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Optimal strategies in the depletion of an uncertain, exhaustible resource

by

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This paper presents explicit, optimal strategies for the intertemporal use of an exhaustible resource stock for a particular class of cost functions. Uncertainty in the estimate of the available resource in-the-ground is described by a Poisson process. Particular attention is given to the behaviour of the scarcity rent, showing that it can rise over time or even decrease, according to the bequest requirements and the (estimated) stock available for the final time. The free horizon time problem is also solved, giving rise to strategies based on disinvestment effect or, alternatively, on conservation. The effect of uncertainty is also studied.

1. Introduction

The economics of exhaustible resources has received wide attention in the literature since Hotelling's paper (1931) on the topic. Recently, the effect of uncertainty has been introduced so that the model has been generalized including the effect for exploration, uncertainty in demand and supply, on the optimal policies (see Gilbert (1979), Desmukh-Pliska (1980, 1983), Arrow-Chang (1980), Pindyck (1980a, 1980b), Duffie-Taksar (1983), and for a review paper Devarajan-Fisher (1981)). A common characteristic of these papers is that they deal with production during the entire life period of the resource, that is up to the (random) date of exhaustion. The idea behind is, of course, to analyze the economics of that resource and since the "right" pricing would avoid the possibility to postpone production[1] (i.e. the producer would be indifferent about when to extract), it should be quite useless to consider shorter time horizons. But, in fact, producers can have shorter horizons, and model their policies according to their bequests for the final time.

Another important point is the behaviour of the scarcity rent, or royalty price, i.e. the component of price that corresponds to the "premium" for scarcity. In

[1] The only paper dealing with possible interruptions in production is Brennan-Schwartz (1983) (see also Tourinho (1983)). In fact, my approach can be a valid alternative to the option pricing methodology they use.

equilibrium, it (or, under uncertainty, its expected value) should rise at the market rate, until the complete depletion of the resource. When, in particular, the cost function depends (inversely) on the stock of the reserve, the rent for scarcity should rise, at the market rate, less then savings in future extraction costs (Dasgupta-Heal (1979), Farzin (1984)).

With a free time horizon and the cost function depending on the stock and the extraction policy, it will be shown that the rent for scarcity may follow different paths, according to the bequest requirements and the (estimated) size of the stock in-the-ground at the final time. This result includes as special cases:

— the case in which the rent declines: the final stock is so large that the resource cannot be considered exhaustible;

— the case in which the rent is shown to rise, during the entire period or from a certain point of time on: this may correspond to various degrees of exhaustability and the eventual depletion is included. This could also suggest a more precise definition of exhaustability.

In this paper, I solve explicitly a free horizon time problem, in a stochastic environment, for a competitive producer of an exhaustible resource, when the price is exogenously given. By this approach, optimal intertemporal strategies for the depletion of a resource with imperfect knowledge of the available endowment are described analytically. The (estimated) evolution of the resource is described by a Poisson process. As the horizon time is free, possibility to extend or interrupt production is given (i.e. the producer can exercise the option to hold back production).

Moreover, through explicit solutions of the problem, we can recognize clearly what the effect can be of not taking into account uncertainty in the estimate; in other words, we can see the bias of using certainty in a situation that requires uncertainty.

Note that, by this approach, I do not impose any hypothesis on the path of the price $p(t)$.

This work can also be seen as part of the well known "cake eating" problem (see e.g. Gilbert (1979)), but in this case the optimal policy is modelled according to the (estimated) "slice" we want to keep for the final time.

In the next section we set forth a theoretical stochastic control model where costs area function of the production and the level of the reserves; optimal solutions and the behaviour of the scarsity rent are also described. The free horizon problem and the effect of uncertainty are considered in section 3. Comments and conclusions are in section 4.

2. The model

The producer's problem is to maximize its expected discounted profit, according to the following:

$$\max_{0 \leq q \leq \bar{q}} E_0 \left[\int_0^T (p(t) - C(Q, q, t))\, q(t) \exp(-rt)\, dt + B(Q, T) \exp(-rT) \right] \qquad (2.1)$$

subject to

$$dQ(t) = -q(t)\,dt - f\,da(t) \tag{2.2}$$

where $p=p(t)$ is the price per unit, $q(t)$ the extraction policy, $C(Q, q, t)$ the cost per unit, r the market rate, B the bequest function. $a(t)$ is a poisson process, i.e. a process in which jumps are allowed in discrete times. $\lambda\Delta t+o(\Delta t)$ is the probability that a jump occurs in $(t, t+\Delta t)$, $1-\lambda\Delta t+o(\Delta t)$ is the probability of no jump, $o(\Delta t)$ is the probability of more than one jump. The amplitude of the jump is given by a random variable S, with compact support $\sum$ and density $g(S)$ and a constant term f with $0 \leqslant f \leqslant Q$. It follows then that the mean amplitude of the jump is $fS\lambda$ where $\bar{S}$ is the expected value of S. When $f=Q$ and $S=1$ w.p.1, we have the case of possible complete depletion in $[0, T]$. When $S=0$ w.p.1, or equivalently, $f=0$, we have no uncertainty about the size of the resource in-the-ground.

The characterization of $Q(t)$ through a Poisson process finds its reason in the fact that estimates of inground reserves are far from being precisely evaluated: "... in most natural resources, oil is an obvious example, the quantity available for eventual production is not known. In many extractive industries geological estimates of the amount of resource "in place" may vary by an order of magnitude of more" (Stewart (1979)), "... The stock of reserve is far from known... But if the stocks are in fact uncertain as evidenced by repeated changes in estimates, then uncertainty should be reflected in the initial planning" (Arrow-Chang (1980)). In this respect, $Q(t)$ is the estimated rather than actual level of reserves.

The cost function is assumed dependent on the level of reserves, the extraction policy and time, and also smooth on its arguments, with $C_0<0$, $C_q>0$, $C_t<0$.

The resource price is assumed to follow a deterministic path and is exogenously given.

Let

$$M(Q, s) = \exp(-rs)\,\mathrm{Max}\,E_s\left[\int_s^T \exp(-rt)(p-C)\,q\,dt + \exp(-rT)\,B(Q, T)\right].$$

If M is differentiable, then $M(Q, s)$ is the solution of the dynamic programming equation

$$-M_t + rM = \underset{q}{\mathrm{Max}}\,((p-C)\,q - qM_Q) + \lambda \int_{\Sigma} (M(Q-Sf) - M(Q))\,g(S)\,dS \tag{2.3}$$

with

$$M(Q(T), T) = B(Q(T), T) \tag{2.4}$$

$$M(0, s) = 0 \tag{2.5}$$

The necessary condition for optimality is

$$0 = p - C - C_q\,q^* - M \tag{2.6}$$

where M_Q is the rent for scarcity. (2.6) is the usual condition that connects price, cost and royalty price.

Let us consider now a particular class of cost functions $C(Q, q, t)=mq/Q$, $m \in R^+$, and for the evolution of the resource, we set $S=s_0$ w.p.1, $0 \leqslant s_0 \leqslant 1$, $f=Q$. It means that it may happen that the producer finds his resource reduced by a revision of the estimates ($s_0>0$). Or, even if properly estimated, the reserve could be reduced by an accident or any geological cause that may prevent extraction. In the case $s_0<0$, we can include revisions "in positive" of the estimates. In this paper I consider only the case of reduction. The extension is obvious.

(2.6) becomes then

$$0=p-2mq^*/Q-M_Q=p-M_Q-2C \tag{2.7}$$

It follows that

$$q^*=Q\,(p-M_Q)/(2m) \tag{2.8}$$

Substituting back (2.8) into (2.3) and rearranging terms we obtain a first order partial differential equation in M:

$$-p^2Q/(4m)=M_t-rM-M_Q\,pQ/(2m)+M_Q^2\,Q/(4m)+ \\ +\lambda\,(M\,(Q-s_0\,Q)-M\,(Q)) \tag{2.9}$$

By putting $M(Q, s)=Qh(s)$, we obtain $M_Q=h(t)$, $M=Qh'(t)$, then, for $Q\neq 0$, (2.9) reduces to

$$-p^2(t)/(4m)=h'(t)-p(t)\,h(t)/(2m)-(r+s_0\,\lambda)\,h(t)+h^2(t)/(4m) \tag{2.10}$$

This is a Riccati differential equation and, in order to solve it, a particular solution is required. (2.10) can be rewritten as

$$h'(t)-(r+s_0\,\lambda)\,h(t)=(-1/4m)\,(p(t)-h(t))^2 \tag{2.11}$$

Note that from (2.8) it follows

$$C(t)=(p(t)-M_Q(t))/2=(p(t)-h(t))/2 \tag{2.12}$$

The right hand expression of (2.11) can therefore be expressed as a function of the cost and since we know that $C_t<0$, (2.11) could be transformed into

$$h'(t)-(r+s_0\,\lambda)\,h(t)=-\exp(-\beta t) \qquad \beta\in R^+ \tag{2.13}$$

The value of β gives the "speed" at which technological innovation can reduce extraction costs. (2.13) corresponds also to the idea that, as time goes to infinity, the scarcity rent should become more and more prevailing as component of the resource price, as it can be seen from (2.7) (see also Dasgupta-Heal (1979), Farzin (1984)).

(2.13) is a linear first order non-homogeneous differential equation, whose solution is

$$h(t)=1/(r+s_0\,\lambda+\beta)\exp(-\beta t)+K\exp((r+s_0\,\lambda)t) \tag{2.14}$$

The condition (2.4) can also be written as

$$M(Q(T), T)=Q(T)\,h(T)=B(Q(T), T) \tag{2.15}$$

It follows then that

$$h(T)=B(Q(T),T)/Q(T) \qquad Q(T)\neq 0 \tag{2.16}$$

With (2.16), (2.14) becomes

$$h(t)=1/(r+s_0\lambda+\beta)\exp(-\beta t)+\big(B(T)/Q(T)-1/(r+ +s_0\lambda+\beta)\exp(-\beta T)\big)\exp\big((r+s_0\lambda)(t-T)\big) \tag{2.17}$$

The dynamics of the shadow price is then given by (2.17); since it depends on Q, at time T, which is subject to a stochastic process, it is necessary to see first how the resource stock develops. It can be observed that $h(t)$ is decreasing for $B/Q<\exp(-\beta T)/(r+s_0\lambda)$, is increasing for $B/Q>\big((r+s_0\lambda)\exp(-\beta T)\big)+\beta\exp\big((r+s_0\lambda)T)\big)/\big((r+s_0\lambda)(r+s_0\lambda+\beta)\big)$, while for $\exp(-\beta T)/(r+s_0\lambda)<B/Q<\big((r+s_0\lambda)\exp(-\beta T)\big)+\exp\big((r+s_0\lambda)T)\big)/\big((r+s_0\lambda)(r+s_0\lambda+\beta)\big)$ there exists $t_0\in(0,T)$ such that $h'(t)<0$ for $t<t_0$, $h'(t_0)=0$, $h'(t)>0$ for $t>t_0$.

It means that, in order to have an increasing scarcity rent, in the entire planning horizon, or at least from a certain point $t_0<T$ on, B/Q should be greater than $\exp(-\beta T)/(r+s_0\lambda)$: this is the discounted value of a function of cost at time T, discounted by the market rate plus the expected jump in the estimate of the stock per unit of time.

The behaviour of the rent for scarcity suggests a more precise definition of exhaustability. As a matter of fact, when B/Q is "sufficiently high" ($Q(T)$ is "sufficiently small"), the rent is rising; when, on the contrary, the stock is estimated to be large enough, the resource cannot be considered exhaustible and the rent declines [2]. The threshold value of the bequest requirement is $\exp(-\beta T)/(r+s_0\lambda)$, that does not depend on Q, but only on the cost. So, once a suitable cost of extraction function is defined, as we did in (2.13), the (estimated) values of $Q(T)$, for which the rent is rising, are found: in this case the resource can be thought of as exhaustible.

The optimal policy is given by (2.8) and (2.17)

$$q^*(t)=Q(t)/(2m)\big[\big(p(t)-1/(r+s_0\lambda+\beta)\exp(-\beta t)-(B/Q+ -1/(r+s_0\lambda+\beta)\exp(-\beta T)\big)\exp\big((r+s_0\lambda)(t-T)\big)\big] \tag{2.18}$$

Since it depends on $Q(t)$, the optimal policy itself is following a stochastic process and, therefore, it should be revised continuously. Production takes place as long as $p(t)>h(t)$, i.e. when the resource price is greater than the scarcity rent [3]

Finally, (see Vorst (1983)), note that $M\big(Q(s),s\big)=\exp(-rs)E_s\big[\int_s^T\exp(-rt)\times \times\big(p(t)-C\big(Q(t),q^*(t),t\big)q^*(t)\,dt+\exp(-rT)B\big(Q(T),T\big)\big)\big]$ is the maximum

[2] It can also be shown that $\delta h(t)/\delta Q(T)<0$ for each t.

[3] In general, a suitable bequest requirement can be assigned in order to produce on all the relevant period. Otherwise, if for instance $p(0)<h(0)$, production will be delayed up to $t_0<T$, such that $p(t_0)>h(t_0)$, and the optimal policy will be determined since then.

expected discounted profit at time $s<T$ when the resource stock is $Q(s)$. Therefore, the maximum expected discounted profit at time 0 is

$$M_T(Q(0), 0) = Q_T(0)\, h_T(0) \tag{2.19}$$

when $h(T)=B(T)/Q(T)$; this value can then be found at the intersection of the h axis with the $h(t)$ curve multiplied by $Q(0)$.

3. The free horizon problem and the effect of uncertainty

Let us now consider a more general case, i.e. the maximization of profit when the horizon is let free. From (2.19), we can, in fact, determine what could be the optimal strategy when T is variable. By computing $\partial h(0)/\partial T$, we see that, in order to get a given bequest, for $B/Q > \exp(-\beta T)/(r+s_0\lambda)$ it is convenient to have a shorter horizon, i.e. to produce less. This suggests a conservation policy, that is, a policy of investment in the inground. The opposite is true for $B/Q < \exp(-\beta T)/(r+s_0\lambda)$: in this case, it is convenient to have a longer horizon, i.e. to choose a policy of disinvestment. This result corresponds perfectly to the idea of exhaustability previously given. Note that the analysis here does not consider the introduction of possible substitutes for the resource. When $B/Q=\exp(-\beta T)/(r+s_0\lambda)$, the planning horizon has no influence on the maximum profit.

Those results can be summarized in the following proposition:

PROPOSITION 1. For a given bequest requirement, it is optimal to follow
— a conservation policy when $B/Q > \exp(-\beta T)/(r+s_0\lambda)$;
— a disinvestment policy when $B/Q < \exp(-\beta T)/(r+s_0\lambda)$;
— $B/Q=\exp(-\beta T)/(r+s_0\lambda)$ is the threshold value of bequest and corresponds to indifference in the planning horizon.

To analyze the effect of uncertainty, we can simply look at the critical value $\exp(-\beta T)/(r+s_0\lambda)$ and see what happens when the uncertainty, i.e. $s_0\lambda$, is increasing. It can be immediately seen that $\partial/\partial(s_0\lambda)\left(\exp(-\beta T)/(r+s_0\lambda)\right)<0$. It means that, as the probability of having the resource stock reduced is increasing, the interval in which the resource may not be considered exhaustible is reducing. Proposition 2 then holds:

PROPOSITION 2. If the optimal policy is given by (2.1) subject to (2.2), but decisions are taken without taking into account the probability of a reduction in the resource, then the producer should decide to disinvest, while in fact it should be optimal to conserve the stock in-the-ground.

4. Comments and conclusions

The problem discussed here regards the optimal policy to be followed by a producer of an exhaustible resource.

Results depend heavily on the choice of the class of cost of production function, in particular on the impact of technological innovations that can reduce future costs.

Conservation or disinvestment strategies are outlined according to the bequest requirements. The resource price $p(t)$ can follow any path and is assumed exogenous. Further analysis should be done in order to analyze the effect of price regulation (government regulation or control by a cartel) and also the case in which the price depends on the stock.

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Optymalne strategie wykorzystania wyczerpywalnych zasobów

Praca dotyczy optymalnych strategii wykorzystania w czasie wyczerpywalnych zasobów naturalnych, dla pewnej szczególnej klasy funkcji kosztów. Niepewność dotycząca oceny osiągalnych zasobów została opisana za pomocą procesu Poisson'a. Szczególną uwagę zwrócono na kształtowanie się ceny przy uzależnieniu jej od tempa wyczerpywania się zasobów. Rozważono również problem przy nieograniczonym horyzoncie czasu. Zbadano efekty uwzględnienia niepewności wielkości zasobów.

Оптимальные стратегии использования исчерпываемых ресурсов

Работа касается оптимальных стратегий использования во времени исчерпываемых природных ресурсов для некоторого особого класса функции стоимости. Неопределенность, касающаяся оценки достигаемых ресурсов, описывается с помощью пуассоновского процесса. Особое внимание обращено на формирование цены при учете ее зависимости от темпа исчерпывания ресурсов. Рассмотрен также случай неограниченного горизонта времени. Исследовались эффекты учета неопределенности величины ресурсов.