Control
and Cybernetics

VOL. 15 (1986) No. 3-4

Geometric aspects of the inclusion principle

by

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One of the recent approaches in large-scale system modelling, stability analysis and decentralized control is, based on the inclusion principle of dynamic systems, the expansion-contraction scheme developed by Siljak and his co-workers. In the paper a detailed analysis of the inclusion concept is presented. Generalized inverses of matrices and the geometric approach to linear time-invariant systems are used to obtain formulas defining expanded and contracted models explicitly.

1. Introduction

For many reasons, practical or conceptual, most of large scale systems are assumed to be composed of interconnected subsystems. In the standard decomposition approach these subsystems appear as disjoint and they are controlled on the basis of locally provided information. However, in many cases, for example in economic systems, traffic or power systems, these subsystems do not appear as disjoint. Some of them can possess certain state variables in common and therefore they overlap.

One of the recent approaches to deal with such systems is, based on the inclusion principle, the expansion-contraction scheme introduced by Siljak and his co-workers [4], [5], [6]. They proposed to expand the original system by linear transformation to a larger state space where subsystems appear as disjoint. If such transformation is performed in a proper way the expanded system includes the original one, i.e. contains all necessary information about the behavior of the original system. Then, subsequent analysis can be carried out for the expanded model using standard disjoint decomposition; resulting conclusions can be contracted to the original system.

Such methodology was applied by Ikeda, Siljak and White to decentralized suboptimal control and to stability analysis of large-scale systems [4], [5]. Its attractiveness was also demonstrated by Titli and Calvet [3]. However, the results Ikeda et al. [4] have obtained define expanded system implicitly by the set of nonlinear matrix equations. This restricts the choice of the expansion-contraction scheme to rather simple and special cases.

In this paper a geometric analysis of the inclusion principle is developed. The formulas we established define explicitly the expanded system and can be easily applied to the overlapping decomposition or further detailed investigation the properties of the expanded system.

2. Inclusion principle, preliminary results

Let us consider a pair of linear time-invariant systems described by

$$S: \dot{x}(t)=Ax(t)+Bu(t), \tag{1}$$

$$y(t)=Cx(t) \tag{2}$$

and

$$\tilde{S}: \dot{\tilde{x}}(t)=\tilde{A}\tilde{x}(t)+\tilde{B}u(t), \tag{3}$$

$$y(t)=\tilde{C}\tilde{x}(t), \tag{4}$$

where $x(t)$, $\tilde{x}(t)$ are respectively an n-dimensional and an $\tilde{n}$-dimensional state vectors, $u(t)$ is an r-dimensional input vector and $y(t)$ is an m-dimensional output vector. The matrices are constant and of appropriate dimensions. It is assumed that the dimensionality of S is smaller or at most equal to that of $\tilde{S}$.

A system $\tilde{S}$ includes a system S (or equivalently a system S is included by a system $\tilde{S}$) if there exists an $\tilde{n}\times n$ matrix T with full column rank such that for any initial state x_0 of S and any fixed input $u(t)$ the choice $\tilde{x}_0=Tx_0$ of the initial state of $\tilde{S}$ implies

$$x(t, x_0, u)=T^+\tilde{x}(t, \tilde{x}_0, u), \tag{5}$$

$$y[x(t)]=y[\tilde{x}(t)] \tag{6}$$

for all $t>0$, where T^+ is a generalized inverse of T. We also say then, that $\tilde{S}$ is an expansion of S or S is a contraction of $\tilde{S}$.

To illustrate the expansion and contraction notion let us consider the following simple example borrowed from [4]. Suppose we are given a system

$$S: \dot{x}(t)=Ax(t), \tag{7}$$

where n-dimensional state vector x is composed of three vector components $x^T=[x_1^T, x_2^T, x_3^T]$, $\dim x_i=n_i$, $n=n_1+n_2+n_3$. The matrix A can be written

$$A=\left[\begin{array}{c:c:c} A_{11} & A_{12} & A_{13} \\ \hdashline A_{21} & A_{22} & A_{23} \\ \hdashline A_{31} & A_{32} & A_{33} \end{array}\right],$$

where the submatrices correspond to the components x_1, x_2, x_3 and have appropriate dimensions. Dashed lines show two overlapping subsystems of S.

Thus, we can decompose the state vector x into two overlapping components $\tilde{x}_1^T=[x_1^T, x_2^T]$, $\tilde{x}_2^T=[x_2^T, x_3^T]$ and define a new expanded vector $\tilde{x}^T=[\tilde{x}_1^T, \tilde{x}_2^T]$, $\dim \tilde{x}= =\tilde{n}=n_1+2n_2+n_3$.

The above procedure is equivalent to the linear transformation $\tilde{x}=Tx$ defined by the matrix

$$T=\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix},$$

where I_1, I_2, I_3 are identity matrices of respective dimensions.

For the expanded system

$$\tilde{S} : \dot{\tilde{x}}(t)=\tilde{A}\tilde{x}(t) \tag{8}$$

Ikeda and Siljak assumed that

$$\tilde{A}=TAT^{+}+M, \tag{9}$$

where M is an $\tilde{n}\times\tilde{n}$ complementary matrix, which can be, for example, equal to

$$M=\begin{bmatrix} 0 & \frac{1}{2}A_{12} & -\frac{1}{2}A_{12} & 0 \\ 0 & \frac{1}{2}A_{22} & -\frac{1}{2}A_{22} & 0 \\ 0 & -\frac{1}{2}A_{22} & \frac{1}{2}A_{22} & 0 \\ 0 & -\frac{1}{2}A_{32} & \frac{1}{2}A_{32} & 0 \end{bmatrix}.$$

Then we obtain

$$\tilde{A}=\left[\begin{array}{cc:cc} A_{11} & A_{12} & 0 & A_{13} \\ A_{21} & A_{22} & 0 & A_{23} \\ \hdashline A_{21} & 0 & A_{22} & A_{23} \\ A_{31} & 0 & A_{32} & A_{33} \end{array}\right].$$

Now, $\tilde{S}$ can be decomposed into two interconnected subsystems as follows

$$\tilde{S}_1 : \dot{\tilde{x}}_1(t)=\tilde{A}_{11}\tilde{x}_1(t)+\tilde{A}_{12}\tilde{x}_2(t),$$

$$\tilde{S}_2 : \dot{\tilde{x}}_2(t)=\tilde{A}_{21}\tilde{x}_1(t)+\tilde{A}_{22}\tilde{x}_2(t),$$

where

$$\tilde{A}_{11}=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \tilde{A}_{22}=\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}$$

are subsystem matrices and

$$\tilde{A}_{12}=\begin{bmatrix} 0 & A_{13} \\ 0 & A_{23} \end{bmatrix}, \quad \tilde{A}_{21}=\begin{bmatrix} A_{21} & 0 \\ A_{31} & 0 \end{bmatrix}$$

are interconnection matrices.

Generally, in order to establish explicit relations between S and $\tilde{S}$ Ikeda and Siljak assumed, similarly as above, that

$$\tilde{A}=TAT^{+}+M, \tag{10}$$

$$\tilde{B}=TB+N, \tag{11}$$

$$\tilde{C}=CT^{+}+L, \tag{12}$$

where M, N and L are complementary matrices of appropriate dimensions. Then, they proved that $\tilde{S}$ includes S if and only if complementary matrices satisfy the set of nonlinear matrix equations

$$\begin{aligned} T^+ M^i T=0, \quad & T^+ M^{i-1} N=0, \\ LM^{i-1} T=0, \quad & LM^{i-1} N=0 \end{aligned} \tag{13}$$

for $i=1, ..., \tilde{n}$.

Next, they distinguished two particular cases. First, called restriction, when any trajectory of $\tilde{S}$ starting at any point $\tilde{x}_0$ from the image of T lies entirely in this subspace. Then, the complementary matrices satisfy

$$MT=0, \quad N=0, \quad LT=0. \tag{14}$$

Second case, called aggregation, is when

$$x(t, T^+ \tilde{x}_0, u)=T^+ \tilde{x}(t, \tilde{x}_0, u). \tag{15}$$

Then, the complementary matrices satisfy

$$T^+ M=0, \quad T^+ N=0, \quad L=0. \tag{16}$$

Because of the complexity of the conditions (14) any attempt to apply expansion-contraction scheme is practically restricted to these two mentioned particular cases. However, at present it is not known how particular are these particular cases. The answer to this question is given in the following section.

3. The main results

We first note that from (5) there follows

$$\dot{x}(t)=T^+ \dot{\tilde{x}}(t)$$

and this is equivalent to

$$Ax(t)+Bu(t)=T^+ \left(\tilde{A}\tilde{x}(t)+\tilde{B}u(t)\right). \tag{17}$$

Moreover, it is well known [2] that solution of (5) takes, for any time $t>0$, the form

$$\tilde{x}(t)=Tx(t)+v(t), \tag{18}$$

where $v(t)$ is an arbitrary vector from the null space of T^+, Ker T^+.

Analysing the evolution of $\tilde{S}$ we state that there are only three possible cases. First is when a trajectory of $\tilde{S}$ starting from any point $\tilde{x}_0 \in ImT$ lies entirely in this subspace. The second case is when a trajectory of $\tilde{S}$ starting from a point $\tilde{x}_0 \in ImT$ lies in a proper subspace of $R^{\tilde{n}}$ containing ImT. In the third case any trajectory of $\tilde{S}$ starting from $\tilde{x}_0 \in ImT$ does not lie, in general, in any proper subspace of $R^{\tilde{n}}$, i.e. $\tilde{x}(t)$ can take any value from $R^{\tilde{n}}$. We study now all three cases.

CASE 1

Since $\tilde{x}(t) \in ImT$ for all $t>0$, $\dot{x}(t) \in ImT$ and in addition $v(t)=0$, it follows from (6) and (17) that

$$T^+ \tilde{A} T = A, \tag{19}$$

$$T^+ \tilde{B} = B, \tag{20}$$

$$\tilde{C} T = C. \tag{21}$$

Because T has full column rank, $T^+ T = I$ and there exist matrices $\tilde{A}, \tilde{B}, \tilde{C}$ satisfying eqs. (19), (20), (21). They are equal to, [2],

$$\tilde{A} = TAT^+ + M, \tag{22}$$

$$\tilde{B} = TB + N \tag{23}$$

$$\tilde{C} = CT^+ + L, \tag{24}$$

where matrices M, N, L satisfy homogeneous equations

$$T^+ MT = 0, \tag{25}$$

$$T^+ N = 0, \tag{26}$$

$$LT = 0. \tag{27}$$

Note, that $M\tilde{x} \in \operatorname{Ker} T^+$ for every $\tilde{x} \in ImT$ and $N\tilde{x} \in \operatorname{Ker} T^+$ for all $\tilde{x} \in R^{\tilde{n}}$. In the case we consider $\tilde{x}(t)$, $\dot{\tilde{x}}(t) \in ImT$ for $t>0$. Thus, we obtain

$$MT = 0, \quad N = 0. \tag{28}$$

It means that

$$\tilde{A} = TAT^+ + Q(I - TT^+), \tag{29}$$

$$\tilde{B} = TB, \tag{30}$$

$$\tilde{C} = CT^+ + R(I - TT^+), \tag{31}$$

for some matrices Q, R of appropriate dimensions.

CASE 2

If $\tilde{x}(t)$ lies in a proper subspace of $R^{\tilde{n}}$, then there exists a matrix T_1 such that $ImT_1 \subset \operatorname{Ker} T^+$ and

$$\tilde{x}(t) = Tx(t) + T_1 T_1^+ v_1(t), \tag{32}$$

where $v_1(t)$ is an arbitrary vector from R^n. Moreover, the subspace $ImT \oplus ImT_1$ is $\tilde{A}$-invariant. It means, that

$$(I - TT^+ - T_1 T_1^+)\tilde{A}(TT^+ + T_1 T_1^+) = 0. \tag{33}$$

Substituting (32) to (17) we obtain that

$$T^+ \tilde{A} T = A \tag{34}$$

$$T^+ \tilde{A} T_1 T_1^+ = 0. \tag{35}$$

Any solution of (33), (34), (35) takes the form (22) where M satisfies (25) and

$$T^+ M T_1 T_1^+ = 0, \tag{36}$$

$$(I - TT^+ - T_1 T_1^+) M (TT^+ + T_1 T_1^+) = 0, \tag{37}$$

so

$$M = Q - (I - T_1 T_1^+) Q (TT^+ + T_1 T_1^+), \tag{38}$$

where Q is a matrix of appropriate dimensions.

Next, since $Im\tilde{B} \subset ImT \oplus ImT_1$ we obtain that $\tilde{B}$ satisfies (20) and

$$(I - TT^+ - T_1 T_1^+) \tilde{B} = 0. \tag{39}$$

Any solution of (20), (39) takes the form (23) where N satisfies (26) and

$$(I - TT^+ - T_1 T_1^+) N = 0, \tag{40}$$

so

$$N = T_1 T_1^+ P \tag{41}$$

for some matrix P.

Substituting (32) to (6) we obtain (21) and

$$\tilde{C} T_1 T_1^+ = 0 \tag{42}$$

It implies that $\tilde{C}$ takes the form (24) where N satisfies (27) and

$$L T_1 T_1^+ = 0, \tag{43}$$

so

$$L = R (I - TT^+ - T_1 T_1^+). \tag{44}$$

Thus, we conclude that

$$\tilde{A} = TAT^+ + Q - (I - T_1 T_1^+) Q (TT^+ + T_1 T_1^+), \tag{45}$$

$$\tilde{B} = TB + T_1 T_1^+ P, \tag{46}$$

$$\tilde{C} = CT^+ + R (I - TT^+ - T_1 T_1^+). \tag{47}$$

CASE 3

Assuming that $\tilde{x}(t)$ can take any value from $R^{\tilde{n}}$ we have

$$\tilde{x}(t) = Tx(t) + (I - TT^+) v_1(t) \tag{48}$$

and it follows from (6) and (17) that $\tilde{A}, \tilde{B}, \tilde{C}$ satisfy (19), (20), (21) and

$$T^+ \tilde{A} (I - TT^+) = 0 \tag{49}$$

$$\tilde{C} (I - TT^+) = 0 \tag{50}$$

Solving these equations we obtain

$$\tilde{A} = TAT^+ + (I - TT^+) Q, \tag{51}$$

$$\tilde{B} = TB + (I - TT^+) P, \tag{52}$$

$$\tilde{C} = CT^+. \tag{53}$$

4. Conclusions

In this paper a geometric analysis of the inclusion principle was used to characterize explicitly the expanded system. The conditions we derived are much simpler than those obtained by Siljak and his co-workers.

By a further analysis of the results we obtained it can be stated that S is in fact an aggregated system for $\tilde{S}$. In aggregation it is usually assumed that input variables can take any value from the input space. However, we can restrict them to some proper subspace of the input space. Such approach was presented by economists for static models and such possibility was also mentioned by Aoki [1], where he suggested to restrict state variables to some proper subspace of the state space.

Clearly, the expansion-contraction scheme for large-scale systems analysis is an open-ended topic. The natural step will be to develop computational algorithms for obtaining expanded systems with couplings between subsystems as small as possible. A fruitful direction of future researches is also to apply the results of this paper to multimodel decentralized control.

References

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Geometryczne aspekty zasady inkluzji

Ostatnio, dla celów analizy i sterowania zdecentralizowanego w systemach złożonych Siljak i jego współpracownicy wykorzystali zasadę inkluzji układów dynamicznych. W pracy, wykorzystując własności geometryczne trajektorii układu oraz uogólnione odwrotności macierzy, została przedstawiona szczegółowa analiza tej zasady dla stacjonarnych układów liniowych.

Геометрические аспекты принципа включения

В работе дан анализ принципа включения динамических систем, на котором основан, предложенный Шиляком один из самых новых подходов к моделированию, анализу и децентрализованному управлению большими системами. Пользуясь псевдообратными матрицами и геометрическими свойствами траектории динамической системы выведены формулы для расширенной системы.