Control
and Cybernetics
VOL. 15 (1986) No. 3-4

A symmetric duality concept for linear goal programming: principal results

by

WŁODZIMIERZ OGRYCZAK
Institute of Informatics
University of Warsaw
PKiN
00-901 Warszawa, Poland

Linear goal programming (GP) is a widely used tool for dealing with problems involving multiple objectives. This paper aims to introduce a new concept of duality for GP. In this concept the dual to a minsum GP problem is a GP problem and the dual to a lexicographic GP problem is a multidimensional lexicographic GP problem. We prove most of typical dual relations including the saddle-point property and the formula for marginal values.

1. Introduction

This paper deals with lexicographic linear goal programming, i.e., with the specific form of linear goal programming wherein one seeks the lexicographic minimum of an ordered set of goal deviations. This approach, also described as preemptive priority based goal programming, is widely used in multiobjective optimization. It covers the minsum goal programming as a special (scalar) case.

Ignizio [1, 2] developed practical sensitivity analysis for lexicographic GP and introduced the so-called multidimensional dual. The Ignizio's dual has, however, some weaknesses which can be summarized as follows:

— dual to the GP problem is not any GP problem,

— dual variables cannot be directly considered as marginal values for several goals.

The purpose of this paper is to present a slightly different duality concept which is free from the above weaknesses. We present such a duality theory in which the dual to a minsum GP problem is also a GP problem and the dual to a lexicographic GP problem is a multidimensional lexicographic GP problem. Moreover, this GP duality covers all the typical LP dual relations including the saddle-point property and the Mills' formula for marginal values.

We shall use throughout this paper the following notation connected with vector inequalities:

$w \leqq v \Leftrightarrow w_i \leqslant v_i$ for each index i,

$w < v \Leftrightarrow w_i < v_i$ for each index i,

$w \preccurlyeq v \Leftrightarrow w_i \leqslant v_i$ or $w_j < v_j$ for some $j < i$,

$w \prec v \Leftrightarrow w \preccurlyeq v$ and $w \neq v$

where w and v are column vectors.

For matrices (or row vectors) the above relations are understood as follows

$Y \text{ r } U \Leftrightarrow Y_i \text{ r } U_i$ for each pair of columns Y_i and U_i.

For any number α

$$(\alpha)_+ = \begin{cases} \alpha & \text{if } \alpha \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

The operator $(\cdot)_+$ applied to a vector or matrix is understand componentwisely. Similarly, for any column vector v

$$(v)_L = \begin{cases} v & \text{if } v \succcurlyeq 0 \\ 0 & \text{otherwise.} \end{cases}$$

For a matrix (or row vector) $(Y)_L$ is defined by

$[(Y)_L]_i = (Y_i)_L$ for each column Y_i of the matrix Y.

2. The primal

The typical formulation of the lexicographic linear GP problem is as follows

Find a vector $x = (x_1, \ldots, x_n)^T$ to lexicographically minimize

$$a = [g_1(d^-, d^+), g_2(d^-, d^+), \ldots, g_K(d^-, d^+)]^T \tag{2.1}$$

subject to

$$\sum_{j \in J} c_{ij} x_j + d_i^- - d_i^+ = b_i \quad \text{for } i \in I \tag{2.2}$$

$$x \geqq 0, d^- \geqq 0, d^+ \geqq 0 \tag{2.3}$$

where

$I = \{1, 2, \ldots, m\}$ — set of goal indices,

$J = \{1, 2, \ldots, n\}$ — set of decision indices,

x_j — j-th decision (structural) variable,

c_{ij} — coefficient of x_j in the i-th goal constraint,

b_i — target for goal i,

d_i^- — negative deviation for goal i,

d_i^+ — positive deviation for goal i,

$g_k(d^-, d^+)$ — linear function of the deviation variables to be minimized at priority k $(k = 1, 2, \ldots, K)$.

Function g_k can be written in the form

$$g_k(d^-, d^+) = \sum_{i \in I} (u_i^{(k)} d_i^- + w_i^{(k)} d_i^+)$$

where

$u_i^{(k)}$ — weight assigned to the variable d_i^- at priority k,

$w_i^{(k)}$ — weight assigned to the variable d_i^+ at priority k.

In this paper we consider a different form of the GP problem with the constraints (2.2) and (2.3) replaced by the following

$$\sum_{j \in J} c_{ij} x_j + d_i^- - d_i^+ = 0 \quad \text{for } i \in I \tag{2.4}$$

$$b_j^- \leqslant x_j \leqslant b_j^+ \quad \text{for } j \in J \tag{2.5}$$

$$d^- \geqslant 0, \; d^+ \geqslant 0 \tag{2.6}$$

where some bounds b_j^+ and b_j^- can take the value ∞ or $-\infty$, respectively.

Note that the constraints (2.2) and (2.3) of any GP problem can be written in the form (2.4)–(2.6). Such a transformation can be performed for instance by introducing additional (logical) variables similarly as in the standard simplex codes. Moreover, many real relations can be easily handled via constraints (2.4)–(2.6). This approach allows to consider some bounds on the original decision variables as well as makes possible to introduce the interval goals (goals with the interval targets). There are many practical reasons for using intervals as targets for some goals (see [5]).

For convenience in discussion, we shall rewrite the GP problem (2.1) and (2.4)–(2.6) in matrix form as shown below.

Find a vector x to lexicographically minimize

$$a = [(u^{(1)} d^- + w^{(1)} d^+), \ldots, (u^{(K)} d^- + w^{(K)} d^+)]^T = Ud^- + Wd^+ \tag{2.7}$$

subject to

$$Cx + I_m d^- - I_m d^+ = 0 \tag{2.8}$$

$$b^- \leqslant x \leqslant b^+ \tag{2.9}$$

$$d^- \geqslant 0, \; d^+ \geqslant 0 \tag{2.10}$$

where

$u^{(k)}$ — $1 \times m$ row vector of weights associated with the negative deviation variables at priority k,

$w^{(k)}$ — $1 \times m$ row vector of weights associated with the positive deviation variables at priority k,

U — $K \times m$ matrix consisted of the rows $u^{(k)}$,

W — $K \times m$ matrix consisted of the rows $w^{(k)}$,

C — $m \times n$ coefficient matrix,

I_m — $m \times m$ identity matrix,

b^- — vector of lower bounds on the decision variables (a column vector, $n\times 1$),
b^+ — vector of upper bounds on the decision variables (a column vector, $n\times 1$),
x — decision variables vector (a column vector, $n\times 1$),
d^- — vector of negative deviation variables (a column vector, $m\times 1$),
d^+ — vector of positive deviation variables (a column vector, $m\times 1$).

Some infinite coefficients in the problem (2.7)–(2.10) are allowed. More precisely, the following can occur:

1° $b_j^- = -\infty$ (x_j has no lower bound),
2° $b_j^+ = \infty$ (x_j has no upper bound),
3° $u_i^{(1)} = u_i^{(2)} = \ldots = u_i^{(K)} = \infty$ ($d_i^- = 0$, i.e., the negative deviation for goal i is forbidden),
4° $w^{(1)} = w^{(2)} = \ldots = w^{(K)} = \infty$ ($d_i^+ = 0$, i.e., the positive deviation for goal i is forbidden).

In what follows we assume that $0\cdot\infty = 0$ and $0\cdot(-\infty) = 0$.

For further simplification we prefer to use an explicit form of the deviations in the GP model. Due to the equality (2.8) the deviation vectors d^- and d^+ can be expressed as

$$d^- = (-Cx)_+ \text{ and } d^+ = (Cx)_+$$

provided that $(d^-)^T d^+ = 0$. The property $(d^-)^T d^+ = 0$ is usually guaranteed for all the optimal solutions by some requirements on the weight coefficients.

Finally, we formulate the GP primal as the lexicographic problem

$$\text{GPP: lexmin } \{U(-Cx)_+ + W(Cx)_+ : b^- \leqslant x \leqslant b^+\} \tag{2.11}$$

A vector x is called to be feasible to the problem GPP if it satisfies the inequality $b^- \leqslant x \leqslant b^+$. A feasible vector x is said to be optimal if it lexicographically minimizes the achievement (vector) function $a = U(-Cx)_+ + W(Cx)_+$ over the whole feasible set, i.e.,

$U(-C\bar{x})_+ + W(C\bar{x})_+ \preccurlyeq U(-Cx)_+ + W(Cx)_+$ for any feasible x.

The problem (2.7)–(2.10) will be referred to as an expanded form of the GP primal (2.11).

Proposition 2.1. If the weight matrices satisfy

$$U + W \succcurlyeq 0, \tag{2.12}$$

then the achievement function is lexicographically convex, i.e.,

$$a(\alpha_1 x^1 + \alpha_2 x^2) \preccurlyeq \alpha_1 a(x^1) + \alpha_2 a(x^2)$$

for any x^1, x^2 and $\alpha_1, \alpha_2 \geqslant 0$, $\alpha_1 + \alpha_2 = 1$.

Proposition 2.2. If the inequality (2.12) holds, then the GP primal and the correspon-ing expanded GP problem are equivalent in the sense of having the same optimal vectors x.

Note that the problem (2.7)–(2.10) is not a well-posed problem if the weight coefficients do not satisfy the requirement (2.12). Similarly, it is unreasonable to consider the GP problems which do not satisfy the requirement

$$b^{+}-b^{-}\geqq 0 \tag{2.13}$$

since they are explicitly infeasible. We say that the GP problem is well-posed if both the inequalities (2.12) and (2.13) are valid. In what follows we shall only consider the well-posed GP problems.

3. The multidimensional GP dual

Consider, at first, the GP problem with the scalar achievement function (i.e., $K=1$). This case, called minsum or weighted GP, is the oldest and simpliest form of goal programming. Any minsum GP problem can be formed as a traditional LP problem with the special structure of the constraints. Thus LP duality leads to some definition for the scalar GP dual. The scalar GP primal takes the form

$$\min \{u(-Cx)_{+}+w(Cx)_{+} : b^{-}\leqq x\leqq b^{+}\} \tag{3.1}$$

where $u=u^{(1)}$ and $w=w^{(1)}$ are rows of the weight coefficients. It can be rewritten in the expanded form as the LP primal:

Find a vector x to minimize

$$a=ud^{-}+wd^{+}$$

subject to the conditions (2.8), (2.9) and (2.10).
The corresponding LP dual is given as follows:

Find a vector y to minimize

$$a^{*}=v^{-}\,b^{-}-v^{+}\,b^{+} \tag{3.2}$$

subject to

$$yC+v^{-}\,I_n-v^{+}\,I_n=0 \tag{3.3}$$

$$-w\leqq y\leqq u \tag{3.4}$$

$$v^{-}\geqq 0,\ v^{+}\geqq 0 \tag{3.5}$$

where

y — $1\times m$ row vector of dual structural variables,
v^{-} — $1\times n$ row vector of dual negative deviation variables,
v^{+} — $1\times n$ row vector of dual positive deviation variables.

The problem (3.2)–(3.5) has a special structure of goal programming. Namely, it is an expanded form of the GP problem

$$\max \{(-yC)_{+}b^{-}-(yC)_{+}b^{+} :\ -w\leqq y\leqq u\} \tag{3.6}$$

It is easy to verify that this problem is a well-posed GP problem whenever the GP primal (3.1) is well-posed. The problem (3.6) will be referred to as the GP dual to the GP primal (3.1). Thus in the scalar case dual to a GP problem is also a GP problem. Using the LP duality theory we can state typical dual relations between the GP primal (3.1) and the GP dual (3.6). Moreover, if the problem (3.6) is treated as a GP primal, then the problem (3.1) is obtained as the GP dual.

The above GP duality concept can be extended to general lexicographic linear GP problems using the idea of the multidimensional dual (see [2, 3]). The multidimensional dual is obtained from the scalar dual by replacing variables y_i by K-dimensional vectors Y_i. Then all the scalar inequalities are understand as lexicographic. Given the GP primal (see (2.11))

$$\text{GPP: lexmin}\,\{U(-Cx)_+ + W(Cx)_+ : b^- \leqslant x \leqslant b^+\},$$

we may write its dual as the multidimensional GP problem

$$\text{GPD: lexmax}\,\{(-YC)_L\, b^- - (YC)_L\, b^+ : -W \preccurlyeq Y \preccurlyeq U\} \qquad (3.7)$$

where

Y is a $K \times m$ matrix of dual variables and each row $y^{(k)}$ ($k=1, 2, \ldots, K$) of the matrix Y is associated with priority level k.

Similarly as in the scalar case, the lexicographic GP dual is a well-posed (multidimensional) GP problem whenever the requirements (2.12) and (2.13) are satisfied.

PROPOSITION 3.1. The achievement function of the multidimensional GP dual is lexicographically concave, i.e.,

$$a^*(\alpha_1 Y^1 + \alpha_2 Y^2) \succcurlyeq \alpha_1 a^*(Y^1) + \alpha_2 a^*(Y^2)$$

for any Y^1, Y^2 and $\alpha_1, \alpha_2 \geqslant 0$, $\alpha_1 + \alpha_2 = 1$.

COROLLARY 3.1. *The optimal set to the multidimensional GP dual is convex.*

The lexicographic GP primal and the multidimensional GP dual satisfy all typical duality relations. More precisely, one can find lexicographic analogues of the principal results in LP duality.

PROPOSITION 3.2. If x is feasible to the lexicographic GP primal and Y is feasible to the multidimensional GP dual, then the following lexicographic inequality holds

$$U(-Cx)_+ + W(Cx)_+ \succcurlyeq -YCx \succcurlyeq (-YC)_L\, b^- - (YC)_L\, b^+ .$$

THEOREM 3.1. *If either the lexicographic GP primal or the multidimensional GP dual posses a finite optimal solution, then the other also does.*

COROLLARY 3.2. *If all the data coefficients are finite, then both the lexicographic GP primal and the multidimensional GP dual have finite optimal solutions and the corresponding optimal values of the achievement (vector) functions are equal.*

The next theorem summarizes the necessary and sufficient optimality conditions. All the typical LP optimality conditions have some analogues in lexicographic GP including even the saddle-point property.

THEOREM 3.2. *The following statements are equivalent:*

1° x^0 *is an optimal solution to the lexicographic GP primal and* Y^0 *is an optimal solution to the multidimensional GP dual;*

2° x^0 *and* Y^0 *are feasible and the corresponding values of the achievement (vector) functions are finite and equal to* $-Y^0 Cx^0$, *i.e.,* $U(-Cx^0)_+ + W(Cx^0)_+ = =(-Y^0 C)_L b^- - (Y^0 C)_L b^+ = -Y^0 Cx^0$;

3° x^0 *and* Y^0 *are feasible and the complementary slackness holds, i.e.,*

$$(Y^0 + W)(Cx^0)_+ = 0,$$

$$(U - Y^0)(-Cx^0)_+ = 0,$$

$$(-Y^0 C)_L (x^0 - b^-) = 0,$$

$$(Y^0 C)_L (b^+ - x^0) = 0;$$

4° *the pair* (x^0, Y^0) *is a lexicographic saddle-point of the vector function* $L(x, Y) = (U - Y)(-Cx)_+ + (Y + W)(Cx)_+ - (-YC)_L (x - b^-) - (YC)_L (b^+ + -x) - YCx$, *that is,*
$L(x^0, Y) \preccurlyeq L(x^0, Y^0) \preccurlyeq L(x, Y^0)$ *for any* $x \in R^n$ *and* $Y \in R^K \times R^m$;

5° x^0 *and* Y^0 *are feasible and the pair* (x^0, Y^0) *is a lexicographic saddle-point of the vector function* $\bar{L}(x, Y) = -YCx$, *i.e.,*

$$-YCx^0 \preccurlyeq -Y^0 Cx^0 \preccurlyeq -Y^0 Cx \text{ for any feasible } x \text{ and } Y.$$

The complementary slackness conditions are usually interpreted by implications: "If a variable in one problem is active, then the corresponding constraint in the other problem must be tight" and "If a constraint in one problem is not tight, then the corresponding variable in the other problem must be on the limit level". The following corollary states such a form of the complementary slackness conditions for the lexicographic GP.

COROLLARY 3.3. *If* x^0 *and* Y^0 *are any optimal solutions to the lexicographic GP primal and multidimensional GP dual, respectively, then the following implications hold*

$$Y_i^0 \succ -W_i \Rightarrow c^i x^0 \leqslant 0$$

$$c^i x^0 > 0 \Rightarrow Y_i^0 = -W_i$$

$$Y_i^0 \prec U_i \Rightarrow c^i x^0 \geqslant 0$$

$$c^i x^0 < 0 \Rightarrow Y_i^0 = U_i$$

$$x_j^0 > b_j^- \Rightarrow Y^0 C_j \succcurlyeq 0$$

$$Y^0 C_j \prec 0 \Rightarrow x_j^0 = b_j^-$$

$$x_j^0 < b_j^+ \Rightarrow Y^0 C_j \preccurlyeq 0$$

$$Y^0 C_j \succ 0 \Rightarrow x_j^0 = b_j^+$$

where c^i *denotes the i-th row of matrix C and* C_j *is the j-th column of matrix C.*

The lexicographic GP primal is, usually, solved as a sequence of scalar GP problems. It turns out that several rows of a dual solution (matrix) correspond to these problems.

THEOREM 3.3. *If Y^0 is an optimal solution to the multidimensional GP dual, then the k-th row of Y^0 ($k=1, 2, \ldots, K$) is a dual solution to the GP primal subproblem to be solved at priority k*

$$P_k: \min \{u^{(k)}(-Cx)_+ + w^{(k)}(Cx)_+ : b^- \leqslant x \leqslant b^+,\ x \in S_{k-1}\}$$

where S_{k-1} is the optimal set to the problem P_{k-1}.

COROLLARY 3.4. *The multidimensional GP dual can be solved as a sequence of scalar GP problems.*

In linear programming, having known dual solution one can calculate the so-called marginal values with respect to some data perturbations. Marginal values for LP problems are given by the Mills' formula (see [6]). Similar analysis can be performed in goal programming. In the scalar case we get a formula for marginal values by using the Mills' formula to the expanded form of GP problem.

For lexicographic GP one may consider the marginal vectors

$$a'_h(H) = \lim_{\alpha \to 0_+} \frac{1}{\alpha}[a(H+\alpha h) - a(H)]$$

where H and h denote the data and their perturbations, respectively. However, the lexicographic minimum is, in general, unstable (see [4]) and therefore some coefficients of the marginal vector are infinite. The instability is caused by specifity of the lexicographic relation. Namely, the dual feasible set defined by lexicographic inequalities is not closed.

PROPOSITION 3.3. If $-w^{(r)} < u^{(r)}$ for some $r < K$, then the feasible set to the multidimensional GP dual is not closed.

Fortunately, the lexicographic GP problem is stable with respect to small perturbations of the bounds coefficients, i.e., if only the bounds coefficients are allowed to be perturbed. Thus it is possible to perform analysis of the bounds coefficients changes and give a formula for the corresponding marginal vectors.

THEOREM 3.4. *Let J_e denote the set of equality bounds, i.e., $J_e = \{j \in J: b_j^- = b_j^+\}$. If the lexicographic GP primal has a finite optimal solution and all the weight coefficients $u_i^{(k)}$ and $w_i^{(k)}$ are finite, then there exist an $\alpha_0 > 0$ such that the perturbed GP problem*

$$\text{lexmin}\{U(-Cx)_+ + W(Cx)_+ : b^- + \beta^- \leqslant x \leqslant b^+ + \beta^+\}$$

is solvable provided that

$$|\beta_j^-| < \alpha_0 \quad \text{and} \quad |\beta_j^+| < \alpha_0 \quad \text{for } j \in J - J_e$$

$$\text{and } \beta_j^- \leqslant \beta_j^+ \quad \text{for } j \in J_e.$$

The corresponding marginal vector is then given by the formula

$$a'_{(\beta^-,\beta^+)}=\underset{Y\in S^*}{lexmax}\,[(-YC)_L\,\beta^- -(YC)_L\,\beta^+]$$

where S^* denotes the optimal set to the multidimensional GP dual.

Typical postoptimality analysis depend on calculating of the marginal values with respect to single entry perturbations. In goal programming one can consider three types of the bound coefficient perturbations:

1° lower bound perturbation

$$\tilde{b}_j^- = b_j^- + \alpha \quad \text{for } j\in J_d=\{j\in J:\ -\infty < b_j^- < b_j^+\},$$

2° upper bound perturbation

$$\tilde{b}_j^+ = b_j^+ + \alpha \quad \text{for } j\in J_u=\{j\in J:\ b_j^- < b_j^+ < +\infty\},$$

3° equality bound perturbation ($b_j^- = b_j^+ = b_j$)

$$\tilde{b}_j^- = \tilde{b}_j^+ = b_j + \alpha \quad \text{for } j\in J_e.$$

Due to Theorem 3.4 the marginal vectors with respect to positive or negative perturbations of the single bound coefficient are given as follows:

1° $a'_{(+b_j^-)}=\underset{Y\in S^*}{\text{lexmax}}\,(-YC_j)_L$, $a'_{(-b_j^-)}=\underset{Y\in S^*}{\text{lexmax}}-(-YC_j)_L$ for $j\in J_d$;

2° $a'_{(+b_j^+)}=\underset{Y\in S^*}{\text{lexmax}}\,-(YC_j)_L$, $a'_{(-b_j^+)}=\underset{Y\in S^*}{\text{lexmax}}\,(YC_j)_L$ for $j\in J_u$;

3° $a'_{(+b_j)}=\underset{Y\in S^*}{\text{lexmax}}-YC_j$, $a'_{(-b_j)}=\underset{Y\in S^*}{\text{lexmax}}\,YC_j$ for $j\in J_e$.

Some bounds b_j^- and b_j^+ represent, usually, targets for goal functions. The corresponding column C_j is then equal to minus unit versor $-e_i$. Thus several multidimensional dual variables prove to be multidimensional shadow prices for the corresponding goal functions.

COROLLARY 3.5. *If a logical variable associated with the goal function f_i is included in the GP primal formulation, then the marginal vectors with respect to perturbations of the corresponding target $[r_i^-;r_i^+]$ are given as follows:*

1° $a'_{(+r_i^-)}=\underset{Y\in S^*}{lexmax}\,(Y_i)_L$ *and* $a'_{(-r_i^-)}=\underset{Y\in S^*}{lexmax}-(Y_i)_L$

for positive or negative perturbation of the lower limit, respectively;

2° $a'_{(+r_i^+)}=\underset{Y\in S^*}{lexmax}-(-Y_i)_L$ *and* $a'_{(-r_i^+)}=\underset{Y\in S^*}{lexmax}\,(-Y_i)_L$

for positive or negative perturbation of the upper limit, respectively;

3° $a'_{(+r_i)}=\underset{Y\in S^*}{lexmax}\,Y_i$ *and* $a'_{(-r_i)}=\underset{Y\in S^*}{lexmax}\,-Y_i$

for positive or negative perturbation of the point target, respectively.

4. Conclusion

This paper has presented a symmetric duality theory for linear goal programming. It definitively disproves the opinion that duality does not exist in linear GP or it is less efficient than in linear programming.

For the scalar (minsum) GP primal

$$\min \{u(-Cx)_+ + w(Cx)_+ : b^- \leqslant x \leqslant b^+\}$$

we get the GP dual

$$\max \{(-yC)_+ b^- - (yC)_+ b^+ : -w \leqslant y \leqslant u\}$$

which is a typical GP problem.

Similarly, for the lexicographic GP primal

$$\operatorname{lexmin} \{U(-Cx)_+ + W(Cx)_+ : b^- \leqslant x \leqslant b^+\}$$

we get the GP dual

$$\operatorname{lexmax} \{(-YC)_L b^- - (YC)_L b^+ : -W \preccurlyeq Y \preccurlyeq U\}$$

which is a lexicographic multidimensional GP problem.

The paper has presented GP analogues to all classic LP duality relations including the saddle-point property and the Mills' formula for marginal values. Moreover, some special GP duality relations have been introduced.

References

[1] Ignizio J. P. Goal Programming and Extensions, Heath, Lexington, MA, 1976.

[2] Ignizio J. P. A note on the multidimensional dual. *European J. Operational Res.* **17** (1984), 116–122.

[3] Iserman H. Lexicographic goal programming: the linear case. In: M. Grauer, A. Lewandowski, A. P. Wierzbicki, Eds., Multiobjective and Stochastic Optimization. IIASA, Laxenburg, 1982, 65–78.

[4] Klepikova M. G. On the stability of lexicographic optimization problems (in russian). *Zh. Vychisl. Mat. i Mat. Fiz.* **25** (1985), 32–44.

[5] Nijkamp P., Spronk J. Goal programming for decision making: an overview and discussion. *Ricerca Operativa*, **12** (1979), 3–49.

[6] Williams A. C. Marginal values in linear programming. *SIAM J. Appl. Math.* **11** (1963), 82–94.

Symetryczna teoria dualności dla liniowych zadań programowania celowego

Programowanie celowe jest powszechnie stosowaną techniką rozwiązywania zagadnień wielokryterialnych. Praca prezentuje kompletną teorię dualności dla liniowych zadań programowania celowego. Problem dualny do skalarnego zadania programowania celowego jest również skalarnym zadaniem programowania celowego. Natomiast problem dualny do leksykograficznego zadania programowania celowego jest wielowymiarowym leksykograficznym zadaniem programowania celowego. Udowodnione są typowe relacje dualne.

Симметричная теория дуальности для линейных задач целевого программирования

Целевое программирование является широко используемым методом решения многокритериальных задач. В работе представлена полная теория дуальности для линейных задач целевого программирования. Дуальная задача в скалярном целевом программировании является также скалярной задачей целевого программирования. В свою очередь дуальная проблема в лексикографической задаче целевого программирования является многомерной лексикографической задачей целевого программирования. Доказаны типичные дуальные отношения.