likeness. Therefore, it occurred to be inadequate for some practical purposes of nature different than that of a network partitioning type. The analysis performed by Stańczak (1986) showed that the basis of classical minimal sets consisted of three elements. They were the method for evaluating "connections" between distinct subsets, the way of ranking these "connections" and the fundamental principle which is described in the final part of this section. That led to the concept of Max-minimal sets, being analogous to the previous one. Moreover, it was conjectured [Stańczak (1986)] that there existed more partitioning techniques similar to that of classical minimal sets and Max-minimal sets. This paper fully confirms the supposition. It is important not only from the theoretical point of view. It occurred that classical minimal sets constituted a valuable tool for solving many practical problems of network partitioning type (e.g., for cooperative facilities Kacprzyk and Stańczak (1975), for the computer Kacprzyk and Stańczak (1970), Nowicki and Stańczak (1980), telecommunication Kaliszewski, Nowicki and Stańczak (1975) and electrical Luccio and Sami (1969) network). Therefore, it can be believed that these analogous approaches give a simple and efficient way for solving problems, where reciprocal "connections" as well as methods of ranking them are defined differently than either in, e.g., Kacprzyk and Stańczak (1976, 1978b) or in Stańczak (1986). Obviously, it is better, if possible, to handle and solve the problem of minimal sets in general and then apply it to real-life models, than to produce a voluminous literature on its specific cases.

This paper is organized as follows. First, the general idea of minimal sets is recalled verbally. Second, in Section 2, it is formalized so as to create a possibility of deriving some of its basic implications. This, though, leads to a clustering technique being, in general, inefficient from the algorithmic point of view. Therefore, some additional assumption is necessary. Its introduction produces a particular case of a generalized minimal set called g-minimal which is discussed in Section 3. Third, some examples of generalized and g-minimal set theories are given and considered. The directions of future research in the minimal set theory and technique are proposed.

Now, let us remind the basic idea of minimal sets (for more detailed discussion see Stańczak (1986). Namely, let an aggregate parameter (or any way of calculating its value) describing either the strength of connections or similarity or dissimilarity, between two disjoint nonempty sets of entities be given in advance. Moreover, let some criterion (or relation) useful for ranking the values of the parameter be defined. This represents the following statement: either "is more strongly connected" or "is more similar" or "is less dissimilar", respectively, and is abbreviated here by "is better to merge". Let us now consider some nonempty subset of a greater set of entities. The former is called a (generalized) minimal set if it is better to merge each of its nonempty proper parts with the remaining part than with the complement of the subset in this greater set. And this is the fundamental principle of minimality. More figuratively, a (generalized) minimal set should be either more strongly connected or more similar or less dissimilar internally than with its environment.

2. Generalized minimal sets

We consider aset $X, |X| > 1$, and a function $g\colon \{\{A, B\}\colon \emptyset \neq A, B \subset X, A \cap B = \emptyset\} \to Y$, where $\{A, B\}$ is an unordered pair, and Y is a nonempty set linearly ordered by some relation $\prec$. The definition yields that g is symmetric with respect to its arguments. Moreover, we assume

G1. $$g(A, C) \prec g(B, D),$$

for each pair of nonempty sets A and C such that $A \subset B$, $C \subset D$, where $B, D \subset X$ are disjoint.

DEFINITION 1. If $g(R, X-S) \precneq g(R, S-R)$ holds (here and further on $a \precneq b$ is equivalent to the pair of relations: $a \prec b$ and $a \neq b$ for $a, b \in Y$) for a nonempty set $S \subset X$, $S \neq X$, and for each its nonempty subset $R \neq S$, then S is called a (generalized) minimal set.

Applying de Morgan rules to Definition 1, we easily obtain.

COROLLARY 1. *Any* $\{x\}$, $x \in X$, *is a minimal set.*

LEMMA 1. *Two minimal sets are either disjoint or one of them is included in the other.*

Proof. Let S and Q be distinct minimal sets. The cases $S \subset Q$ and $Q \subset S$ are evident, therefore we can suppose $S \not\subset Q$ and $Q \not\subset S$. Let $S \cap Q = T \neq \emptyset$. For convenience, we denote $P = Q-T$, $R = S-T$ and $H = X-(S \cup Q)$. Obviously, P and R are nonempty. By Definition 1, we get $g(T, H \cup R) = g(T, X-Q) \precneq g(P, T)$. Thus, using G1, we obtain $g(R, T) \precneq g(P, T)$. Applying the same arguments for $g(T, H \cup P)$ we derive $g(P, T) \precneq g(R, T)$, which contradicts the previous relation. Therefore, we have $T = \emptyset$. ■

By $S(X, g, \prec)$ we denote the collection consisting of all minimal sets for X, g and $\prec$ given as above. Moreover, let M be a nonempty set of indices, and $\{H_m : m \in M\}$ be a family of subsets taken from X. From now on $\bigcup_{m \in M} H_m$ is denoted by H_M. Furthermore, $S_M = \{S_m : m \in M\} \subset S(X, g, \prec)$ denotes a collection of pairwise disjoint minimal sets.

Combining Corollary 1 and Lemma 1, we get

PROPOSITION 1. $S(X, g, \prec)$ is nonempty and is partially ordered by inclusion.

PROPOSITION 2. If S is a minimal set, then there exists S_M (in particular, $|M|=1$) such that $S = S_M$.

In other words, Propositions 1 and 2 say that there exist algorithms generating minimal sets (for a finite X) which can be considered as particular cases of so called the hierarchical clustering technique. Moreover, they are agglomerative.

Let the image of g (denoted here by (Y/g)) be lower bounded. Hence, $\inf(Y/g) \prec g(R, X-S)$ for any nonempty $R, S \subset X$ such that $R \cap (X-S) = \emptyset$. Therefore, by Definition 1, we get.

COROLLARY 2. *If S is a minimal set and (Y/g) is lower bounded, then* $\inf(Y/g) \precneq g(R, S-R)$ *for each nonempty subset R of S, $R \neq S$.*

If $\inf(Y/g) \in (Y/g)$, then some specific situations may exist so that the search for minimal sets can be decomposed into several separate subprocesses. The case occurs when there exists a nonempty set $H \subset X$, $H \neq X$, such that

$$g(H, X-H) = \inf(Y/g) \tag{1}$$

Namely, let P and R be nonempty, $P \subset H$, $R \subset X-H$ and $P \cup R \neq X$. Then, by G1, we have $\inf(Y/g) \prec g(P, R) \prec g(H, X-H) = \inf(Y/g)$, i.e. $g(P, R) = \inf(Y/g)$, due to the transitivity of $\prec$. Therefore, Corollary 2 implies that $P \cup R$ is not a minimal set. Using Propositions 1 and 2, we easily get.

PROPOSITION 3. Let (1) be satisfied for some nonempty set $H \subset X$, $H \neq X$. Then, there exists S_M (in particular, $|M|=1$) such that $H = S_M$ and, moreover, $H \cap S = \emptyset$ for any $S \in \{x : x \in S(X, g, \prec),\ x \not\subset S_M\}$.

In other words, Proposition 3 says that each H fulfilling (1) can be examined separately which is the desired result.

3. Some properties of g-minimal sets

The features of $S(X, g, \prec)$ derived in Section 2 are insufficient to give a basis for any efficient algorithm seeking minimal sets, even for a finite X. Namely, although the set of candidates for minimal sets is restricted to some specific subsets of X (see Propositions 1, 2 and 3), no constructive way for verifying their minimality is known. It resides, in fact, on Definition 1 only, i.e. one should examine exactly $2^{|S|}-2$ relations for any finite S, $|S|>1$. To improve the way of verification we need to say more about the function g.

Let us assume that there exists a binary operation

$$\circ : Y \times Y \to Y,$$

defined by

G2. $$g(A, B \cup C) = g(A, B) \circ g(A, C),$$

where A, B, C is any triple of pairwise disjoint nonempty subsets of X. From G1 and G2 the following rules immediately result

01. $a \circ b = b \circ a$, (commutativity)
02. $a \circ (b \circ c) = (a \circ b) \circ c$, (associativity)
03. $a \prec a \circ b$, (weak monotonicity)
04. if $b \prec c$, then $a \circ b \prec a \circ c$, (weak monotonicity)
05. if $a \preccurlyeq b$, then $a \preccurlyeq a \circ b$, (strong monotonicity)
06. if $a \preccurlyeq c$ and $b \preccurlyeq c$, then $a \circ b \preccurlyeq a \circ c$, (strong monotonicity)

for $a, b, c, \in (Y/g)$.

Let us now assume that $a \circ b \preccurlyeq a \circ c$. Supposing $c \prec b$, we get $a \circ c \prec a \circ b$, according to 04, which contradicts our assumption.
Thus,

07. if $a \circ b \preccurlyeq a \circ c$, then $b \preccurlyeq c$, for $a, b, c \in (Y/g)$.

By using appropriate examples, it can be easily shown that $a \circ b \preccurlyeq a \circ c$ does not, in general, imply $a \prec c$. Moreover, to avoid any confusion, we introduce.

Definition 2. If g satisfies G2, then each $S(-S(X, g, \prec)$ is called g-minimal.

Evidently, each result serived for minimal sets remains in force for g-minimal ones. Moreover, it is convenient to make no changes in the notations introduced in Section 2. Furthermore, let some S_M be given, and sets $R_m \subset S_m$ be chosen so that $R_m \neq S_m$, $m \in M$. We denote $R_M = \{R_m : m \in M\}$ and $K(R_M) = \{j : j \in M, R_j \neq \emptyset\}$. Now, we are in a position to prove.

Lemma 2. *S is g-minimal if and only if*

$$g(S, X-S) \preccurlyeq g(R, X-R) \tag{2}$$

holds for each nonempty $R \subset S$, $R \neq S \neq X$.

Proof. We consider a set $S \subset X$, $S \neq X$, and its nonempty subset $R \neq S$, and denote $P = S - R$, $H = X - S$. P and H are, obviously, nonempty and disjoint and, moreover, $X - R = P \cup H$. By G2, we have

$$g(R, X-R) = g(R, H) \circ g(P, R) \quad \text{and} \quad g(S, X-S) = g(R, H) \circ g(P, H) \tag{3}$$

Supposing that S is g-minimal and using Definitions 1 and 2, we obtain

$$g(P, H) \preccurlyeq g(P, R) \quad \text{and} \quad g(R, H) \preccurlyeq g(P, R) \tag{4}$$

These relations yield

$$g(R, H) \circ g(P, H) \preccurlyeq g(R, H) \circ g(P, R), \tag{5}$$

by 06, which, combined with (3), gives (2).

Assuming that (2) holds (for R and S described as above (3)) and applying (3), we obtain (5). Thus, using 07, we reach (4). Since R is any nonempty proper subset of S, then the condition in Definition 1 is satisfied. ■

Lemma 3. *Let S be g-minimal, R be its nonempty subset, $R \neq S$, and V be a nonempty set chosen so that $S \cap V = \emptyset$. Then, $g(V, X-V) \prec g(R \cup V, X-(R \cup V))$.*

Proof. Since $X - V = H \cup R$, where $H = X - (R \cup V)$, then

$$g(V, X-V) = g(V, H) \circ g(R, V), \tag{6}$$

$$g(R \cup V, X-(R \cup V)) = g(V, H) \circ g(R, H), \tag{7}$$

by G2. Taking into account that $S \cap V = \emptyset$ implies $X - S \supset V$ and $H \supset S - R$ (because $X - V \supset S$), we obtain

$$g(R, S-R) \prec g(R, H), \tag{8}$$

$$g(R, V) \prec g(R, X-S), \tag{9}$$

according to G1. Let us now suppose that $g(R \cup V, X-(R \cup V)) \preccurlyeq g(V, X-V)$ Therefore, using (6), (7) and 07, we get $g(R, H) \preccurlyeq g(R, V)$. Combining it with (8) and (9), we have $g(R, S-R) \preccurlyeq g(R, X-S)$, which violates the assumption that S is g-minimal. ■

Lemma 3 plays the crucial role in proving of

PROPOSITION 4. Let I be a nonempty and finite set of indices, S_I, R_I, $K=K(R_I)$ be defined as before, and $Q\subset X$ be an arbitrary set such that $S_I\cap Q=\emptyset$. If $|Q|\,|K|>0$ and/or $|K|\geqslant 2$ is satisfied, then $P=Q\cup R_I$ is not g-minimal. Moreover, the inequality $|Q|>0$ implies

$$g(Q, X-Q)\prec g(P, X-P) \tag{10}$$

Proof. The assumptions yield $|K|\geqslant 1$. Therefore, without any loss of generality, we can suppose that K is an initial and finite segment of the set of natural numbers, since $K\subset I$ and I is finite. We denote $K(i)=\{t: t\leqslant i\}$, $i=1, 2, \ldots, |K|$ and $P_i=Q\cup R_{K(i)}$. The proof proceeds by induction.

Let Q be nonempty. For $|K|=1$ the assertion holds by Lemma 3. Let us now assume that

$$g(Q, X-Q)\prec g(P_j, X-P_j) \tag{11}$$

is satisfied for each $j=i$, $1\leqslant i<|K|$. Taking $V=P_i$ in Lemma 3, we obtain $g(P_i, X-P_i)\prec g(P_{i+1}, X-P_{i+1})$. The latter, combined with (11) for $j=i$, gives (11) again, but now for $j=i+1$, i.e. the second step of induction is accomplished.

According to Lemma 2, (10) means that P is not g-minimal, which finishes the proof for the case $Q\neq\emptyset$.

If $Q=\emptyset$, then, by assumptions, $|K|\geqslant 2$. Lemma 1 yields that P_2 is not g-minimal. For $|K|>2$, we can take $V=P_{|K|-1}$, $R=R_{|K|}$, and then use Lemmas 3 and 2. ∎

Before we formulate the subsequent assertion, it should be pointed out that the operation $\max=\max_{\prec}$ is here performed subject to the relation $\prec$ applied to a finite set. In other words it may be, in general, not the same as the ordinary $\max_{\leqslant}$, where the latter refers to the usual $\leqslant$ (i.e. "not less than") defined for any nonempty subset of real numbers. The analogous remark can be stated for the operation $\min=\min_{\prec}$, which is used further.

PROPOSITION 5. Let I, S_I, R_I and K have the same meaning as in Proposition 4. Then, $K\neq\emptyset$ implies $\max\{g(S_i, X-S_i): i\in K\}\preccurlyeq g(R_I, X-R_I)$.

Proof. The proof proceeds by induction. If $|K|=1$, then the assertion is implied by Lemma 2. Let us suppose

$$\max\{g(S_t, X-S_t): t\in T\}\preccurlyeq g(R_T, X-R_T), \tag{12}$$

for each i, $1\leqslant i<|K|$, where $T=K(i)$, and the assumptions relating to K and the definition of $K(i)$ are the same as in the proof of Proposition 4. Due to Proposition 4, we have (10a) where $P=R_{K(i+1)}$ and $Q=R_{K(i)}$ in (10) and, second (10b) where $P=R_{K(i+1)}$ again and $Q=R_{i+1}$ in (10). The latter form of (10) i.e. (10b), combined with Lemma 2 (applied to S_{i+1} and R_{i+1}), gets

$$g(S_{i+1}, X-S_{i+1})\preccurlyeq g(P, X-P) \tag{13}$$

The former form of (10) (i.e. (10a)), (13) and (12) yield (12) again, but now for $T=K(i+1)$, which accomplishes the second step of induction, and thus the whole proof. ■

PROPOSITION 6. Let I and S_I be as in Proposition 4 and, moreover, $S_I \neq X$. If S_J is not g-minimal for any $J \subset I$ such that $|J|>1$, then $\min\{g(S_i, X-S_i): i\in I\} \prec g(S_I, X-S_I)$.

Proof. Since S_J, $|J|>1$, is not g-minimal, then Lemma 2 yields the existence of a nonempty set $P=P(J)\subset S_J$, $P\neq S_J$, which satisfies

$$g(P, X-P) \prec g(S_J, X-S_J) \tag{14}$$

Evidently, in general, $P=R_T \cup S_V$, where $T=T(J)$ and $V=V(J) \neq J$ are disjoint, $\emptyset \neq V\cup T\subset J$ and, moreover, $T=K(R_T)$. Therefore,

$$|V|<|J| \tag{15}$$

CASE 1. $T\neq\emptyset$ and $V=\emptyset$, Hence, $P=R_T$, i.e. we obtain (12), due to Proposition 5. Thus, using (14) and the evident properties of operations min and max, we immediately get the assertion.

CASE 2. $T\neq\emptyset$ and $V\neq\emptyset$. Taking $Q=S_V$ and using Proposition 4, we obtain (10), which, combined with (14), gives Case 3, as described below.

CASE 3a. $T=\emptyset$ and $V\neq\emptyset$. If $|V|=1$, then $P=S_V=S_j$, for some $j\in I$, and the assertion is implied by the obvious properties of the operation min. Otherwise, we have Case 3b.

CASE 3b. $T=\emptyset$ and $|V|>1$. Now we arrive again in the situation described at the very beginning of the proof, where here we handle V instead of J.

In the other words, either we reach Case 1 or Case 3a which finishes the proof, else we successively return to Case 3b, perhaps via Case 2. The latter situation causes the iterative process which consists in constructing consecutive P's, as described at the preliminary part of the proof and checking up whether Cases 1 and/or 3a occur. If so, then the process terminates. Otherwise, we construct the next P, etc. Due to (15), the cardinality of the current V diminishes in each step which proves that the process must terminate (in the worst case it converges to Case 3a). Furthermore, (14) holds for each its step. Therefore, by using the transitivity of $\prec$ and the remarks about Cases 1 and 3a, the proof is accomplished.

Now, we are in a position to prove the main result of this section.

THEOREM 1. *Let I, $|I|>1$, be a finite set of indices, and S_I be a collection of pairwise disjoint g-minimal sets such that $S_I\neq X$. Let S_J be not g-minimal for any $J\subset I$, $1<|J|<|I|$. S_I is g-minimal if and only if*

$$g(S_I, X-S_I) \preccurlyeq \min\{g(S_i, X-S_i): i\in I\} \tag{16}$$

P r o o f. The necessity immediately results from Lemma 2.

Sufficiency. Let us consider any $P=P(I)$ constructed as described below (14). If $T\neq\emptyset$ and $V=\emptyset$, then

$$g(S_I, X-S_I)\preccurlyeq g(P, X-P), \tag{17}$$

according to Proposition 5 and (16). If $T=\emptyset$ and $V\neq\emptyset$, then we get (17), by Proposition 6 and (16). It remains to consider the case $T\neq\emptyset$ and $V\neq\emptyset$. Taking $Q=S_V$, we obtain (10), due to Proposition 4. By the assumptions S_V is not g-minimal, because $1<|V|<|I|$. Therefore, applying Proposition 6, combining the result with the previously obtained (10) and using (16), we get (17) again, due to the transitivity of $\prec$. Therefore, for each nonempty $P\subset S_I$, $P\neq S_I$, the relation (17) holds, which proves that S_I is g-minimal, due to Lemma 2. ■

Theorem 1 suggests how to verify the g-minimality in a more efficient and constructive way than that based on Definitions 1 and 2 or on Lemma 2. It can be easily shown by using a simple example. Namely, let a candidate for being g-minimal have the form $S_{\{1,2,3,4\}}$, where, e.g., $|S_i|=50$, $i=1, 2, 3, 4$. Hence, the method described in the initial part of this section requires $2^{200}-2\approx 1.606937\cdot 10^{60}$ examinations while the way proposed in Theorem 1 needs only 4 verifications.

4. Examples and final remarks

In the Max-minimal [Stańczak (1986)] and classical minimal sets theory [Kacprzyk and Stańczak (1976, 1978b)] the assumption was adopted that X is finite which is omitted here in Definitions 1 and 2. In fact, it depends upon the intrinsic nature of g whether this assumption is necessary or not. For instance g_i's, $i=1, 2, 6$, mentioned below need it, and for $i=3, 4, 5$ it is needless. On the other hand, this generalization, however interesting, is of less practical meaning. First, since we rather rarely consider a real-life decomposition problem defined for an infinte set of entities. Second, the paper gives no method for checking up even the g-minimality for infinite candidates, and it is, in fact, unknown, in general.

Let us now restrict our considerations to the partitioning problems assuming finite X. The definition of generalized minimality is, in general, no constructive from the algorithmic point of view, even for such a case. To overcome this drawback some additional assumptions should be adopted, as, e.g., it was done in Section 3 and has led to the concept of g-minimality. Considering G2 in more detail and taking into account that the arguments of g, say A and B, are now finite (as subsets of a finite set), we easily get

$$g(A,B)=\left.\underbrace{\begin{array}{l} w(x_1,y_1)\circ w(x_2,y_1)\circ\ldots\circ w(x_{|A|},y_1)\\ \circ\, w(x_1,y_2)\quad w(x_2,y_2)\circ\ldots\circ w(x_{|A|},y_2)\\ \ldots\qquad\quad \ldots\qquad\quad \ldots\\ \circ\, w(x_1,y_{|B|})\circ w(x_2,y_{|B|})\circ\ldots\circ w(x_{|A|},y_{|B|})\end{array}}_{A}\right\} B, \tag{18}$$

by induction, for $w(x_i, y_j)=g(\{x_i\}, \{y_j\})$ being the elementary strength of connection, similarity or dissimilarity, etc., between the x_ith and y_jth entities, Kacprzyk and Stańczak (1976, 1978b), $A=\{x_i: i=1, 2, ..., |A|\}$, $B=\{y_j: j=1, 2, ..., |B|\}$ and $A \cap B=\emptyset$. It means that the function g and the operation $\circ$ are strongly dependent on each other or one of them is generated, if possible, by the other. This rule is built in into the definitions of Max-minimal (Stańczak (1986)) and classical minimal sets. For instance, g_i' s, $i=2, 3, 4, 5, 6, 7, 8$, mentioned below satisfy G2, and thus possess the property (18) for the operation $\circ$ as indicated in brackets.

The relation $\prec$ also strongly depends upon the nature of g and, if exists, of $\circ$. It is implied, first, by the preliminary assumption that $\prec$ linearly orders the image of g, second, by G1 and third, by the rules 03–07. Moreover, it should have an intuitive meaning which is necessary for an adequate interpretation of results given by the partitioning process.

Now, we give some simple examples of functions g, their images and relations $\prec$ implied by the form of g. The appropriate operations $\circ$, if exist, are indicated in brackets. The statement $a: =b$ used below denotes that a is defined by b. Moreover, the symbols being on the right-hand side of $: =$, e.g. $: +, ., \leqslant$ (not less than) and $\geqslant$ (not greater than) have their usual sense.

In geographical (regional) analysis the mapping often used is

$$g_1(A, B):=(\sum_{x\in A}\sum_{y\in B}|w(x, y)|^p)^q, \text{ for } p, q>0, \quad Y_1: =[0, +\infty), \quad \prec: =\leqslant$$

Its particular form, i.e.

$$g_2(A, B): =g_1(A, B)_{|p=q=1}, \quad Y_2: =[0, +\infty), \quad \prec: =\leqslant, \quad (\circ_2: =+),$$

is taken into account in the classical minimal sets theory. The system

$$g_3(A, B): =\sup\{w(x, y): x\in A, y\in B\}, \quad Y_3: =(-\infty, +\infty), \quad \prec: =\leqslant, \quad (\circ_3: =\sup),$$

restricted to a finite X (therefore, sup was substituted by max), gave a basis for Max-minimal sets. Moreover, for instance, we can take

$$g_4(A, B): =\inf\{w(x, y): x\in A, y\in B\}, \quad Y_4: =(-\infty, +\infty), \quad \prec: =\geqslant, (\circ_4: =\inf),$$

$$g_5(A, B): =\prod_{x\in A}\prod_{y\in B} w(x, y), \quad Y_5: =[0, 1], \prec: =\geqslant, (\circ_5: =\cdot),$$

$$g_6(A, B): =\prod_{x\ni A}\prod_{y\in B} w(x, y), \quad Y_6: =[1, +\infty), \prec: =\leqslant, (\circ_6: =\cdot),$$

Furthermore, let $\bar{u}: (-\infty, +\infty)\to(-\infty, +\infty)$ and $\underline{u}: (-\infty, +\infty)\to(-\infty, +\infty)$ be an upper and lower, respectively, cutting function defined by

$$\bar{u}(a)=\begin{cases} a & \text{for } a\leqslant u_{01} \\ u_{01} & \text{for } a>u_{01} \end{cases}, \quad \underline{u}(a)=\begin{cases} u_{02} & \text{for } a<u_{02} \\ a & \text{for } a\geqslant u_{02} \end{cases},$$

where u_{01} and u_{02} are some real constants given in advance.

It can be easily shown that the system

$$g_7(A,B):=\bar{u}(g_i(A,B)),\ Y_7:=\{y: y\leqslant u_{01}\}\cap Y_i\neq\emptyset,\ \prec:=\leqslant,\ \left(\circ_7:=\bar{u}(\circ_i(\cdot,\cdot))\right),$$

for $i=2, 3, 6$, has the required properties, where $\bar{u}(\circ_i(\cdot,\cdot))$ denotes the superposition, i.e. $\bar{u}(\circ_i(a,b)):=\bar{u}(a\circ_i b)$, as usual. The analogous features hold for

$$g_8(A,B):=\underline{u}(g_i(A,B)),\ Y_8:=\{y: y\geqslant u_{02}\}\cap Y_i\neq\emptyset,\ \prec:=\geqslant,\ \left(\circ_8:=\underline{u}(\circ_i(\cdot,\cdot))\right),$$

for $i=4, 5$.

The functions g_1 and g_2 (taken with their image and the ordering relation) are of a strenght of connections-type (see, e.g., Kacprzyk and Stańczak (1975, 1978a), Kaliszewski, Nowicki and Stańczak (1975), Luccio and Sami (1969) and Nowicki and Stańczak (1980)). The function g_3 has a sense of similarity, g_4 — of dissimilarity, and g_5 can be understood in terms of probability. Furthermore, g_7 and g_8 have the analogous interpretations as the previous g_i' s used in their definitions. Hence, it is obvious that the g-minimality has a wide range of interpretations, and thus, that it can be considered as a useful tool for solving many real-life partitioning problems.

It remains to say something about future researches which should be made in the generalized and/or g-minimal set theories. Namely, a polynomial-type procedure for seeking g-minimal sets is needed. It can be conjectured that, first, such a procedure exists and, second, probably it has a structure similar to that derived by Stańczak (1984) for classical minimal sets (however, some simpler and more efficient procedures can exist for particular forms of g). Moreover, the g-minimal sets technique is a tool oriented to solve practical, real-life problems. Thus, the values of g are obtained from approximate formulae and/or measurements. That yields either a fuzziness or even inexactness in the problem statement which may give, in fact, a more adequate approach to the reality than the sharp model discussed in the paper. Therefore, the theory of generalized and g-minimal sets could be developed so as to introduce some fuzziness or inexactness. The third problem, interesting as well from the theoretical as the practical point of view, lies in establishing relations between the technique of g-minimal sets and another clustering approaches. Some preliminary remarks about this subject can be found in Owsiński (1981).

The author would like to express thanks to the anonymous referee for his remarks, most of which could be taken into account in this paper. Some other ones shall be taken into consideration in the furthcoming paper of this author.

References

[1] Kacprzyk J., Stańczak W. Zastosowanie metody zespołów minimalnych do podziału grupy przedsiębiorstw na podgrupy. *Archiwum Automatyki i Telemechaniki*, **20** (1975) 4, 513–526.

[2] Kacprzyk J., Stańczak W. On an extension of the method of minimally interconnected subnetworks. *Control a. Cybernetics*, **5** (1976) 4, 61–77.

[3] Kacprzyk J., Stańczak W. Partitioning a computer network into subnetworks and allocation of distributed data bases. [in:] Optimization Techniques, J. Stoer (ed.), Berlin–Heidelberg–New York, Springer Verlag 1978, p. 464–472.

[4] Kacprzyk J., Stańczak W. On a further extension of the method of minimally interconnected subnetworks. *Control a. Cybernetics*, 7 (1978) 2, 17–31.

[5] Kaliszewski I., Nowicki T., Stańczak W. O dekompozycji struktury sieci telefonicznej międzycentralowej metodą zespołów minimalnych. *Rozprawy Elektrotechniczne*, **21** (1975) 2, 573–580.

[6] Luccio F., Sami M. On the decomposition of networks in minimally interconnected subnetworks. *IEEE Trans. Circuit* **CT-16** (1969) 2, 184–188.

[7] Nowicki T., Stańczak W. O metodzie wstępnego projektowania struktur topologicznych i konfiguracji sieci teleprzetwarzania. Warszawa–Łódź, PWN, 1980.

[8] Owsiński J. Intuition vs. formalisation in clustering: local and global criteria of grouping. *Control a. Cybernetics*, **10** (1981) 1–2, 73–87.

[9] Stańczak W. An efficient algorithm for partitioning a network into minimally interconnected subnetworks. *Control a. Cybernetics*, **13** (1984) 1–2, 97–112.

[10] Stańczak W. An introduction to Max-minimal sets. *Control and Cybernetics*. **15** (1986) 1.

Pewna uogólniona struktura generowana przez ideę zespołów minimalnych

W poprzednich pracach autora i in. (patrz np. literatura) rozpatrywano pewne szczególne przypadki teorii i techniki tzw. zespołów minimalnych. Obecnie pokazano, że podstawowa idea zespołów minimalnych dotyczy większej liczby problemów podziału. Opierając się na wspomnianej koncepcji zdefiniowano uogólnione zespoły minimalne oraz zespoły g-minimalne i wyprowadzono ich właściwości. Podano szereg przykładów.

Общая структура генерирована концепцей минимально связанных множеств

В предыдущих статьях автора и др. (см. указатель литературы) рассмотрены некоторые особенные случаи теории и техники т. наз. минимально связанных множеств. В настоящее время доказывается, что фундаментальная концепция минимально связанных множеств касается более широкого круга проблем декомпозиции. Базируя на приведенной идеи определяются общее и g-минимально связанные множества и виводятся их свойства. Приводятся некоторые примеры.