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The unique solution of a diffusion-consumption problem with hysteresis*

by

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A class ot nonlinear diffusion-consumption problems with hysteresis effects is considered. The problems under consideration refer to biochemical systems, with parabolic equations of evolution such that the terms which represent volumetric cources are dependent on some switching on-off functional. This functional depends on history of the system via a hysteretic nonlinearity. Presence of such nonlinearity essentially complicates an analysis of questions concerning uniqueness of solutions and character of their continuous dependence on initial conditions. In this paper, a result on the uniqueness and continuous dependence on initial data is formulated and proved in the case of a problem in one space dimension,

1. Introduction

The phenomenon of "hysteresis" is taken here in a generic sense to refer to situations in which a physical process governing the development of a system may unfold in any one of a finite number of modes, with the mode which is selected dependent on the previous history of the system. Thus, in a sense the allowance of hysteresis in the formulation of a problem is really a consequence of an economy effected in representing the essential dynamics of the system, since one may give an equivalent description of the problem in which the overt dependence of the mode of evolution on the past development is eliminated by the following expedient: Adding to the system representation a dependent variable, the values of which at any time have the sole effect of determining the mode according to which the system develops, and adding an equation for the evolution of this dependent variable in terms of the other variables describing the system.

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Systems containing hysteresis occur naturally in the mathematical modeling of numerous phenomena. The system in which the hysteresis occurs may be governed by hyperbolic or parabolic equations. Examples of the former type are afforded by the equations describing elastic-plastic flow and some electromagnetic phenomena. Our concern in this paper will be the presence of hysteresis in systems governed by parabolic equations. A rather pristine model containing a parabolic equation with hysteresis arises in the theory of thermostats due to Glashoff and Sprekels [3]. In this theory, the system evolves in two possible modes, according to whether or not the thermostat is switched on or off. More complicated models have been used to describe chemical and biological phenomena. In particular, systems of equations with hysteresis have been proposed to govern chemical and biological processes exhibiting pattern formation, in an attempt to explain this phenomenon [4–7].

Typically, as in the biomathematical papers of Jäger and Hoppensteadt [4, 5], one encounters coupled parabolic equations for the concentrations of species in the system, and the source terms for the various species depend on some sort of switching functional, which is dependent on the previous history of the system. In a chemical example, the model proposed by Keller and Rubinow to describe the formation of Liesegang rings [6, 7], the notion of hysteresis does not appear directly. However, if one eliminates the concentration of precipitate from the list of dependent variables, one obtains a set of coupled reaction-diffusion equations for three species, with a term representing the rate of precipitation which takes two forms, according to whether or not precipitation has taken place at a locale in the past. The occurrence of precipitation is itself triggered by the crossing of a threshold for one of the three interacting species.

In all the examples referred to above, a salient feature of the hysteresis is the discontinuous dependence of a term representing "sources" or "sinks" of densities on the past histories of those densities. The presence of such hysteretic discontinuities severely complicates the proof of the uniqueness of solutions of initial value problems for the equations modeling the system, as Lipschitz-continuous dependence of the source term on the dependent variables is a standard prerequisite for establishing the uniqueness of solutions of ordinary or partial differential equations. As of this writing, very little has been established with regard to the uniqueness of solutions of any of the systems referred to above [3–7].

For the equations modeling a physical process, one is usually interested in proving not only the uniqueness of solutions of an initial value problem, but also their continuous dependence on the initial data. A simple example may clarify our remarks. Consider the following equation modeling a diffusion-consumption problem with hysteresis:

$$\begin{cases} u_t = \Delta u - \begin{cases} 1, & u(x, t') > 1 \quad \text{for some } t' \in (0, t] \\ 0, & \text{otherwise} \end{cases}, & (x, t) \in \mathbf{R}^N \times \mathbf{R}^+, \\ u(x, 0) = u_0(x), \quad x \in \mathbf{R}^N. \end{cases} \tag{1.1}$$

The "hysteresis" occurs in the possibility of the "consumption" term having either of the values 0 or 1. The same problem could be represented by the solution of a pair of equations without hysteresis:

$$\begin{cases} u_t = \Delta u - \begin{cases} 0, & v>0 \\ 1; & v=0, \end{cases} (x,t)\in \mathbf{R}^N\times \mathbf{R}^+, \\ v_t=(u-1)_+, \quad (x,t)\in \mathbf{R}^N\times \mathbf{R}^+, \end{cases} \tag{1.2a}$$

with

$$\begin{matrix} u(x,0)=u_0(x), & x\in \mathbf{R}^N, \\ v(x,0)=0, & x\in \mathbf{R}^N. \end{matrix} \tag{1.2b}$$

Among the class of solutions of the initial value problem (1.1) will be the spatially-independent ones:

$$\begin{cases} \dfrac{du}{dt} = -\begin{cases} 1, & u(t')>1 \quad \text{for some } t'\in(0,t] \\ 0, & \text{otherwise} \end{cases}, \quad t>0, \\ u(0)=u_0. \end{cases} \tag{1.3}$$

It is clear that solutions of (1.3) depend discontinuously on the initial data:

$$u(t)=\begin{cases} u_0-t, & u_0>1 \\ u_0 \ , & u_0\leqslant 1 \end{cases}, \quad t>0. \tag{1.4}$$

Thus, if we are to prove continuous dependence of solutions of (1.1) on the initial data, we must somehow exclude initial data with $u(x,0)=1$ for all x in some set of positive measure. On the other hand, we do not want to restrict ourselves to the situations

$$u_0(x)>1 \quad \forall x\in \mathbf{R}^N \tag{1.5a}$$

or

$$u_0(x)\leqslant 1 \quad \forall x\in \mathbf{R}^N, \tag{1.5b}$$

as the corresponding problems do not exhibit the interesting characteristics associated with hysteretic phenomena. Appropriate classes of initial data on which solutions of (1.1) will depend continuously will be presented in the sequel.

Let us note the difference between the diffusion-consumption problem with hysteresis, (1.1), and a standard diffusion-consumption problem [1, 8]:

$$\begin{cases} u_t=\Delta u-g(u), & (x,t)\in \mathbf{R}^N\times \mathbf{R}^+, \\ u(x,0)=u_0(x), & x\in \mathbf{R}^N, \end{cases} \tag{1.6a}$$

where

$$g(u)=\begin{cases} 1 & u>1 \\ 0 & u\leqslant 1. \end{cases} \tag{1.6b}$$

The problem (1.6) is a semilinear problem. Although the consumption term $g(u)$ does not depend continuously on u, it may be obtained as the limit, as $\varepsilon\to 0^+$, of a family of Lipschitz-continuous consumption terms $g_\varepsilon(u)$, to each of which there corresponds a solution $u_\varepsilon(x, t)$ which is continuous in its dependence on $u_0(x)$, uniformly in ε [8]. Moreover, the solutions $u_\varepsilon(x, t)$ depend monotonically on both ε and $u_0(x)$, and their limit $u(x, t)$ depends monotonically on $u_0(x)$. Accordingly, one can prove the continuous dependence of solutions of (1.6) on $u_0(x)$. As an example, in the case of spatial independence of $u_0(x)$, the solution of (1.6) is

$$u(t)=\begin{cases} u_0-t, & u_0>1, \quad t\leqslant u_0-1 \\ 1, & u_0>1, \quad t>u_0-1, \\ u_0, & u_0\leqslant 1 \end{cases} \tag{1.7}$$

which does not display the discontinuous dependence on u_0 that appeared in (1.4). In the problem with hysteresis, it is the possible persistence, for all time, of the discontinuity in the evolution of the system that can lead to discontinuous dependence on the initial data, as in (1.4), when they are not appropriately chosen, and which leads, in general, to a loss of monotone dependence on the initial data.

For the remainder of this paper, our attention will be focused on solutions of (1.1). This is an extremely simple equation, but we feel that the essential difficulties in treating the questions of uniqueness and continuous dependence are encountered here in a form free of unnecessary encumbrances. In the form (1.2), the initial value problem bears a resemblance to the treatment of the formation of Liesegang rings by Keller and Rubinow [6, 7]. v would take the place of the "precipitate" in the Keller-Rubinow theory, and u would be the concentration of one of the reacting and diffusing species (silver dichromate). The threshold value at which precipitation begins, denoted by c^* in the Keller-Rubinow treatment, is $u=1$ here.

Naturally, equation (1.1) will not possess the richness of the systems proposed by Jäger and Hoppensteadt, and by Keller and Rubinow, with regard to the study of pattern formation. In these systems there will be reversible (reaction) and irreversible (diffusion) processes, with the possibility of oscillatory behavior. However, it is our belief that a careful treatment of (1.1) can then be extended to fuller systems, such as would be obtained, for example, if (1.1) were replaced by a reaction-diffusion semilinear system in which all the nonlinear terms had a Lipschitz-continuous dependence on the dependent variables, except for one hysteresis term like that on the right-hand side of (1.1). Solutions of the fuller system would be obtained by taking the limit, as the time step in a time discretization goes to 0, of approximate solutions which evolve from one discrete time to the next by combining solutions, over the intervening time step, of the reaction-diffusion system without hysteresis, with solutions of (1.1) over the same time interval. This approach is currently being pursued by Eid, who is extending the results of this paper to the Keller-Rubinow model of Liesegang rings, as well as obtaining numerical solutions thereof [2].

Equation (1.1) is similar to, but simpler than, a parabolic equation with hysteresis studied by Visintin [9]. In equation (1.1), the consumption term is "switched on" when the "concentration" u exceeds 1. In Visintin's work, there is in addition a lower threshold at which the consumption is "switched off". In the system studied by Jäger and Hoppensteadt [4, 5], there are also thresholds for switching on and switching off of a growth term. It appears to us that the extension of the results of this paper to an equation with two thresholds is quite feasible, and it is our intention to study the fuller system of Jäger and Hoppensteadt with this goal in mind.

In the next section we will obtain a proof of the continuous dependence of solutions of (1.1) on the initial data for one-dimensional problems ($N=1$), provided that the initial data are suitably restricted. Work on extending the result to higher-dimensional problems is continuing.

2. A special one-dimensional case

We will study the problem

$$\begin{cases} u_t = u_{xx} - \begin{cases} 1, & u(x,t')>1 \text{ for some } t' \in (0,t] \text{ or } x \in I(0) \\ 0, & \text{otherwise} \end{cases}, & (x,t) \in \boldsymbol{R}\times\boldsymbol{R}^+, \quad (2.1a) \\ u(x,0) = u_0(x), \quad x \in \boldsymbol{R}, & (2.1b) \end{cases}$$

where

$$I(0) = (-\infty, 0). \tag{2.1c}$$

We will restrict our analysis to the case that $u_0(x)$ is a differentiable function satisfying the following conditions:

$$\begin{gathered} u_0(x) \leqslant 1 \quad \text{for} \quad x \geqslant 0, \\ u_0'(0) < 0, \\ \sup_{x\in R} |u_0'(x)| \leqslant M_1 < \infty, \\ |u_0'(x) - u_0'(y)| \leqslant K(\alpha)\,|x-y|^\alpha, \quad (x,y) \in R\times R, \quad \alpha \in (0,1), \\ u_0(x) \leqslant 1 - \eta(\delta) \quad \text{for } x \geqslant \delta, \quad \text{where } \eta(\delta) > 0 \quad \text{for } \delta > 0. \end{gathered} \tag{2.2}$$

We will compare solutions, if any, of (2.1) with solutions, if any, of

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} - \begin{cases} 1, & \tilde{u}(x,t')>1 \text{ for some } t' \in (0,t] \text{ or } x \in \tilde{I}(0) \\ 0, & \text{otherwise} \end{cases}, & (x,t) \in \boldsymbol{R}\times\boldsymbol{R}^+, \quad (2.1a') \\ \tilde{u}(x,0) = \tilde{u}_0(x), \quad x \in \boldsymbol{R}, & (2.1b') \end{cases}$$

where

$$\tilde{I}(0) \supset \{x \mid \tilde{u}_0(x) > 1\}. \tag{2.1c'}$$

We can find numbers ε_0, X_0 such that

$$\begin{gathered} \sup_{x\in R} |u_0(x) - \tilde{u}_0(x)| \leqslant \varepsilon_0, \\ |\tilde{I}(0) - I(0)| + |I(0) - \tilde{I}(0)| \leqslant X_0. \end{gathered} \tag{2.3}$$

Our procedure will be to construct functions $u^\pm(x,t)$ such that

$$u^-(x,t) \leqslant \min\big(u(x,t), \tilde{u}(x,t)\big) \leqslant \max\big(u(x,t), \tilde{u}(x,t)\big) \leqslant u^+(x,t) \tag{2.4}$$

whenever solutions $u, \tilde{u}$ of (2.1) and (2.1′) exist, and such that

$$u^-(x,t) \leqslant u^+(x,t), \tag{2.4′}$$

in any case. We shall show that suitable functions $u^{\pm}(x,t)$ may be found for which the difference $u^+(x,t) - u^-(x,t)$ goes to 0 as X_0 and ε_0 go to 0. It will be apparent from the construction of $u^{\pm}$ that they satisfy a pair of coupled differential equations. Upon taking the limit $X_0 \to 0$ and $\varepsilon_0 \to 0$ in which u^+ and u^- converge to the same function, one will find that they are solutions of (2.1). Thus, the existence of solutions to (2.1) will be established at the same time that their continuous dependence on the initial data is demonstrated.

Before we state our uniqueness theorem, we will dispense with some notation. Given a set E and a function $v(x,t)$, we define

$$I\{v(t); E\} = E \cup \{x | v(x,t') > 1 \quad \text{for some } t' \in (0,t)]\}. \tag{2.5}$$

The sets I have two obvious monotonicity properties:

$$I\{v(t_1); E\} \supset I\{v(t_2); E\} \quad \text{if } t_1 \geqslant t_2 \tag{2.6}$$

and

$$I\{v_1(t); E_1\} \supset I\{v_2(t); E_2\} \quad \text{if } E_1 \supset E_2 \quad \text{and } v_1 \geqslant v_2 \ \forall t' \in (0,t]. \tag{2.7}$$

$S(t)$ is used to denote the semigroup generated by Δ in $\boldsymbol{R}^N$:

$$S(t) = e^{t\Delta}, \tag{2.8a}$$

$$(S(t)v)(x) = \frac{1}{(4\pi t)^{N/2}} \int_{R^N} e^{-(x-y)^2/4t} v(y)\, dy. \tag{2.8b}$$

$S(t)$ has the familiar monotonicity property:

$$S(t)v_1 \geqslant S(t)v_2 \quad \text{if } v_1 \geqslant v_2. \tag{2.9}$$

The principal result of this section is stated by the following theorem.

THEOREM 2.1. *Let u and $\tilde{u}$ be solutions of* (2.1) *and* (2.1′), *and let $u_0(x)$ satisfy the conditions listed in* (2.2). *Given any time $T > 0$ and ε_0, X_0 in* (2.3) *sufficiently small, there exists a constant $C(T)$ such that, for*

$$0 \leqslant t \leqslant T, \sup_{x \in R} |u(x,t) - \tilde{u}(x,t)/1 + \sqrt{T}\, [|I\{u(t); I(0)\} - I\{\tilde{u}(t); \tilde{I}(0)\}| +$$
$$+ |I\{\tilde{u}(t); \tilde{I}(0)\} - I\{u(t); I(0)\}|] \leqslant C(T)(\varepsilon_0 + \sqrt{T}\, X_0). \tag{2.10}$$

The first step in the proof of the theorem is construction of the functions $u^{\pm}$ of (2.4). Following this, we will state some lemmas which will enable us to bound $u^+ - u^-$.

To construct the subfunction $u^-(x,t)$ and superfunction $u^+(x,t)$, we begin by setting

$$u_0^- = \min(u_0, \tilde{u}_0) \quad \text{and} \quad u_0^+ = u_0 + \varepsilon_0. \tag{2.11}$$

On account of (2.3), we have

$$u_0^+ \geqslant \max(u_0, \tilde{u}_0) \tag{2.12}$$

Likewise,

$$\sup_{x\in R} (u_0^+ - u_0^-) \leqslant 2\varepsilon_0 . \tag{2.13}$$

In addition, we define

$$I^-(0) = I(0) \cap \tilde{I}(0) . \tag{2.14}$$

We will also define a set $I^+(0)$. For the moment, we require only that

$$I^+(0) \supset I(0) \cup \tilde{I}(0) . \tag{2.15}$$

$I^+(0)$ will be determined with more definiteness in the sequel. See equation (2.32). It follows from (2.3) that

$$|I^+(0) - I^-(0)| \leqslant X_0 + |I^+(0) - I(0) \cup \tilde{I}(0)| . \tag{2.16}$$

Next, we construct the functions

$$u^{(0)}(x, t) = S(t)\, u_0^+ - \int_0^t S(t-t')\, \chi(I^-(0))\, dt' \tag{2.17a}$$

and, for $i \geqslant 0$,

$$u^{(2i+1)}(x, t) = S(t)\, u_0^- - \int_0^t S(t-t')\, \chi(I\{u^{(2i)}(t'); I^+(0)\})\, dt' , \tag{2.17b}$$

$$u^{(2i+2)}(x, t) = S(t)\, u_0^+ - \int_0^t S(t-t')\, \chi(I\{u^{(2i+1)}(t'); I^-(0)\})\, dt' . \tag{2.17c}$$

Here $\chi(E)$ is the characteristic function of E.

Note that the solutions of (2.1) and (2.1′) must satisfy, if they exist,

$$u(x, t) = S(t)\, u_0 - \int_0^t S(t-t')\, \chi(I\{u(t'); I(0)\})\, dt' \tag{2.18}$$

and

$$\tilde{u}(x, t) = S(t)\, \tilde{u}_0 - \int_0^t S(t-t')\, \chi(I\{\tilde{u}(t'); \tilde{I}(0)\})\, dt' . \tag{2.18′}$$

On account of (2.11), (2.12), (2.14), and (2.15), and the monotonicity properties (2.6), (2.7), and (2.9), we obtain the inequalities

$$\begin{aligned} &u^{(2i)}(x, t) \geqslant u^{(2i+2)}(x, t) \quad , \ i \geqslant 0, \\ &u^{(2i+1)}(x, t) \leqslant u^{(2i+3)}(x, t) , \ i \geqslant 0, \\ &u^{(2i)}(x, t) \geqslant u^{(2j+1)}(x, t) \quad , \ i \geqslant 0,\ j \geqslant 0. \end{aligned} \tag{2.19}$$

Similarly, if solutions of (2.18) and (2.18′) exist, it follows from the same inequalities that

$$u^{(2i+1)}(x,t)\leqslant \min\big(u(x,t),\tilde u(x,t)\big)\leqslant \max\big(u(x,t),\tilde u(x,t)\big)\leqslant$$
$$\leqslant u^{(2i)}(x,t),\quad i\geqslant 0. \tag{2.20}$$

An immediate consequence of (2.19) is that the functions

$$u^{+}(x,t)=\lim_{i\to\infty} u^{(2i)}(x,t) \quad \text{and}\quad u^{-}(x,t)=\lim_{i\to\infty} u^{(2i+1)}(x,t) \tag{2.21}$$

exist. Likewise, (2.4′) and (2.4), when applicable, follow from (2.19) and (2.20), when applicable. Taking the limit of (2.17) as $i\to\infty$, we obtain

$$\begin{aligned} u^{+}(x,t)&=S(t)\,u_0^{+}-\int_0^t S(t-t')\,\chi\big(I\{u^{-}(t');I^{-}(0)\}\big)\,dt',\\ u^{-}(x,t)&=S(t)\,u_0^{-}-\int_0^t S(t-t')\,\chi\big(I\{u^{+}(t');I^{+}(0)\}\big)\,dt'. \end{aligned} \tag{2.22}$$

Our purpose is now to bound $u^{+}-u^{-}$ in terms of X_0 and ε_0. To this end, we introduce the majorizing functions

$$X(t)=\sup_{0\leqslant t'\leqslant t} |I\{u^{+}(t');I^{+}(0)\}-I\{u^{-}(t');I^{-}(0)\}| \tag{2.23a}$$

and

$$\varepsilon(t)=\sup_{x\in R,\,0\leqslant t'\leqslant t}\big(u^{+}(x,t')-u^{-}(x,t')\big). \tag{2.23b}$$

First, let us use the conditions (2.2) to find a relatively simple function which serves as an upper bound on $u_0(x)$ for $x\geqslant 0$. Clearly,

$$u_0'(x)\leqslant u_0'(0)+K(\alpha)\,x^{\alpha}\leqslant 0 \quad \text{for } 0\leqslant x\leqslant \delta_0, \tag{2.24}$$

where

$$\delta_0=(-u_0'(0)/K(\alpha))^{1/\alpha}. \tag{2.25}$$

Integration of (2.24) yields

$$u_0(x)\leqslant 1+\alpha x\,u_0'(0) \quad \text{for } 0\leqslant x\leqslant \delta_0. \tag{2.26}$$

Since

$$u_0(x)\leqslant 1-\eta(\delta_0) \quad \text{for } x\geqslant \delta_0, \tag{2.27}$$

we can write

$$u_0(x)\leqslant \begin{cases} 1-\beta_0 x & 0\leqslant x\leqslant \delta_0\\ 1-\beta_0\delta_0 & x\geqslant \delta_0 \end{cases}, \tag{2.28}$$

where

$$\beta_0=\min\left(-\alpha u_0'(0),\ \frac{\eta(\delta_0)}{\delta_0}\right). \tag{2.29}$$

Note that, on account of (2.11),

$$u_0^+(x) \leqslant \begin{cases} 1-\beta_0(x-a) & 0 \leqslant x-a \leqslant \delta_1 \\ 1-\beta_0\,\delta_1 & x-a \geqslant \delta_1 \end{cases} \tag{2.30}$$

where

$$a = \frac{\varepsilon_0}{\beta_0} \quad \text{and} \quad \delta_1 = \delta_0 - a. \tag{2.31}$$

We may now give $I^+(0)$ with less ambiguity than appears in (2.15):

$$I^+(0) = (-\infty, a). \tag{2.32}$$

From (2.11) and (2.2) it follows that

$$\sup_{x \in R} |(u_0^+)'(x)| \leqslant M_1 \tag{2.33a}$$

and

$$|(u_0^+)'(x) - (u_0^+)'(y)| \leqslant K(\alpha)\,|x-y|^\alpha, \quad (x, y) \in \boldsymbol{R} \times \boldsymbol{R}. \tag{2.33b}$$

From (2.33) and (2.22), a similar sort of regularity can be deduced for $u^+(x, t)$. For this purpose we use the following lemma.

LEMMA 2.1. *Let u satisfy the following initial value problem:*

$$\begin{cases} u_t = \Delta u + f \quad , \ (x, t) \in \boldsymbol{R}^N \times \boldsymbol{R}^+, \\ u(x, 0) = u_0(x), \ x \in \boldsymbol{R}^N, \end{cases} \tag{2.34}$$

with the conditions

$$\begin{aligned} &\sup_{(x,t) \in R^N \times R^+} |f(x, t)| \leqslant M, \\ &\sup_{x \in R^N} |\nabla u_0(x)| \leqslant M_1, \end{aligned} \tag{2.35}$$

and

$$|\nabla u_0(x) - \nabla u_0(y)| \leqslant K(\alpha)\,|x-y|^\alpha, \ (x, y) \in \boldsymbol{R}^N \times \boldsymbol{R}^N, \quad \alpha \in (0, 1).$$

Then, given $T > 0$, the following regularity results hold for times $t \in [0, T]$:

$$|\nabla u(x, t)| \leqslant M_1(T), \ x \in \boldsymbol{R}^N, \ t \in [0, T], \tag{2.36a}$$

$$|\nabla u(x, t) - \nabla u(y, t)| \leqslant K(\alpha, T)\,|x-y|^\alpha, \ (x, y) \in \boldsymbol{R}^N \times \boldsymbol{R}^N, \ t \in [0, T], \tag{2.36b}$$

$$|u(x, s) - u(x, t)| \leqslant K_1(T)\,|s-t|^{1/2}, \ x \in \boldsymbol{R}^N, \ (s, t) \in [0, T] \times [0, T], \tag{2.36c}$$

where

$$M_1(T) = M_1 + 2M\sqrt{T}\,\frac{1}{\pi^{N/2}} \int_{R^N} |\xi|\, e^{-\xi^2}\, d\xi, \tag{2.37}$$

$$K(\alpha, T)=K(\alpha)+\frac{2MT^{\frac{1}{2}-\frac{\alpha}{2}}}{(1-\alpha)\,\pi^{N/2}}\left[3\int_{R^N}|\xi|^{1-\alpha}e^{-\xi^2}\,d\xi+\right.$$

$$\left.+\left(\frac{3}{2}\right)^{N+2}\int_{R^N}|\xi|^{3-\alpha}e^{-\xi^2}\,d\xi\right], \tag{2.38}$$

and

$$K_1(T)=2M_1\int_{R^N}|\xi|\,e^{-\xi^2}\,d\xi+M\sqrt{T}\left(3+\frac{N}{2}+\frac{1}{\pi^{N/2}}\int_{R^N}\xi^2\,e^{-\xi^2}\,d\xi\right). \tag{2.39}$$

P r o o f. The solution of (2.34) is written as

$$u(t)=S(t)\,u_0+\int_0^t S(t-t')\,f(t')\,dt' \tag{2.40}$$

Using (2.8) and carefully bounding the terms in $\nabla u(x, t)$, $\nabla u(x, t)-\nabla u(y, t)$, and $u(x, s)-u(x, t)$ corresponding to the two terms on the right-hand side of (2.40), we obtain the desired result. ■

Observe that it follows from the continuity of u in time, as given by (2.36c), that $I\{u(t); \emptyset\}\supset I(0)$ for $t>0$ if $u(t)$ is a solution of (2.1) and $u_0(x)>1$ for $x<0$. With that restriction, solutions of (2.1) will also be solutions of (1.1) for the case $N=1$.

We proceed with the task of bounding $\varepsilon(t)$ and $X(t)$. From (2.13),

$$\varepsilon(0)\leqslant 2\varepsilon_0\,, \tag{2.41a}$$

and from (2.32) and (2.16),

$$X(0)\leqslant X_0+a\,. \tag{2.41b}$$

The next lemma enables us to bound $\varepsilon(t)$ in terms of $X(t)$.

LEMMA 2.2. *With* $\varepsilon(t)$ *and* $X(t)$ *given by* (2.23),

$$\varepsilon(t)\leqslant 2\varepsilon_0+\sqrt{\frac{t}{\pi}}\;X(t)\,. \tag{2.42}$$

P r o o f. From (2.22),

$$u^+(x, t)-u^-(x, t)=S(t)\,(u_0^+-u_0^-)+$$

$$+\int_0^t S(t-t')\,\chi\left(I\{u^+(t'); I^+(0)\}-I\{u^-(t'); I^-(0)\}\right)dt'\,. \tag{2.43}$$

$S(t)\,(u_0^+-u_0^-)$ is bounded by $2\varepsilon_0$, from the maximum principle. With regard to a bound on the other term on the right-hand side of (2.43), we use the definition of $X(t)$ in (2.23a) and we use (2.8b) with $N=1$ to bound

$$S(t-t')\,\chi\left(I\{u^+(t'); I^+(0)\}-I\{u^-(t'); I^-(0)\}\right)$$

by $X(t)/\sqrt{4\pi(t-t')}$. This bound is then integrated over t', and (2.42) follows immediately. ■

It remains to bound $X(t)$ in terms of $\varepsilon(t)$ and the characteristic parameters ε_0 and X_0 of the initial data. Let

$$x^+(t)=\sup\{x|u^+(x,t)>1\}, \tag{2.44a}$$

$$\xi^+(t)=\max(a, \sup_{0\leq t'\leq t} x^{+(t')}), \tag{2.44b}$$

and

$$x^-(t)=\sup\{x|u^-(x,t)>1\}, \tag{2.45a}$$

$$\xi^-(t)=(\sup_{0\leq t'\leq t} x^-(t'))_+. \tag{2.45b}$$

Clearly,

$$\xi^+(0)=a \quad \text{and} \quad \xi^-(0)=0. \tag{2.46}$$

From (2.23a),

$$X(t)\leq X_0+\sup_{0\leq t'\leq t}(\xi^+(t')-\xi^-(t')). \tag{2.47}$$

If

$$\sup_{0\leq t'\leq t}(\xi^+(t')-\xi^-(t'))=a, \tag{2.48}$$

then

$$X(t)\leq X_0+a. \tag{2.49}$$

Otherwise,

$$\sup_{0\leq t'\leq t}(\xi^+(t')-\xi^-(t'))>a, \tag{2.50}$$

$t>0$, and we can define the non-empty set $\sigma(t)$ by

$$\sigma(t)=\{t'\in(0,t]\,|\lim_{\eta\to 0}[\sup_{t'-\eta\leq t''\leq\min(t'+\eta,t)}(\xi^+(t'')-\xi^-(t''))]= = \sup_{0\leq t'\leq t}(\xi^+(t')-\xi^-(t'))\} \tag{2.51a}$$

and the time $t^*(t)\geq 0$ by

$$t^*(t)=\inf\{t'|t'\in\sigma(t)\}. \tag{2.51b}$$

We must have

$$u^+(\xi^+(t),t')\leq 1 \quad \text{for } 0\leq t'\leq t. \tag{2.52}$$

For, if $u^+(\xi^+(t),t')>1$, it would follow from the differentiability with respect to x of $u^+(x,t')$ proved in lemma 2.1 that the condition (2.44) defining $\xi^+(t)$ would be violated.

On the other hand, we must have

$$u^+(\xi^+(t^*(t)), t^*(t)) \geq 1 \tag{2.53}$$

when (2.50) holds. First of all, if (2.50) holds, it follows from (2.46) that, in order to have $t^*(t)=0$, it would be necessary that $\xi^+(t)$ be discontinuous at $t=0$. This can be shown to be false by examining solutions of (2.22) with the initial data u_0^+ satisfying (2.30). If $\xi^+(t)$ were discontinuous at $t=0$, it would follow that $u^+(x, t)$ was not continuous in t, in violation of the result of lemma 2.1. Consequently, $t^*(t)>0$. If $u^+\big(\xi^+(t^*(t)), t^*(t)\big)<1$, we must have, on account of the continuity of $u^+(x, t)$ in t, $u^+\big(\xi^+(t^*(t)), t'\big)\leq 1$ for $t^*(t)-\eta\leq t'\leq t^*(t)+\eta$ and some $\eta>0$. But then, from (2.52), $u^+\big(\xi^+(t^*(t)), t'\big)\leq 1$ for $0\leq t'\leq t^*(t)+\eta$. Since u^+ satisfies (2.22) with u_0^+ bounded by (2.30), one concludes from the maximum principle for $u_t^+\leq u_{xx}^+$ that $\xi^+\big(t^*(t)+\eta\big)=\xi^+\big(t^*(t)\big)$ for some $\eta>0$. Furthermore, when $u^+\big(\xi^+(t^*(t)), t^*(t)\big)<1$ and $\xi^+\big(t^*(t)\big)>a$, it follows from (2.44) and the continuity of $u^+(x, t)$ in x and t that $u^+\big(\xi^+(t^*(t)), t_1\big)=1$ for some $t_1<t^*(t)$. Then $\xi^+(t_1)=\xi^+\big(t^*(t)\big)$. On the other hand, because of the monotone dependence of $\xi^-(t)$ on t, $\xi^-(t_1)\leq\xi^-\big(t^*(t)\big)$. Hence $\xi^+(t_1)-\xi^-(t_1)\geq\xi^+\big(t^*(t)\big)-\xi^-\big(t^*(t)\big)=$ $=\xi^+\big(t^*(t)+\eta\big)-\xi^-\big(t^*(t)\big)$, which contradicts the definition on $t^*(t)$ given by (2.51). Thus, (2.53) must hold. Having established (2.52) and (2.53), we are in a position to prove the following lemma.

LEMMA 2.3. *Let* $u^+(x, t)$ *be given by* (2.22) *with the bound* (2.30) *on* u_0^+. *Define* $\xi^+(t)$ *and* $t^*(t)$ *by* (2.44) *and* (2.51). *Then, for* $0\leq t\leq T$,

$$u_x^+\big(\xi^+(t^*(t)), t^*(t)\big)\leq -\frac{\beta_0}{\sqrt{\pi T}}\,\delta_1\, e^{-\delta_1^2/4t} \tag{2.54}$$

Proof. W note that

$$\begin{gathered} u_s^+=u_{xx}^+, \quad x\in\big(\xi^+(t^*(t)), \infty\big), \quad s\in\big(0, t^*(t)\big), \\ u_0^+(x, 0)\leq\begin{cases}1-\beta_0\big(x-\xi^+(t^*(t))\big) & 0\leq x-\xi^+\big(t^*(t)\big)\leq\delta_1, \\ 1-\beta_0\,\delta_1 & x-\xi^+\big(t^*(t)\big)\geq\delta_1,\end{cases} \\ u^+\big(\xi^+(t^*(t)), s\big)\leq 1, \quad 0\leq s\leq t^*(t), \\ u^+\big(\xi^+(t^*(t)), t^*(t)\big)\geq 1. \end{gathered} \tag{2.55}$$

A decrease in $u_0^+(x, 0)$ or $u^+\big(\xi^+(t^*(t)), s\big)$ for $0\leq s<t^*(t)$, and an increase in $u^+\big(\xi^+(t^*(t)), t^*(t)\big)$ all have the effect of reducing $u_x^+\big(\xi^+(t^*(t)), t^*(t)\big)$. Thus

$$u_x^+\big(\xi^+(t^*(t)), t^*(t)\big)\leq V_y\big(0, t^*(t)\big),$$

where

$$\begin{gathered} V_s=V_{yy}, \quad y\in(0, \infty), \quad s\in\big(0, t^*(t)\big), \\ V(y, 0)=\begin{cases}1-\beta_0\, y, & 0\leq y\leq\delta_1, \\ 1-\beta_0\,\delta_1, & y\geq\delta_1,\end{cases} \\ V(0, s)=1, \quad 0\leq s\leq t^*(t). \end{gathered} \tag{2.56}$$

Solving (2.56) and noting that $t^*(t) \leq t \leq T$, one obtains the bound

$$V_y(0, t^*(t)) \leq -\frac{\beta_0}{\sqrt{\pi T}} \delta_1 e^{-\delta_1^2/4T},$$

and accordingly (2.54) is proven.

To obtain a bound for $X(t)$ in terms of $\varepsilon(t)$, we use lemmas 2.1 and 2.3: For $x < \xi^+(t^*(t))$,

$$u_x^+(x, t^*(t)) \leq -\frac{\beta_0}{\sqrt{\pi T}} \delta_1 e^{-\delta_1^2/4T} + \left(\xi^+(t^*(t)) - x\right)^\alpha K(\alpha, T).$$

Integrating this and using (2.53), we get, for $x < \xi^+(t^*(t))$,

$$u^+(x, t^*(t)) \geq 1 + \frac{\beta_0}{\sqrt{\pi T}} \delta_1 e^{-\delta_1^2/4T} \left(\xi^+(t^*(t)) - x\right) +$$

$$-\frac{\left(\xi^+(t^*(t)) - x\right)^{\alpha+1}}{\alpha+1} K(\alpha, T) \geq 1 + \frac{\beta_0}{2\sqrt{\pi T}} \delta_1 e^{-\delta_1^2/4T} \left(\xi^+(t^*(t)) - x\right) \tag{2.57}$$

when

$$\xi^+(t^*(t)) - x \leq \delta_2, \tag{2.58a}$$

where

$$\left(\delta_2 = \frac{\beta_0}{2\sqrt{\pi T}} \frac{(\alpha+1)\delta_1 e^{-\delta_1^2/4T}}{K(\alpha, T)}\right)^{1/\alpha}. \tag{2.58b}$$

Hence

$$u^+(x, t^*(t)) > 1 + \varepsilon(t) \tag{2.59}$$

if

$$\xi^+(t^*(t)) - x - \frac{\varepsilon(t) 2\sqrt{\pi T}}{\beta_0 \delta_1 e^{-\delta_1^2/4T}} \in (0, \eta) \quad \text{for some } \eta > 0 \tag{2.60}$$

and

$$\varepsilon(t) \leq \frac{\beta_0 \delta_1 e^{-\delta_1^2/4T}}{2\sqrt{\pi T}} \delta_2. \tag{2.61}$$

It follows from (2.59) and (2.23b) that $u^-(x, t^*(t)) > 1$ when (2.60) and (2.61) are satisfied. But then, from (2.45),

$$\xi^-(t^*(t)) \geq \xi^+(t^*(t)) - \frac{\varepsilon(t) 2\sqrt{\pi T}}{\beta_0 \delta_1 e^{-\delta_1^2/4T}} \tag{2.62}$$

From the inequalities (2.55), we see that $u^+(x, t^*(t)) < 1$ for $x > \xi^+(t^*(t))$. Accordingly, $\xi^+(t^*(t) + n) - \xi^+(t^*(t)) \to 0$ as $\eta \to 0^+$, since otherwise there would be a violation of the continuity of $u^+(x, t)$ in t. It follows from (2.51) that

$$\sup_{0 \leq t' \leq t} \left(\xi^+(t') - \xi^-(t')\right) \leq \xi^+(t^*(t)) - \xi^-(t^*(t)) \tag{2.63}$$

So finally, when (2.50) holds and (2.61) is satisfied, (2.47), (2.63), and (2.62) give

$$X(t) \leq X_0 + \max\left(\frac{2\sqrt{\pi T}}{\beta_0 \delta_1 e^{-\delta_1^2/4T}} \varepsilon(t), a\right). \tag{2.64}$$

Insert (2.64) into (2.42):

$$\begin{aligned} \varepsilon(t) &\leq 2\varepsilon_0 + \sqrt{\frac{t}{\pi}} X_0 + \frac{2\sqrt{T}}{\beta_0 \delta_1 e^{-\delta_1^2/4T}} \sqrt{t}\, \varepsilon(t) \leq \\ &\leq 2\left(2\varepsilon_0 + \sqrt{\frac{\tau}{\pi}} X_0\right) \equiv \varepsilon_1 \end{aligned} \tag{2.65}$$

if $t \leq \tau$, where

$$\tau = \left(\frac{\beta_0 \delta_1 e^{-\delta_1^2/4T}}{4\sqrt{T}}\right)^2. \tag{2.66}$$

(2.65) and (2.66) may be inserted into (2.64) to yield, for $0 \leq t \leq \tau$,

$$X(t) \leq 2X_0 + 2\sqrt{\frac{\pi}{\tau}} \varepsilon_0 \equiv X_1. \tag{2.67}$$

Similarly, for $\tau \leq t \leq 2\tau$, we obtain

$$\begin{aligned} \varepsilon(t) &\leq 2\left(\varepsilon_1 + \sqrt{\frac{\tau}{\pi}} X_1\right), \\ X(t) &\leq 2X_1 + \sqrt{\frac{\pi}{\tau}} \varepsilon_1. \end{aligned} \tag{2.68}$$

For $i\tau \leq t \leq (i+1)\tau$, $i \geq 1$,

$$\begin{aligned} \varepsilon(t) &\leq 2\left(\varepsilon_i + \sqrt{\frac{\tau}{\pi}} X_i\right), \\ X(t) &\leq 2X_i + \sqrt{\frac{\pi}{\tau}} \varepsilon_i, \end{aligned} \tag{2.69}$$

where, for $i \geq 2$,

$$\begin{aligned} \varepsilon_i &= 2\left(\varepsilon_{i-1} + \sqrt{\frac{\tau}{\pi}} X_{i-1}\right), \\ X_i &= 2X_{i-1} + \sqrt{\frac{\pi}{\tau}} \varepsilon_{i-1} \leq 2\left(X_{i-1} + \sqrt{\frac{\pi}{\tau}} \varepsilon_{i-1}\right). \end{aligned} \tag{2.70}$$

Thus, for $i \geq 2$,

$$\varepsilon_i + \sqrt{\frac{\tau}{\pi}} X_i \leq 4\left(\varepsilon_{i-1} + \sqrt{\frac{\tau}{\pi}} X_{i-1}\right), \tag{2.71}$$

and for $0 \leqslant t \leqslant T$,

$$\varepsilon(t) + \sqrt{\frac{\tau}{\pi}}\, X(t) \leqslant 4^{[T/\tau]+1}\left(2\varepsilon_0 + \sqrt{\frac{\tau}{\pi}}\, X_0\right). \tag{2.72}$$

These relations hold as long as the constraint (2.61) is valid:

$$\varepsilon(t) \leqslant \frac{2}{\sqrt{\pi}} \sqrt{\tau}\, \delta_2. \tag{2.73}$$

From (2.72), we see that this is the case if the initial data satisfy

$$2\varepsilon_0 + \sqrt{\frac{\tau}{\pi}}\, X_0 \leqslant \frac{\delta_2}{2} \sqrt{\frac{\tau}{\pi}}\, 4^{-[T/\tau]} \tag{2.74}$$

It is now a simple matter to go from the bound (2.72), when the constraint (2.74) applies, to the result (2.10) expressed in theorem 2.1. ∎

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Jednoznaczność rozwiązania zagadnienia dyfuzji--konsumpcji z histerezą

W pracy rozważana jest klasa nieliniowych zagadnień dyfuzji konsumpcji z efektem histerezy. Rozważane zagadnienia odnoszą się do systemów biochemicznych, z parabolicznymi równaniami ewolucji, w których składniki reprezentujące źródła przestrzenne są zależne od pewnego funkcjonału przełączającego. Funkcjonał ten zależy od historii systemu, co jest wyrażone poprzez nieliniowość typu histerezy. Obecność nieliniowości tego rodzaju w istotny sposób komplikuje analizę problemów jednoznaczności rozwiązań oraz ich ciągłej zależności od warunków początkowych. W pracy zostaje sformułowane i udowodnione twierdzenie o jednoznacznej zależności rozwiązań od danych początkowych w przypadku zagadnienia przestrzennie jednowymiarowego.

Единственность решения проблемы диффузии--консумпции гистерезисом

В работе рассуждается класс нелинейных проблем диффузии—консумпции с эффектом гистерезиса. Рассуждаемые проблемы принадлежат к математическим моделям биохимических систем, с параболическими уравнениями эволюции, в которых слагаемые представляющие собой пространственные источники зависит от некоторого переключающего функционала. Завесимость этого функционала от истории системы выражена нелинейностью гистерезисного типа. Нелинейность такого рода существенно усложняет анализ проблем единственности решений и характера их непрерывной зависимости от начальных условий. В работе формулируется и доказывается теорема о единственности решений и их непрерывной зависимости от начальных условий в случае одномерной геометрии проблемы.

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