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On extending linear quadratic control theory to non-symmetric policy objectives

by

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A linear econometric model is considered in a state-space form, which relates deviations of targets and instruments from a reference trajectory in a presence of noise. A non-quadratic criterion has been suggested in order to account for risk and asymmetry of economic policy. It has been shown that a nonlinear feedback policy rule involving a density function $f(x)$, can be applied to solve the formulated optimization problem. The solution is optimal when $f(x)$ is normally distributed and suboptimal otherwise. If the system is stabilizable in the sense of linear-quadratic theory, it is also stabilizable with the introduced non-quadratic approach.

1. Introduction

As the basis for a quantitative theory of economic policy we shall consider an econometric model in state space form relating deviations of targets x and intruments u from a reference trajectory in presence of noise w

$$x(t+1)=Ax(t)+Bu(t)+w(t)$$

where x and w are n-vectors, u is an m-vector, A and B conformable matrices, and t is discrete time. A and B are assumed known from estimation, and w is assumed to represent random shocks over a control period $0 \leqslant t \leqslant T$. The goal of economic policy is to neutralize the effects of shocks by a proper manoeuvre of the instruments. So formulated the problem is often cast into linear quadratic format

$$\min_{u(\cdot)} E \sum_{t=0}^{T} \{x'(t)\,Qx(t)+u'(t)\,Ru(t)\} \tag{1}$$

$$x(t+1)=Ax(t)+Bu(t)+w(t) \tag{2}$$

where E is the expected value operator and x' is a transposition of x

Under the assumptions

i. Q and R are positive semidefinite

ii. $x(0)$ is known

iii. $Ew(t)=0$; $Ew(t)\,w(s)=W\delta(t-s)$ (δ, Kronecker delta) a unique solution exists and has the well known feedback form (Kwakernaak 1972)

$$u(t)=-G(t)\,x(t) \tag{3}$$

where $G(t)$, the optimal policy rule, is found by solving the matrix Riccati equation

$$G(t)=[B'\,P(t)\,B+R]^{-1}\,B'\,P(t)\,A \tag{4}$$

$$P(t)=Q+A'\,P(t+1)\,A-A'\,P(t+1)\,B\,[B'\,P(t+1)\,B+R]^{-1}\,B'\,P(t+1)\,A \tag{5}$$

$$P(T)=Q \tag{6}$$

backwards in time.

When (3) is adopted in (2), the resulting closed loop model becomes

$$x(t+1)=[A-BG(t)]\,x(t)+w(t) \tag{7}$$

and under the further assumption

iv. $[A,B]$ is stabilizable

asymptotic stability of (7) is ensured.

In the stochastic framework this implies boundedness of the covariance evolution

$$V(t+1)=A(t)\,V(t)\,A'(t)+W, \qquad V(0)=I \tag{8}$$

where

$$V(t)=Ex(t)\,x'(t) \qquad A(t)=A-BG(t)$$

This formulation has both attractive features and weaknesses.

Firstly, assumption iv. has an immediate implication for economic theory. In the deterministic case ($w(t)=0$) iv. implies that R can be chosen in (1) so that any point in target space is attained by a bounded manoeuvre $u(1)\;u(2)\;u(3)\ldots$ of length less or equal n. In the stationary stochastic case, the desired state is attained up to an unbiased disturbance of minimal variance. Therefore, stabilizability can be regarded as the dynamic stochastic extension of Tinbergen's theory of economic policy in the sense that the required matching of linearly independent instruments to the number of targets is softened by the possibility of repeatedly using a smaller number of instruments over a greater number of periods. For more details see for instance (Kunstman 1984, Petit 1985).

Secondly, the solution in feedback form provides simple and useful economic indications which may extend beyond the particular problem. For instance, the elements of G can be interpreted as cross elasticities of instruments with respect to target deviations and facilitate exercises of comparative statics or dynamics, as for instance in (Karakitsos 1985). Moreover, the linear structure of (3) indicates that optimal elasticities (in the sense of (1)) should be independent of target deviations, a result of immediate empirical relevance both from a normative and an analytical viewpoint.

Despite these advantages, linear quadratic theory is prone to criticism in a number of economic situations involving risk and asymmetry in policy objectives.

Firstly, recall the known result of certainty equivalance which under iii. holds in the form

$$\arg\min_{u(\cdot)} \{EJ(u) \mid f(x, u, w)=0\} = \arg\min_{u(\cdot)} \{J(u) \mid f(x, u, Ew)=0\}$$

whenever J is quadratic and f linear, as in (1–2). This implies that the optimal policy rule (3) is the same irrespective of the uncertainty in the exogenous shocks, i.e. the covariance matrix W. As uncertainty affects costs, the suggested policy is insensitive to the variability of the cost function. In this sense it has been observed that (1) cannot represent a Von Neumann-Morgenstern utility function, as it essentially embodies a risk neutrality assumption (Sharpe 1970, Hughes Hallet 1984). For given uncertainty W, one should optimise a criterion which considers first and higher moments of the cost function. When this is done, however, one looses the quadraticity of the minimand and must resort to mathematical programming techniques. These, in turn, seldom result in feedback solutions, when they do not guarantee asymptotic stability and, in any case, destroy the elegance and the simplicity of linear quadratic control, as noted in (Hughes Hallet 1984).

Secondly, quadratic criteria fail to capture nonsymmetric effects that deviations above or below the target may produce on costs. When monetary authorities announce a target inflation rate of 4%, they certainly do not regard 2% as equally costly as 6%. In terms of expected utility theory this means that our criterion should consider at least the third moment of the random cost function.

The approach taken in this paper is aimed at extending the simple and useful linear quadratic technique to handle more general criteria of the kind

$$\min_{u(\cdot)} E \sum_{t=0}^{T} \{f'(x(t))\, Qf(x(t)) + u'(t)\, Ru(t)\} \tag{9}$$

where $f(\cdot)$ is assumed to be a globally invertible vector function from R^n to R^n, such that

$$f(0)=0$$

$$x>y \quad \text{implies} \quad f(x)>f(y)$$

$$f(x)>0 \quad \text{iff} \quad x>0$$

$$f(x) \quad \text{not equal} \quad -f(-x)$$

In this respect our approach differs from (Hughes Hallet 1984) in that the asymmetry problem is explicitly addressed. Specifically, the functions (defined componentwise)

$$f(x)=x+m|x| \quad \text{and} \quad f(x)=.5x\,(1+\exp(mx))$$

have been considered. Notice that for $m=0$, (9), reduces to (1) so that, for these functions, (9) can be regarded as a perturbed version of (1), m being the perturbation parameter. Notice also that deviations above the target are costlier than below for $m>0$, and conversely for $m<0$.

2. Statement of the problem

Consider the problem

$$\min_{u(\cdot)} E \sum_{t=0}^{T} \{f'(x(t))\, Q f(x(t)) + u'(t)\, R u(t)\} \tag{9}$$

$$x(t+1) = Ax(t) + Bu(t) + w(t) \tag{10}$$

$$x(0) \quad \text{known} \quad Ew(t)=0 \quad Ew(t)\, w(s) = W\delta(t-s) \tag{11}$$

Next introduce variables $z=f(x)$ and rewrite (9–10)

$$\min_{u(\cdot)} E \sum_{t=0}^{T} \{z'(t)\, Qz(t) + u'(t)\, Ru(t)\} \tag{12}$$

$$z(t+1) = f[Af^{-1}(z(t)) + Bu(t) + w(t)] \tag{13}$$

$$z(0) = f(x(0))$$

Equation (13) is a nonlinear stochastic difference equation. As we are interested in a feedback control law, we shall put in (13)

$$u(t) = -G(t)\, z(t) \tag{14}$$

$$z(t+1) = f[Af^{-1}(z(t)) - BG(t)\, z(t) + w(t)] \tag{15}$$

and seek a minimizing sequence $G(0)\ G(1)\ G(2) \dots$ for (12).

3. Approximate solution method

Following a statistical linearization technique originally suggested by (Sunahara 1970), and subsequently developed by others (Iwan 1980, Beaman 1981, Beaman 1985), we shall replace (15) by an "equivalent" time varying linear system

$$z(t+1) = [A(t) - BG(t)]\, z(t) + C(t)\, w(t) \tag{16}$$

$$z(0) = f(x(0))$$

where matrices $A(t)-BG(t)$, $C(t)$ are assumed to be a function of the instantaneous statistical properties of the solution $z(t)$. Since these are not known a priori, it is customary in the equivalent linearization approach to assume a Gaussian probability density function for $z(t)$, as for instance in (Iwan 1980). In particular, we assume for $z(t)$ a zero mean gaussian distribution with covariance $Ez(t)z'(t)=V(t)$. Indicating by $D(t)$ the difference between (15) and (16)

$$D(t)=F\big(G(t), z(t), w(t)\big)-[A(t)-BG(t)]\,z(t)-C(t)\,w(t)$$

where we put for simplicity

$$F\big(G(t), z(t), w(t)\big)=f\big(Af^{-1}\big(z(t)\big)-BG(t)\,z(t)+w(t)\big)$$

we choose $A(t)-BG(t)$, $C(t)$ so as to minimize

$$\sum_{t=0}^{T} ED'(t)\,D(t)$$

where the expectation is taken with respect to the joint probability density functions of $z(t)$ and $w(t)$. Since these are independent random vectors and the latter is uniformly distributed, this is simply the product of a constant times a Guassian density function.

The solution obtained from the appropriate Euler equation is

$$A(t)=EF\big(G(t), z(t), w(t)\big)\,z'(t)\,V^{-1}(t)+BG(t) \tag{17}$$

$$C(t)=EF\big(G(t), z(t), w(t)\big)\,w'(t)\,W^{-1} \tag{18}$$

where

$$V(t)=Ez(t)\,z'(t)$$

It is well known that the solutions to (15) and (16) coincide up to second order, i.e. they are random processes whose mean and covariance evolutions are identical, provided the expectations in (17–18) are taken with respect to the exact probability density function (Beaman 1981). Within this approximation, the optimizing sequence $G(0)\,G(1)\,...$ for problem (12, 14, 16) is found by solving the matrix Riccati equation

$$G(t)=[B'P(t)B+R]^{-1}\,B'P(t)\,A(t) \tag{19}$$

$$P(t)=Q+A'(t)P(t+1)A(t)-A'(t)P(t+1)B[B'P(t+1)\times \\ \times B+R]^{-1}B'P(t+1)A(t) \tag{20}$$

$$P(T)=Q \tag{21}$$

Since $A(t)$ is a function of the covariance matrix $V(t)$ the covariance evolution must be simultaneously solved. The appropriate form of eq. (8) becomes

$$V(t+1)=[A(t)-BG(t)]\,V(t)\,[A(t)-BG(t)]'+C(t)\,WC'(t) \tag{22}$$

$$V(0)=C(0)\,WC'(0)$$

Therefore, in order to obtain the optimal sequence $G(0)\,G(1)\,...$ it is necessary to solve the two point boundary value problem defined by (20–22).

To this end we adapted to our needs an algorithm suggested by (Yoshida 1984)

1. Set initially $A(0)=A \quad C(0)=I$, the identity matrix.
2. Solve Riccati equation (20) backwards in time and find the $P(t)$ sequence. For each $P(t)$ get a $G(t)$ from (19).
3. Substitute $P(t)$ obtained in 2. into the covariance equation (22), with $A(t)=A$ and $C(t)=I$ and solve it forward in time. At this stage we have the sequences $V(t)$ and $G(t)$.
4. Use $V(t)$ and $G(t)$ to compute $A(t)$ and $C(t)$ with eqs. (17–18).
5. Substitute $A(t)$ into Riccati equation and obtain $P(t)$ from (20) and $G(t)$ from (19).
6. Substitute $A(t)$, $C(t)$ into covariance equation (22) and solve it for $V(t)$.
7. Iterate steps 4, 5, 6 until $V(t)$ and $G(t)$ converge.

The steady state properties of this control strategy are obtained by letting T go to infinity. The algorithm is unchanged except time dependence is dropped from all variables and (20, 22) become algebraic equations. Concerning convergence of the algorithm, we shall confine ourselves to just a few remarks on the steady state case. Assume that when $A(n)$ is computed (step 4, iteration n) the pair $(A(n), B)$ is stabilizable. (If not, we could replace B by $B(n)$ in eqs. (16–23) and solve (17) for $(A(n), B(n))$ so that this condition holds). Then it is known that the Riccati equation converges to a steady state solution and the resulting closed loop system is asymptotically stable. Therefore, the covariance equation also converges, (notice that $C(t)$ is bounded for bounded W from eq. (18)) and the closed loop system (16) is asymptotically stable whenever the algorithm converges, in fact whenever a bounded solution exists for (20–22).

A non linear feedback solution in terms of the original target variables is thus obtained

$$u(t)=-G(t)f(x(t)) \tag{23}$$

and its adoption into model (2) results in a closed loop model

$$x(t+1)=Ax(t)-BG(t)f(x(t))+w(t) \tag{24}$$

which, in the sense of moments, inherits the boundedness of the response of (16).

Notice that the nonlinearity appearing in the feedback law is the same as the one appearing in the criterion (9). While this might have been intuitively surmised, our analysis shows that this is indeed the optimal solution when $f(x)$ is normally distributed. Notice that optimal elasticities in the sense of (9) are no longer independent of target deviations.

4. Properties of the solution and risk aversion

Comparing the optimal decision rules (3) and (23) the following remarks are in order

1. whereas in the quadratic case the optimal policy is insensitive to the noise covariance W, (as already observed) in the non-quadratic case the optimal policy

rule $G(t)$ depends explicitly on the covariance $V(t)$, eqs. (19, 20, 17), whose evolution is driven by W, eq. (22). Thus (23) is no longer risk neutral. Indeed our criterion (9) generalizes usual characterizations of risk aversion in that it depends on all moments of the target variables distribution, and it appears in consonance with the criteria for the definition of "greater riskiness" proposed in (Rothschild 1970).

2. whereas in the quadratic case (3) describes a symmetric reaction function, (23) need no longer be symmetric and $f(x)$ can be chosen to yield the desired effect. This feature has to be contrasted with the solution suggested in (Hughes Hallet 1984) where although a risk sensitive policy is obtained, the feedback law is still linear in the target deviations.

3. consider the case in which $f(x)$ is chosen in a one-parameter class, as for instance

$$f(x)=0.5x\,(1+\exp mx) \tag{25}$$

where m is the parameter.

Then in the m-parametrized class risk neutrality obtains only for $m=0$, the linear-quadratic case, and a quantitative assessment of risk seems naturally offered by the usual measures of risk aversion. Although there is no suggestion to interpret directly $-f(x)$ as a utility function, comparison of optimal risky prospects could still be made on the basis, for instance, of the Arrow-Pratt measure of risk aversion, which in the case of (25) would yield

$$R=f''(x)/f'(x)=m\,(2+mx)/((1+mx)+\exp -mx)$$

Since $m=R$ for $x=0$, parameter m is readily interpreted as the $A-P$ measure of risk aversion at the desired level of the target variables.

5. Examples

Performance of the criterion has been numerically experimented on simple mock-up systems of low dimension. An application to the control of the term structure of interest rates for the Italian economy is also being considered. Exercises are currently under way and, due to their incompleteness, they are not reported here. However, the results obtained so far are rather encouraging. For more details, interested readers are invited to contact the author.

6. Concluding remarks

A linear econometric model has been assumed to describe the economic system in terms of targets and instruments. A non-quadratic criterion has been suggested in order to account for riskiness and asymmetry of economic policy. It has been shown that a nonlinear feedback policy rule involving $f(x)$, the same nonlinearity appearing in the criterion, can be considered a good candidate to solve the underlying optimization problem. The solution obtained is optimal whenever $f(x)$ is normally distributed, suboptimal otherwise. In the latter case, the approximation amounts to replacing the original stochastic constraint by one that differs from it by moments of order 3 and higher.

If the system is stabilizable in the sense of linear-quadratic theory, it is also stabilizable with the present non-quadratic apparoach.

The policy rule is dependent on the noise intensity W and in fact it takes into account riskiness in a rather general form. An assessment of the risk aversion embodied in the criterion can be made by usual $A-P$ indices.

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Uogólnienie zadania sterowania liniowo-kwadratowego dla przypadku niesymetrii we wskaźnikach jakości

Rozpatrzono liniowy model ekonometryczny sformułowany w przestrzeni stanów, w którym rozpatruje się odchylenia od zadanej trajektorii. W modelu uwzględniono zakłócenia losowe. Zaproponowano kryterium nieliniowe, które uwzględnia ryzyko i niesymetrię odchyleń. Pokazano, że do rozwiązania sformułowanego zadania można zastosować regulator ze sprzężeniem zwrotnym, w którym występuje zmienna o rozkładzie $f(x)$. Gdy $f(x)$ ma rozkład normalny, otrzymane rozwiązanie jest optymalne, w przypadkach innych rozkładów jest ono suboptymalne. Wprowadzenie powyższej nieliniowości nie narusza stabilizowalności układu.

Обобщение задачи линейно-квадратичного управления в случае асимметрии показателей качества

Описана линейная эконометрическая модель, определенная в пространстве состояний в которой рассматриваются отклонения от заданной траектории. В модели учтены случайные помехи. Предложен нелинейный критерий, который учитывает риск и асимметрию отклонений. Показано, что для решения поставленной задачи можно применить регулятор с обратной связью, в которой используется переменная с распределением $f(x)$. В случае, если $f(x)$ имеет нормальное распределение, получаемое решение является оптимальным, а для других распределений оно является субоптимальным. Введение выше указанной нелинейности не нарушает устойчивости системы.