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# „Soft" consensus measures for monitoring real consensus reaching processes under fuzzy preferences

by

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Some new „soft" consensus measures are developed, The point of departure is a set of individual fuzzy preference relations which give a degree of preference, between all the options in question as felt by each individual. As the proposed „soft" consensus measure, a degree to which $Q1$ (most, almost all, etc.) pairs of individuals agree as to their preferences between $Q2$ (most, almost all, etc.) relevant options" is propoced. A fuzzy logic based calculus of linguistically quantified proposition is employed. The measures better reflect a practical human perception of the nature of consensus, and may speed up procedures to obtain an acceptable consensus.

## 1. Introduction

Group decision making concerns virtually choice processes, and their properties (see, e.g., Arrow, 1963 and Kelly, 1978), whose purpose is to find an option (or a set of options) that is "best" acceptable by a considered group of individuals whose testimonies are expressed by individual preferences over a set of options. Various solution concepts are possible — in principle, to a solution there belong options preferred by some majority of individuals (see, e.g., Farris and Sage, 1975).

Fuzzy sets theory had been considered a useful tool in group decision making for a long time. For instance, Blin (1974) and Blin and Whinston (1973) proposed to employ fuzzy preference relations to represent group preference, Fung and Fu (1975) discussed aggregation of individual preferences into group preference. Bezdek, Spillman and Spillman (1977, 1978, 1979) discussed ways to determine group preference and derived a scalar measure of consensus. Kuzmin (1982) and Kuzmin and Ovchinnikov (1980a, 1980b) studied group decision making in terms of a distance in the space of fuzzy preference relations. Tanino (1984) considered ways to obtain a group fuzzy preference on the basis of individual fuzzy preferences. Kacprzyk (1984, 1985b, 1986, 1987) and Nurmi (1981) discussed various solution concepts.

In virtually all the above cited works the point of departure was a set of individual fuzzy preference relations. If $S=\{s_1, ..., s_n\}$ is a set of options in question and $K=\{1, ..., m\}$ is a set of individuals involved, then a testimony provided by individual $k$ is assumed to be given as his or her individual fuzzy preference relation $R_k$ whose membership function is

$$\mu_{R_k}: S \times S \rightarrow [0, 1] \tag{1}$$

such that $\mu_{R_k}(s_i, s_j) \in [0, 1]$ denotes preference for $s_i$ over $s_j$ as perceived by individual $k$:

$$\mu_{R_k}(s_i, s_j) = \begin{cases} 1 & \text{if } s_i \text{ is definitely preferred over } s_j \\ c \in (0.5, 1) & \text{if } s_i \text{ is slightly preferred over } s_j \\ 0.5 & \text{if there is no preference between } s_i \text{ and } s_j \\ & \text{(i.e., indifference)} \\ d \in (0, 0.5) & \text{if } s_j \text{ is slightly preferred over } s_i \\ 0 & \text{if } s_j \text{ is definitely preferred over } s_i. \end{cases} \tag{2}$$

Obviously, such a fuzzy preference relation can more adequately represent the real human preferences than nonfuzzy preference relations conventionally used.

If card $S$ is small enough, as we assume here, $R_k$ may be conveniently represented by a matrix $R_k=[r_{ij}^k]$, $r_{ij}^k=\mu_{R_k}(s_i, s_j)$; $i, j=1, ..., n$; $k=1, ..., m$. It is generally assumed, also here, that $R_k$ is reciprocal in the sense that $r_{ij}^k+r_{ij}^k=1$, and — by definition — $r_{ii}^k=0$, for all $i, j, k$.

Consensus is virtually a major goal of group decision making (see, e.g., Goodwin and Restle, 1974; Hare, 1952; Kline, 1972; Knutson, 1972). Attempts to precisely define and quantify the notion of consensus have however met both conceptual and empirical difficulties. Roughly speaking, consensus is usually meant as a full and unanimous agreement. This is clearly by no means a well defined and clear-cut definition, hence a variety of approaches and techniques for dealing with various consensus concepts (see, e.g., Farris and Sage, 1975).

However, consensus — as a full and unanimous agreement — is often a utopia. First, in nontrivial practical situations groups rarely arrive at such a consensus due to some inherent differences in value systems, flexibility, etc. of their members. Second, even if so, a (dynamic) consensus reaching process may be too long.

From a pragmatic point of view it would therefore make more sense to speak about a distance from, or a degree of consensus. In our context, that is with fuzzy preference relations, the seminal contributions are here the following. Spillman, Spillman and Bezdek (1977, 1978), and Bezdek, Spillman and Spillman (1978), determine a distance from consensus as a difference between some average preference matrix and one of several possible consensus preference matrices. Spillman, Bezdek and Spillman (1979) derive some measure of attitudinal similarity between individuals that is an extension of the classical Tanimoto coefficient. Spillman, Spillman and Bezdek (1980) derive a consensus measure based on $\alpha$-cuts of the respective individual fuzzy preference matrices.

In the above approaches the consensus measures introduced are in a sense "hard" because they indicate (full) consensus (=1) only in case of a complete agreement, i.e. of all the individuals as to their preferences between all the options. In practice, however, this may be often seen counterintuitive because we may be already fully satisfied with some "partial" consensus if, e.g., most of the individuals agree as to their preferences between, e.g., most of the options. Such an attitude is usually fully acceptable in most practical cases.

New "softer" consensus measures were proposed by Kacprzyk (1987). To deal with linguistic quantifiers (e.g., most) involved in those measures, a fuzzy-logic-based calculus of linguistically quantified propositions was employed that had proved to be useful to "soften" a large spectrum of multiobjective (Yager, 1983; Kacprzyk and Yager, 1984a, 1984b; Kacprzyk, 1985a, etc.), multistage (Kacprzyk, 1983a, 1985a) or group (Kacprzyk, 1984, 1985b, 1986, 1987) decision making models.

In this paper we further advance the above idea (Kacprzyk, 1987) to use fuzzy linguistic quantifiers for deriving "softer" consensus measures. In particular, we allow for a different (degree of) relevance — between the definite relevance (=1) and the definite irrelevance (=0), through all intermediate values — of the particular options. Roughly speaking, we derive a degree to which, e.g., most of the individuals agree as to their preferences between, e.g., most of the relevant options.

Let us notice that we are concerned here only with some tools for monitoring real consensus reaching processes, i.e. in fact for evaluating degrees ofr consensus temporally evolving during such processes (hopefully converging to a sufficiently good consensus). We do not deal here with how the fuzzy preference relations are to be subsequently changed to eventually arrive at consensus. For an approach to this problem, see Ragade (1976, 1977); further details can be found in Fedrizzi (1984) and Fedrizzi and Ostasiewicz (1984).

In the next section we briefly present a calculus of linguistically quantified propositions to be used in the sequel. In Section 3, to provide a point of departure, we review the derivation of the "soft" consensus measures without accounting for relevance of the options. In Section 4 we propose the new "soft" consensus measures in which relevance of the options is accounted for.

Our notation will be standard. A fuzzy set $A$ in $X$, $A \subseteq X$, will be represented by, and often informally equated with its membership function $\mu_A: X \to [0, 1]$; $\mu_A(x) \in \in [0, 1]$ is the grade of membership of $x$ in $A$. Moreover, $a \wedge b = \min(a, b)$. For further notation, see Kacprzyk (1983b).

## 2. A calculus of linguistically quantified propositions

In this section we sketch a calculus of linguistically quantified propositions to be used in the sequel.

A linguistically quantified proposition is exemplified by "most experts are convinced". In general, it may be written as

$$QY\text{'s are } F \tag{3}$$

where $Q$ is a linguistic quantifier (e.g., most), $Y=\{y\}$ is a set of objects (e.g., experts) and $F$ is a property (e.g., convinced).

Importance $B$ may be introduced into (3) yielding

$$QBY\text{'s are } F \tag{4}$$

e.g., "most ($Q$) of the important ($B$) experts ($Y$'s) are convinced ($F$)".

The problem is basically to find the (degree of) truth of (3) or (4), i.e. truth ($QY$'s are $F$) or truth ($QBY$'s are $F$), respectively, knowing all truth ($y_i$ is $F$), $y_i \in Y$.

The conventional two-valued predicate calculus makes it possible to find the above truths for crisp quantifiers only, mainly for "all" and "at least one". The class of quantifiers used in practice is however much richer, e.g., "few", "a couple of", "most", "almost all", etc.. We will present below a fuzzy-logic-based calculus of linguistically quantified propositions to deal with such quantifiers.

In the classical method proposed by Zadeh (1983) a linguistic quantifier $Q$ is assumed to be a fuzzy set in[0, 1], $Q \subsetneq [0, 1]$. For instance, $Q=$"most" may be given as

$$\begin{aligned} \mu_{\text{"most"}}(x) &= 1 && \text{for } x \geqq 0.8 \\ &= 2x-0.6 && \text{for } 0.3<x<0.8 \\ &= 0 && \text{for } x \leqq 0.3 \end{aligned} \tag{5}$$

Throughout the paper we will use only the so-called proportional linguistic quantifiers exemplified by "most", "almost all", etc. since they are more appropriate in our context. For the so-called absolute linguistic quantifiers as, e.g., "about 5", "much more than 10", etc. the reasoning is similar but they are defined as fuzzy sets in $R$, the real line.

Particularly important in our context are the so-called nondecreasing fuzzy quantifiers defined as

$$x'>x'' \Rightarrow \mu_Q(x') \geqq \mu_Q(x'') \quad \text{for each} \quad x', x'' \in [0, 1] \tag{6}$$

"Most" given by (5) is evidently nondecreasing.

Property $F$ is defined as a fuzzy set in $Y$, $F \subsetneq Y$. If $Y=\{y_1, ..., y_p\}$, then truth ($y_i$ is $F$)$=\mu_F(y_i)$, $i=1, ..., p$.

The calculation of truth ($QY$'s are $F$) is based on the (nonfuzzy) cardinalities,

$\Sigma$ Counts, of the respective fuzzy sets (see. e.g., Zadeh, 1979, 1983) and proceeds as follows:

STEP 1. Calculate

$$r=\Sigma\,\text{Count}(F)/\Sigma\,\text{Count}\,(Y)=\frac{1}{p}\sum_{i=1}^{p}\mu_F\,(y_i) \tag{7}$$

STEP 2. Calculate

$$\text{truth}\ (QY\text{'s are } F)=\mu_Q\,(r) \tag{8}$$

Importance is introduced as follows. $B=$"important" is defined as a fuzzy set in $Y, B\subseteq Y$, such that $\mu_B\,(y_i)\in[0,1]$ is a degree of importance of $y_i$: the higher $\mu_B\,(y_i)$ the more important $y_i$.

We rewrite "$QBY$'s are $F$" as "$Q$ ($B$ and $F$) are $B$" which leads to the following counterparts of (7) and (8), respectively:

STEP 1. Calculate

$$r'=\Sigma\,\text{Count}\ (B \text{ and } F)/\Sigma\,\text{Count}\ (B)=$$

$$=\sum_{i=1}^{p}(\mu_B\,(y_i)\wedge\mu_F\,(y_i))/\sum_{i=1}^{p}\mu_B\,(y_i) \tag{9}$$

("$\wedge$" may be replaced by, e.g., a $t$-norm; see Kacprzyk and Yager, 1984b).

STEP 2. Calculate

$$\text{truth}\ (QBY\text{'s are } F)=\mu_Q\,(r') \tag{10}$$

EXAMPLE 1. Let: $Y=$"experts"$=\{A, B, C\}$; $F=$"convinced"$=0.1/A+0.6/B+0.8/C$; $Q=$"most" be given by (5); $B=$"important"$=0.2/A+0.5/B+0.6/C$.

Then, on the one hand,

$$r=\frac{1}{3}\,(0.1+0.6+0.8)=0.5$$

and

$$\text{truth ("most experts are convinced")}=2\cdot 0.5-0.6=0.4$$

while, on the other hand,

$$r'=(0.1+0.5+0.6)/(0.2+0.5+0.6)=1.2/1.3$$

and

$$\text{truth ("most of the important experts are convinced")}=1.$$

The method presented may be viewed to provide a consensory-like aggregation of the pieces of evidence "$y_i$ is $F$" (for details, see Kacprzyk, 1983a or Yager, 1983).

## 3. Soft measures of consensus without accounting for relevance of options

As mentioned in Section 1, to overcome some "rigidness" of a variety of consensus measures developed in a line of papers by Bezdek, Spillman and Spillman (1977, 1978, 1979), Spillman, Bezdek and Spillman (1979), and Spillman, Spillman and Bezdek (1979, 1980) in which the full consensus (=1) occurs only when "everybody agrees as to everything", some "softer" consensus measures have been proposed in Kacprzyk (1987). Basically, they are equivalent to finding "a degree to which, e.g., most of the individuals agree as to their preferences between, e.g., most pairs of options". We will show now how to derive such consensus measures.

For our purposes the derivation of those "soft" measures (degrees) of consensus may be viewed as a hierarchical pooling process shown in Fig. 1 to be meant as follows. First, for each pair of individuals we derive a degree of agreement as to their preferences between all the pairs of options, next we pool (aggregate) these degrees to obtain a degree of agreement of each pair of individuals as to their preferences between $Q1$ (a linguistic quantifier as, e.g., "most") pairs of options, and, finally, we pool these degrees to obtain a degree of agreement of $Q2$ (a linguistic quantifier as, e.g., "almost all") pairs of individuals as to their preferences between $Q1$ pairs of options. This is meant to be the degree of consensus sought. Notice that, roughly speaking, $Q1$ and $Q2$ are "all" for the conventional ("hard") notions of consensus.

We will now develop the above mentioned degrees of consensus using the calculus of linguistically quantified propositions outlined in Section 2.

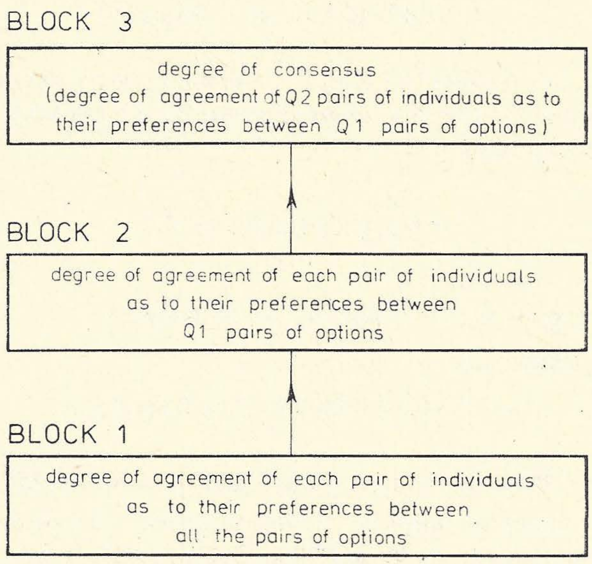


**Fig. 1. Derivation of a degree of consensus as a hierarchical pooling process**

We start with the degree of strict agreement between individuals $k1$ and $k2$ as to their preference between options $s_i$ and $s_j$ as

$$v_{ij}(k1, k2) = \begin{cases} 1 & \text{if } r_{ij}^{k1} = r_{ij}^{k2} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Notice that $k1 = 1, \ldots, m-1$; $k2 = k1+1, \ldots, m$, because $v_{ij}(k1, k2) = v_{ij}(k2, k1)$ and $v_{ij}(k1, k1)$ are irrelevant. Moreover, since the fuzzy preference relations $R_k$ are reciprocal and $r_{ii}^k = 0$, then $i = 1, \ldots, n-1$; $j = i+1, \ldots, n$.

Next, the degree of agreement between individuals $k1$ and $k2$ as to their preferences between all the pairs of options is

$$v(k1, k2) = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} v_{ij}(k1, k2) \tag{12}$$

and the degree of agreement between individuals $k1$ and $k2$ as to their preferences between $Q1$ pairs of options is (cf. (8))

$$v_{Q1}(k1, k2) = \mu_{Q1}(v(k1, k2)) \tag{13}$$

In turn, the degree of agreement of all the pairs of individuals as to their preferences between $Q1$ pairs of options is

$$v_{Q1} = \frac{2}{m(m-1)} \sum_{k1=1}^{m-1} \sum_{k2=k1+1}^{m} v_{Q1}(k1, k2) \tag{14}$$

and, finally, the degree of agreement of $Q2$ pairs of individuals as to their preferences between $Q1$ pairs of options, called the degree of $Q1/Q2$-consensus, is

$$\text{con}(Q1, Q2) = \mu_{Q2}(v_{Q1}) \tag{15}$$

Since (11), i.e. the strict agreement, may be viewed too rigid and restrictive, we 'soften' it by consecutively introducing the notions of a sufficient and strong agreement (cf. Kacprzyk, 1984, 1985b, 1986, 1987).

First, the degree of sufficient (at least to degree $1-\alpha$) agreement of individuals $k1$ and $k2$ as to their preferences between options $s_i$ and $s_j$ is defined as

$$v_{ij}^{\alpha}(k1, k2) = \begin{cases} 1 & \text{if } |r_{ij}^{k1} - r_{ij}^{k2}| \leqq 1 - \alpha \leqq 1 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

Then, following (12)–(14), we obtain

$$v^{\alpha}(k1, k2) = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} v_{ij}^{\alpha}(k1, k2) \tag{17}$$

$$v_{Q1}^{\alpha}(k1, k2) = \mu_{Q1}(v^{\alpha}(k1, k2)) \tag{18}$$

$$v_{Q1}^{\alpha} = \frac{2}{m(m-1)} \sum_{k1=1}^{m-1} \sum_{k2=k1+1}^{m} v_{Q1}^{\alpha}(k1, k2) \tag{19}$$

and, finally, the degree of sufficient (at least to degree $1-\alpha$) agreement of $Q2$ pairs of individuals as to their preferences between $Q1$ pairs of options, called the degree of $\alpha/Q1/Q2$ — consensus, is

$$\operatorname{con}^{\alpha}(Q1, Q2)=\mu_{Q2}(v^{\alpha}_{Q1}) \tag{20}$$

It is easy to notice that $\operatorname{con}^1(Q1, Q2)=\operatorname{con}(Q1, Q2)$.

Second, we can explicitly introduce the strength of agreement into (11) and define the degree of strong agreement of the individuals $k1$ and $k2$ as to their preferences between the options $s_i$ and $s_j$, e.g., as

$$v^s_{ij}(k1, k2)=s(|r^{k1}_{ij}-r^{k2}_{ij}|) \tag{21}$$

where $s$: $[0, 1]\to[0, 1]$ is some function representing a degree of strong agreement as, e.g.,

$$\begin{aligned} s(x)&=1 && \text{for } x\leqq 0.05 \\ &=-10x+1.5 && \text{for } 0.05<x<0.15 \\ &=0 && \text{for } x\geqq 0.15 \end{aligned} \tag{22}$$

such that $x'<x''\Rightarrow s(x')\geqq s(x'')$ and $s(x)=1$ for some $x\in[0, 1]$.

For some other definitions of $v^s_{ij}(k1, k2)$, see Kacprzyk (1987).

Then, following (12)–(14), and (17)–(19), we obtain

$$v^s(k1, k2)=\frac{2}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} v^s_{ij}(k1, k2) \tag{23}$$

$$v^s_{Q1}(k1, k2)=\mu_Q(v^s(k1, k2)) \tag{24}$$

$$v^s_{Q1}=\frac{2}{m(m-1)}\sum_{k1=1}^{m-1}\sum_{k2=k1+1}^{m} v^k_{Q1}(k1, k2) \tag{25}$$

and, finally, the degree of strong agreement of $Q2$ pairs of individuals as to their preferences between $Q1$ pairs of options, called the degree of $s/Q1/Q2$ — consensus, is

$$\operatorname{con}^s(Q1, Q2)=\mu_{Q2}(v^s_{Q1}) \tag{26}$$

EXAMPLE 2. Let the individual fuzzy preference relations of individuals $k=1, 2, 3, 4$ be, respectively:

$$R^1=\begin{bmatrix} 0 & 0.4 & 0.7 & 0.1 \\ 0.5 & 0 & 0.8 & 0.2 \\ 0.3 & 0.2 & 0 & 0.7 \\ 0.9 & 0.8 & 0.3 & 0 \end{bmatrix} \quad R^2=\begin{bmatrix} 0 & 0.4 & 0.5 & 0 \\ 0.6 & 0 & 0.8 & 0.2 \\ 0.5 & 0.2 & 0 & 0.7 \\ 1 & 0.8 & 0.3 & 0 \end{bmatrix}$$

$$R^3=\begin{bmatrix} 0 & 0.4 & 0.6 & 0.3 \\ 0.4 & 0 & 0.8 & 0.2 \\ 0.6 & 0.2 & 0 & 0.7 \\ 0.7 & 0.8 & 0.3 & 0 \end{bmatrix} \quad R^4=\begin{bmatrix} 0 & 0.4 & 0.7 & 0.1 \\ 0.6 & 0 & 0.7 & 0.1 \\ 0.3 & 0.3 & 0 & 0.7 \\ 0.9 & 0.9 & 0.3 & 0 \end{bmatrix}$$

Then, for $Q1=Q2=$"most" given by (5):

con ("most", "most") $=0.61$

$\text{con}^{0.9}$ ("most", "most")$=1$

$\text{con}^{s}$ ("most", "most") $=0.92$

Let us finally point out some important properties of the proposed degrees of consensus whose proofs can be found in Kacprzyk (1987).

PROPOSITION 1. For any nondecreasing fuzzy quantifiers $Q1, Q2 \subseteq [0, 1]$, any $s(x)$ of type (22), and any $\alpha \in (0, 1]$, we have

$$\text{con}^{\alpha}(Q1, Q2) \geqq \text{con}(Q1, Q2) \tag{27}$$

and

$$\text{con}^{s}(Q1, Q2) \geqq \text{con}(Q1, Q2) \tag{28}$$

PROPOSITION 2. For any nondecreasing $Q1, Q2 \subseteq [0, 1]$ and any $\alpha \in [0,1]$, we have

$$\alpha' > \alpha'' \Rightarrow \text{con}^{\alpha'}(Q1, Q2) \leqq \text{con}^{\alpha''}(Q1, Q2) \tag{29}$$

## 4. „Soft" measures of consensus with accounting for relevance of options

In this section we extend the "soft" measures of consensus presented in Section 3 to cover the case when the particular options may be of different relevance. The derivation of such new measures of consensus may also be portrayed as a hierarchical pooling process shown in Fig. 1 with a natural replacement of Block 2 by Block 2' shown in Fig. 2. Thus, we derive first for each pair of individuals a degree of agreement as to their preferences between all the pairs of options. Next, we pool (aggregate) these degrees to obtain a degree of agreement of each pair of individuals as to their preferences between $Q1$ relevant pairs of options, and finally we pool these degrees to obtain a degree of agreement of $Q2$ pairs of individuals as to their preferences between $Q1$ relevant pairs of options. This is meant to be the degree of consensus sought.

Relevance is assumed to be a fuzzy set defined in the set of options $S=\{s_1, ..., s_n\}$, i.e. $R \subseteq S$, and $\mu_R(s_i) \in [0, 1]$ is a degree of relevance of option $s_i$: from 0 standing

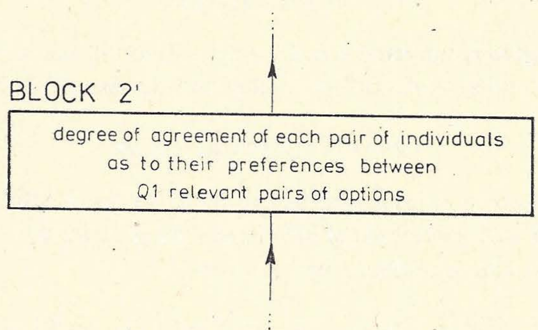


Fig. 2. Block 2' replacing Block 2 in Fig. 1 in case of accounting for relevance of the options

for "definitely irrelevant" and 1 standing for "definitely relevant", through all intermediate values.

Relevance of a pair of options, say $(s_i, s_j) \in S \times S$, can be defined in various ways among which

$$b_{ij}^R = \frac{1}{2}\left(\mu_R(s_i) + \mu_R(s_j)\right) \tag{30}$$

is certainly the most straightforward; obviously, $b_{ij}^R = b_{ji}^R$ and $b_{ii}^R$ do not matter, for all $i, j, k$.

Now we are in a position to derive counterparts of the consensus measures introduced in Section 3 taking into account relevance of the options.

We start with the degree of strict agreement between individuals $k1$ and $k2$ as to their preferences between options $s_i$ and $s_j$ given as (11), i.e.

$$\begin{aligned} v_{ij}(k1.\, k2) &= 1 \quad \text{if} \quad r_{ij}^{k1} = r_{ij}^{k2} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

The ranges of $k1, k2, i$, and $j$ are as for (11).

Next, the degree of agreement between individuals $k1$ and $k2$ as to their preferences between all the relevant pairs of options is (cf. (9))

$$v_R(k1, k2) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(v_{ij}(k1, k2) \wedge b_{ij}^R\right) / \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} b_{ij}^R \tag{31}$$

The degree of agreement between individuals $k1$ and $k2$ as to their preferences between $Q1$ relevant pairs of options is

$$v_{Q1.R}(k1, k2) = \mu_{Q1}\left(v_R(k1, k2)\right) \tag{32}$$

In turn, the degree of agreement of all the pairs of individuals as to their preferences between $Q1$ relevant pairs of options is

$$v_{Q1.R} = \frac{2}{m(m-1)} \sum_{k1=1}^{m-1} \sum_{k2=k1+1}^{m} v_{Q1.R}(k1, k2) \tag{33}$$

and, finally, the degree of agreement of $Q2$ pairs of individuals as to their preferences between $Q1$ relevant pairs of options, called the degree of $Q1/Q2/R$-consensus, is

$$\text{con}_R(Q1, Q2) = \mu_{Q2}(v_{Q1.R}) \tag{34}$$

Since the strict agreement (11) may be too rigid, we can use the degree of sufficient (at least to degree $1-\alpha$) agreement of individuals $k1$ and $k2$ as to their preferences between options $s_i$ and $s_j$ defined by (16), i.e.

$$\begin{aligned} v_{ij}^{\alpha}(k1, k2) &= 1 \quad \text{if} \quad |r_{ij}^{k1} - r_{ij}^{k2}| \leqq 1 - \alpha \leqq 1 \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{35}$$

Then, following (31)–(33), we obtain

$$v_R^{\alpha}(k1, k2)=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(v_{ij}^{\alpha}(k1, k2)\wedge b_{ij}^{R})/\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} b_{ij}^{R} \tag{36}$$

$$v_{Q1,R}^{\alpha}(k1, k2)=\mu_{Q1}(v_R^{\alpha}(k1, k2)) \tag{37}$$

$$v_{Q1,R}^{\alpha}=\frac{2}{m(m-1)}\sum_{k1=1}^{m-1}\sum_{k2=k1+1}^{m} v_{Q1,R}^{\alpha}(k1, k2) \tag{38}$$

and, finally, the degree of sufficient (at least to degree $1-\alpha$) agreement of $Q2$ pairs of individuals as to their preferences between $Q1$ relevant pairs of options, called the degree of $\alpha/Q1/Q2/R$-consensus is

$$\mathrm{con}_R^{\alpha}(Q1, Q2)=\mu_{Q2}(v_{Q1,R}^{\alpha}) \tag{39}$$

We can also use (21), i.e. the degree of strong agreement of the individuals $k1$ and $k2$ as to their preferences between the options $s_i$ and $s_j$, that is

$$v_{ij}^{s}(k1, k2)=s(|r_{ij}^{k1}-r_{ij}^{k2}|)$$

where $s$: $[0, 1]\to[0, 1]$ is a function of type (22).

Then, following (31)–(33), and (36)–(38), we obtain

$$v_R^{s}(k1, k2)=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(v_{ij}^{s}(k1, k2)\wedge b_{ij}^{R})/\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} b_{ij}^{R} \tag{40}$$

$$v_{Q1,R}^{s}(k1, k2)=\mu_{Q1}(v_R^{s}(k1, k2)) \tag{41}$$

$$v_{Q1,R}^{s}=\frac{2}{m(m-1)}\sum_{k1=1}^{m=1}\sum_{k2=k1+1}^{m} v_{Q1,R}^{s}(k1, k2) \tag{42}$$

and, finally, the degree of strong agreement of $Q2$ pairs of individuals as to their preferences between $Q1$ relevant pairs of options, called the degree of $s/Q1/Q2/R$--consensus is

$$\mathrm{con}_R^{s}(Q1, Q2)=\mu_{Q2}(v_{Q1,R}^{s}) \tag{43}$$

**Example 3.** For the same individual fuzzy preference relations as in Example 2, $Q1=Q2=$"most" given by (5), let relevance of the particular pairs of options be

$$b_R^{12}=0.5 \quad b_R^{13}=0.4 \quad b_R^{14}=0.2 \quad b_R^{23}=0.6 \quad b_R^{24}=0.4 \quad b_R^{34}=0.3$$

Now, due to (31)

$$\begin{array}{lll} v_R(1, 2)=0.75 & v_R(1, 3)=0.75 & v_R(1, 4)=0.58 \\ v_R(2, 3)=0.75 & v_R(2, 4)=0.33 & v_R(3, 4)=0.33 \end{array}$$

and (32) implies

$$\begin{array}{lll} v_{Q1,R}(1, 2)=0.9 & v_{Q1,R}(1, 3)=0.9 & v_{Q1,R}(1, 4)=0.56 \\ v_{Q1,R}(2, 3)=0.9 & v_{Q1,R}(2, 4)=0.6 & v_{Q1,R}(3, 4)=0.06 \end{array}$$

and (33) yields

$$v_{Q1,R}=0.56$$

and, finally, by (34),

$$\mathrm{con}_R\,(Q1, Q2)=0.52$$

Second, with $\alpha=0.9$, from (36)

$$v_R^{0.9}\,(1,2)=0.75 \quad v_R^{0.9}\,(1,3)=0.75 \quad v_R^{0.9}\,(1,4)=1$$
$$v_R^{0.9}\,(2,3)=0.66 \quad v_R^{0.9}\,(2,4)=0.58 \quad v_R^{0.9}\,(3,4)=0.5$$

and (37) implies

$$v_{Q1,R}^{0.9}\,(1,2)=0.9 \quad v_{Q1,R}^{0.9}\,(1,3)=0.9 \quad v_{Q1,R}^{0.9}\,(1,4)=1$$
$$v_{Q1,R}^{0.9}\,(2,3)=0.72 \quad v_{Q1,R}^{0.9}\,(2,4)=0.56 \quad v_{Q1,R}^{0.9}\,(3,4)=0.4$$

and (38) yields

$$v_{Q1,R}^{0.9}=0.75$$

and, finally, by (39)

$$\mathrm{con}_R^{0.9}\,(Q1, Q2)=0.9$$

Third, from (40)

$$v_R^s\,(1,2)=0.86 \quad v_R^s\,(1,3)=0.75 \quad v_R^s\,(1,4)=0.96$$
$$v_R^s\,(2,3)=0.92 \quad v_R^s\,(2,4)=0.79 \quad v_R^s\,(3,4)=0.71$$

and from (41)

$$v_{Q1,R}^s\,(1,2)=1 \quad v_{Q1,R}^s\,(1,3)=0.9 \quad v_{Q1,R}^s\,(1,4)=1$$
$$v_{Q1,R}^s\,(2,3)=1 \quad v_{Q1,R}^s\,(2,4)=0.99 \quad v_{Q1,R}^s\,(3,4)=0.82$$

and (42) yields

$$v_{Q1,R}^s=0.95$$

and, finally, by (43),

$$\mathrm{con}_R^s\,(Q1, Q2)=1$$

## 5. Concluding remarks

The proposed fuzzy-quantifier-based measures of consensus do considerably "soften" the traditional rigid notion of consensus assumed to be a full and unanimous agreement. Thus, on a conceptual level, they may be viewed as a step to obtain a formal characterization of consensus which would be closer to real human perception of its very essence. On the other hand, the use of these new measures can help quicker obtain an acceptable consensus in practical consensus reaching processes.

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## „Miękkie" miary consensusu w procesach osiągania consensusu w przypadku preferencji rozmytych

W pracy podano pewne nowe „miękkie" miary stopnia consensusu. Punktem wyjścia jest zbiór indywidualnych rozmytych relacji preferencji, które podają dla każdej pary opcji stopień preferencji jednej opcji w stosunku do drugiej. Jako „miękką" miarę consensusu zaproponowano stopień w jakim $Q1$ (większość, prawie wszystkie itp.) par osobników jest zgodnych co do swoich preferencji między $Q2$ (większość, prawie wszystkie itp.) parami istotnych opcji". Zastosowano rachunek zdań z kwantyfikatorami lingwistycznymi oparty na logice rozmytej. Zaproponowane miary consensusu lepiej oddają praktyczną percepcję istoty consensusu i mogą przyśpieszyć procedury prowadzące do osiągnięcia zadawalającego consensusu.

## „Мягкие" меры согласия в процессах достижения решения для случая размытых предпочтений

В работе приведены некоторые новые „мягкие" меры степени согласия. Исходной точкой является множество индивидуальных размытых отношений предпочтений, которые дают для каждой пары выбора степень предпочтения одного решения по отношению к другому.

В качестве мягкой меры согласия предлагается степень, в которой $Q1$ (большинство, почти все и т.п.) пар лиц согласны в отношении своих предпочтений между $Q2$ (большинство, почти все и т.п.) парами существенных выборов". Используется исчисление высказываний с лингвистическими кванторами, основанное на размытой логике. Предлагаемые меры согласия лучше отображают практическое восприятие сути согласия и могут ускорить процедуры, ведущие к достижению удовлетворяющего согласия.