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The method of generalized Heaviside expansion*)

by

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The paper presents the general method of evaluation of the integral of weighted squared dynamic error for control systems with time delay. The analytical results obtained in the paper enable to find optimal values of the parameters of the controller, and to investigate the influence of the weighted functions. For ilustration a simple example is also considered.

1. Introduction

In this paper the evaluation is considered of the integral of weighted squared dynamic error for systems with time-delay.

(Systems without time-delay, or without weighted functions are particular cases).

$$I_{2,r} = \int_0^\infty t^r e^2(t)\, dt \qquad r = 0, 1, 2, \ldots \tag{1}$$

$e(t)$ — dynamic error

t — time

2. Derivation of the formula for dynamic error $e(t)$

Basic assumptions:

The method of solution is shown for the class of control systems whose error transforms may be represented by the following relation:

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$$E(s) = \frac{U(s)}{s^v \cdot V(s)} \tag{2}$$

where

$$\begin{aligned} &V(s) = A(s) + C(s) \exp(-hs) \\ &V(o) \neq 0 \end{aligned} \tag{3}$$

is a quasipolynomial and $A(s)$, $C(s)$ are polynomials with real coefficients and

$$\deg A(s) = n - v \tag{4}$$

$$\deg C(s) \leqslant n - v \tag{5}$$

$h > 0$ — is the time-delay.

$$v = a(1+b) \tag{5a}$$

where a denotes the coefficient of existence of the zero pole of $E(s)$

$$a = \begin{cases} 1 \\ 0 \end{cases} \tag{6}$$

and $v = 0, 1, 2, \ldots$ — multiplicity of the zero pole.

$$U(s) = B(s) + D(s) \exp(-hs) \tag{6a}$$

is a quasipolynomial, and $B(s)$, $D(s)$ are polynomials with real coefficients and

$$\deg B(s) \leqslant n - 1 \tag{7}$$

$$\deg D(s) \leqslant n - 1 \tag{8}$$

For the existence of the integral (1) it is necessary that

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = 0 \tag{9}$$

Taking into account (5) and (9) we obtain as the conclusion that in particular for

$$v = 1 \text{ there must be } U(o) = B(o) + D(o) = 0 \tag{10}$$

$$v = 2 \text{ there must be } U(o) = 0 \quad \text{and}$$

$$U'(o) = B'(o) + D'(o) - hD(o) = 0 \tag{11}$$

and in general for

$$v = N \text{ there must be } U(o) = 0,\ U'(o) = 0, \ldots, U^{(N-1)}(o) = 0 \tag{12}$$

It is proved (Bellman, Cooke 1963) that the transcendental characteristic equation

$$V(s) = A(s) + C(s) \exp(-hs) = 0 \tag{13}$$

has infinite number of roots $s_1, s_2, \ldots$.

For the sake of simplicity we assume that equation (13) has different roots only, in consequence the derivative of $V(s)$ with respect to s has no common root with $V(s)$:

$$V'(s_i) = A'(s_i) + [C'(s_i) - hC(s_i)] \exp(-hs_i) \neq 0 \quad \text{for} \quad i = 1, 2, \ldots \tag{14}$$

For the nontrivial systems with time-delay

$$C(s_i) \neq 0 \qquad i = 1, 2, \ldots \tag{15}$$

and we can use in the following for elimination purposes the relation obtained from (13)

$$\exp(-hs_i) = -\frac{A(s_i)}{C(s_i)} \qquad i = 1, 2, \ldots \tag{16}$$

It is proved (Wright 1949, Hahn 1956, 1956, 1957) that the inverse transform for (2) exists and is given by the generalized Heaviside formula

$$e(t) = \sum_{i=1}^{\infty} \frac{U(s_i) \exp(s_i t)}{s_i^v V'(s_i)} \tag{17}$$

(remark: components from zero poles according to (10)–(12) are equal to zero). According to (6a) and (14), using the relation (16) we can write:

$$\frac{U(s_i)}{V'(s_i)} = \frac{B(s_i) + D(s_i) \exp(-hs_i)}{A'(s_i) + [C'(s_i) - hC(s_i)] \exp(-hs_i)} =$$

$$= \frac{B(s_i) C(s_i) - A(s_i) D(s_i)}{hA(s_i) C(s_i) + A'(s_i) C(s_i) - A(s_i) C'(s_i)} = \frac{L(s_i)}{M(s_i)} \tag{18}$$

where $L(s_i)$, $M(s_i)$ are polynomials with respect to s_i and

$$\deg s_i^v M(s_i) \leqslant 2n - v \tag{19}$$

$$\deg L(s_i) \leqslant 2n - v - 1 \qquad \deg L(s_i) < \deg s_i^v M(s_i) \tag{20}$$

Finally, using notation (18) we can write relation (17) as follows:

$$e(t) = \sum_{i=1}^{\infty} \frac{L(s_i) \exp(s_i t)}{s_i^v M(s_i)} \tag{21}$$

3. Evaluation of the integral J_{2r}

We have mentioned that for the existence of the integral (1) it is necessary that the error $e(t)$ vanish when time tends to infinity (9).

The requirement (9) can be fulfilled if all the zeros of the quasipolynomial $V(s)$ determined by (13) have negative real parts, and their multiplicity is finite (14)

$$\operatorname{Re} s_i < 0 \qquad i = 1, 2, \dots \tag{22}$$

It is also assumed that in the particular case when the time-delay

$$\left.\begin{aligned} h &= 0 \\ V(s_i) &= A(s_i) + C(s_i) = 0 \end{aligned}\right\} \tag{23}$$

the polynomial obtained from the general formula (13) has only zeros with negative real parts:

$$\operatorname{Re} s_i < 0 \qquad i = 1, 2, \dots, n-v \tag{24}$$

Starting from the relation (1) and using for $e(t)$ the relation (21), having in mind the conditions (22) we can write:

$$J_{2r} = \int_0^\infty t^r \left[\sum_{i=1}^{\infty} \frac{L(s_i)\exp(s_i t)}{s_i^v M(s_i)}\right]\left[\sum_{j=1}^{\infty} \frac{L(s_j)\exp(s_j t)}{s_j^v M(s_j)}\right] dt \tag{25}$$

or

$$J_{2r} = \sum_{i=1}^{\infty} \frac{L(s_i)}{s_i^v M(s_i)} \left\{\sum_{j=1}^{\infty} \frac{L(s_j)}{s_j^v M(s_j)} \left[\int_0^\infty t^r \exp(s_i + s_j)\, t\, dt\right]\right\} \tag{26}$$

Multiple integration by parts of the integral in (26) yields the relation:

$$\int_0^\infty t^r \exp(s_i + s_j)\, t\, dt = (-1)^{r+1} \frac{r!}{(s_i + s_j)^{r+1}} \qquad i, j = 1, 2, \dots \infty$$

$$r = 0, 1, 2, \dots \tag{27}$$

and $\operatorname{Re}(s_i + s_j) < 0$ because we have the mode assumption (22). Returning to (26) gives

$$J_{2r} = (-1)^{r+1} \cdot r! \sum_{i=1}^{\infty} \frac{L(s_i)}{s_i^v M(s_i)} \sum_{j=1}^{\infty} \frac{L(s_j)}{s_j^v M(s_j)(s_i + s_j)^{r+1}} =$$

$$= (-1)^{r+1} \cdot r! \sum_{i=1}^{\infty} \frac{L(s_i)}{s_i^v M(s_i)} S_i(s_i) \tag{28}$$

where the inner sum

$$S_i(s_i) = \sum_{j=1}^{\infty} \frac{L(s_j)}{s_j^v M(s_j)(s_j + s_i)^{r+1}} \tag{29}$$

In the calculations which follow we must compute the sum of the form of (29) using partial fraction method. To achieve this goal we consider the following general problem: Calculate the sum determined by the relation:

$$S=\sum_{j=1}^{\infty}\frac{L(s_j)\,P(s_j)}{s_j^v\,M(s_j)\,Q^{1+r}(s_j)} \tag{30}$$

where the function $P(s_j)$ is introduced for calculation of more general sums than (29).
We assume that

$$\deg P(s_j)\leqslant m-1 \tag{31}$$

and

$$\deg Q^{1+r}(s_j)=m \tag{32}$$

We denote the roots of the polynomials, respectively

$$M(p_k)=0 \qquad k=1,2,\ldots,2n-v \tag{33}$$

$$Q(q_k)=0 \qquad k=1,2,\ldots,\frac{m}{1+r} \tag{34}$$

Let us consider the function under the sign $\sum$ in (30)

$$\Phi_1(s)=\frac{L(s)\cdot P(s)}{s^{a(1+b)}\,M(s)\,Q^{1+r}(s)} \tag{35}$$

(remark: $v=a(1+b)$, see (5a)).
Using the partial fraction method we can write that

$$\Phi_1(s)=a\sum_{k=0}^{b}\frac{1}{k!}\left[\frac{L(s)\cdot P(s)}{M(s)\,Q^{1+r}(s)}\right]_0^{(k)}\cdot\frac{1}{s^{1+b-k}}+$$

$$+\sum_{k=1}^{2n-a(1+b)}\frac{L(p_k)\,P(p_k)}{p_k^{a(1+b)}\cdot M'(p_k)\,Q^{1+r}(p_k)}\,\frac{1}{s-p_k}+$$

$$+\sum_{k=1}^{\frac{m}{1+r}}\sum_{l=0}^{r}\frac{1}{l!}\cdot\left[\frac{L(s)\,P(s)\,(s-q_k)^{1+r}}{s^{a(1+b)}\,M(s)\,Q^{1+r}(s)}\right]_{s=q_k}^{(l)}\frac{1}{(s-q_k)^{1+r-l}} \tag{36}$$

The original for the function $\Phi_1(s)$ is

$$\varphi_1(t)=a\sum_{k=0}^{b}\frac{1}{k!}\left[\frac{L(s)\,P(s)}{M(s)\,Q^{1+r}(s)}\right]_{s=0}^{(k)}\frac{1}{(b-k)!}\,t^{b-k}+$$

$$+\sum_{k=1}^{2n-a(1+b)}\frac{L(p_k)\,P(p_k)}{p_k^{a(1+b)}\,M'(p_k)\,Q^{1+r}(p_k)}\cdot\exp(p_k\,t)+$$

$$+\sum_{k=1}^{m/(1+r)}\sum_{l=0}^{r}\frac{1}{l!}\left[\frac{L(s)\,P(s)\,(s-q_k)^{1+r}}{s^{a(1+b)}\,M(s)\,Q^{1+r}(s)}\right]_{s=q_k}^{(l)}\cdot t^{r-l}\cdot\exp(q_k\,t) \tag{37}$$

In the investigated problem the assumption that

$$\deg\left[s^{a(1+b)}M(s)\cdot Q^{1+r}(s)\right] > \deg\left[sL(s)\,P(s)\,(s-q_k)^{1+r}\right] \tag{38}$$

is always true.

Taking into account that

$$\varphi_1(o) = \lim_{s\to\infty} s\Phi_1(s) = 0$$

if (38) holds, we obtain finally from (37) the useful relation

$$a\frac{1}{b!}\left[\frac{L(s)\,P(s)}{M(s)\,Q^{1+r}(s)}\right]^{(b)}_{s=0} + \sum_{k=1}^{2n-a(1+b)} \frac{L(p_k)\,P(p_k)}{p_k^{a(1+b)}M'(p_k)\,Q^{1+r}(p_k)} + $$

$$+ \sum_{k=1}^{m/(1+r)} \frac{1}{r!}\left[\frac{L(s)\,P(s)\,(s-q_k)^{1+r}}{s^{a(1+b)}M(s)\,Q^{1+r}(s)}\right]^{(r)}_{s=q_k} = 0 \tag{40}$$

Using relation (36) we can calculate the sum (30):

$$S_1 = \sum_{j=1}^{\infty} \Phi_1(s_j) = a\sum_{k=0}^{b}\frac{1}{k!}\left[\frac{L(s)\,P(s)}{M(s)\,Q^{1+r}(s)}\right]^{(k)}_{s=0}\cdot\sum_{j=1}^{\infty}\frac{1}{s_j^{1+b-k}} +$$

$$+ \sum_{k=1}^{2n-a(1+b)} \frac{L(p_k)\,P(p_k)}{p_k^{a(1+b)}M'(p_k)\,Q^{1+r}(p_k)} \sum_{j=1}^{\infty}\frac{1}{s_j-p_k} +$$

$$+ \sum_{k=1}^{m/(1+r)}\sum_{l=0}^{r}\frac{1}{l!}\left[\frac{L(s)\,P(s)\,(s-q_k)^{1+r}}{s^{a(1+b)}M(s)\,Q^{1+r}(s)}\right]^{(l)}_{s=q_k}\cdot\sum_{j=1}^{\infty}\frac{1}{(s_j-q_k)^{1+r-l}} \tag{41}$$

Further progress may be attained if we calculate the infinite sums in (41). It is proved (Levin 1956, Chapter 5) that for the entire function of class A determined by the relation (3), the following Weierstrass decomposition into a product holds:

$$V(s) = e^{\varkappa s}\prod_{j=1}^{\infty}\left(1-\frac{s}{s_j}\right) \tag{42}$$

where $\varkappa$ is a real number.

Taking logarithm from both sides of the relation (42) we obtain

$$\ln V(s) = \varkappa s + \sum_{j=1}^{\infty}\left(1-\frac{s}{s_j}\right) \tag{43}$$

The derivative of this function with respect to s is

$$\frac{d}{ds}\left[\ln V(s)\right] = \frac{V'(s)}{V(s)} = \varkappa - \sum_{j=1}^{\infty}\frac{1}{s_j-s} \tag{44}$$

similarly

$$\left[\frac{V'(s)}{V(s)}\right]' = -\sum_{j=1}^{\infty}\frac{1}{(s_j-s)^2} \tag{45}$$

and in general

$$\left[\frac{V'(s)}{V(s)}\right]^{(k)} = -k!\sum_{j=1}^{\infty}\frac{1}{(s_j-s)^{k+1}}$$

In particular, we will use the relations which follow from (44)–(45):

$$\sum_{j=1}^{\infty}\frac{1}{s_j^{1+r-k}} = \varkappa - \frac{1}{(r-k)!}\left[\frac{V'(s)}{V(s)}\right]^{(r-k)}_{s=0}, \qquad \varkappa = 0 \text{ for } k<r \tag{47}$$

$$\sum_{j=1}^{\infty}\frac{1}{(s_j+s_i)^{r+1-k}} = \varkappa - \frac{1}{(r-k)!}\left[\frac{V'(s)}{V(s)}\right]^{(r-k)}_{s=-s_i} \qquad \varkappa = 0 \text{ for } k<r \tag{48}$$

$$\sum_{j=1}^{\infty}\frac{1}{(s_j-p_k)^{r+1-k}} = \varkappa - \frac{1}{(r-k)!}\left[\frac{V'(s)}{V(s)}\right]^{(r-k)}_{s=p_k} \qquad \varkappa = 0 \text{ for } k<r \tag{49}$$

Application of these relations to (41) gives:

$$\begin{aligned}
S_1 = \varkappa\Bigg\{ & a\frac{1}{b!}\left[\frac{L(s)\,P(s)}{M(s)\,Q^{1+r}(s)}\right]^{(b)}_{s=0} + \sum_{k=1}^{2n-a(1+b)}\frac{L(p_k)\,P(p_k)}{p_k^{a(1+b)}M'(p_k)\,Q^{1+r}(p_k)} + \\
& + \sum_{k=1}^{m/(1+r)}\frac{1}{r!}\left[\frac{L(s)\,P(s)\,(s-q_k)^{1+r}}{s^{a(1+b)}M(s)\,Q^{1+r}(s)}\right]^{(r)}_{s=q_k}\Bigg\} + \\
& + a\sum_{k=0}^{b}\frac{1}{k!}\left[\frac{L(s)\,P(s)}{M(s)\,Q^{1+r}(s)}\right]^{(k)}_{s=0}\frac{1}{(b-k)!}\left[\frac{V'(s)}{V(s)}\right]^{(b-k)}_{s=0} + \\
& + \sum_{k=1}^{2n-a(1+b)}\frac{L(p_k)\,P(p_k)}{p_k^{a(1+b)}M'(p_k)\,Q^{1+r}(p_k)}\cdot\frac{V'(p_k)}{V(p_k)} + \\
& + \sum_{k=1}^{m/(1+r)}\sum_{l=0}^{r}\left[\frac{1}{l!}\frac{L(s)\,P(s)\,(s-q_k)^{r+1}}{s^{a(1+b)}M(s)\,Q^{r+1}(s)}\right]^{(l)}_{s=q_k}\cdot\frac{1}{(r-l)!}\left[\frac{V'(s)}{V(s)}\right]^{(r-l)}_{s=q_k}
\end{aligned} \tag{50}$$

The term in brackets is equal to that in (40) and is equal to zero. Calculation of the expression $\dfrac{V'(p_k)}{V(p_k)}$ using (3), (18) and (33) gives

$$\begin{aligned}
\frac{V'(p_k)}{V(p_k)} &= \frac{A'(p_k)+[C'(p_k)-hC(p_k)]\exp(-p_k h)}{A(p_k)+C(p_k)\exp(-p_k h)} = \\
&= \frac{A'_{pk}+[-A'C+A'C+AC'-hCA]_{p_k}\exp(-p_k h)}{A[A+C\exp(-p_k h)]_{p_k}} = \\
&= \frac{AA'_{p_k}+[-M+A'C]_{p_k}\exp(-p_k h)}{A_{p_k}[A+C\exp(-p_k h)]_{p_k}}
\end{aligned} \tag{51}$$

But from (33) we have $M(p_k)=0$ so finally we find that

$$\frac{V'(p_k)}{V(p_k)}=\frac{A'(p_k)}{A(p_k)} \tag{52}$$

The relation (50) can be written in a simpler form. An application of the Leibniz formula for the b-th derivative of product of two functions gives:

$$\left[\frac{L(s)\,P(s)}{M(s)\,Q^{r+1}(s)}\cdot\frac{V'(s)}{V(s)}\right]^{(b)}=\sum_{k=0}^{b}\binom{b}{k}\left[\frac{L(s)\,P(s)}{M(s)\,Q^{r+1}(s)}\right]^{(k)}\left[\frac{V'(s)}{V(s)}\right]^{(b-k)} \tag{53}$$

or equivalently

$$\sum_{k=0}^{b}\frac{1}{k!}\left[\frac{L(s)\,P(s)}{M(s)\,Q^{r+1}(s)}\right]^{(k)}\cdot\frac{1}{(b-k)!}\left[\frac{V'(s)}{V(s)}\right]^{(b-k)}=$$
$$=\frac{1}{b!}\left[\frac{L(s)\,P(s)}{M(s)\,Q^{r+1}(s)}\cdot\frac{V'(s)}{V(s)}\right]^{(b)} \tag{54}$$

Taking into account (40), (52) and (54) in the relation (50) gives

$$-S_1=a\frac{1}{b!}\left[\frac{L(s)\,P(s)}{M(s)\,Q^{r+1}(s)}\cdot\frac{V'(s)}{V(s)}\right]^{(b)}_{s=0}+$$
$$+\sum_{k=1}^{2n-a(1+b)}\frac{L(p_k)\,P(p_k)\,A'(p_k)}{p_k^{a(1+b)}M'(p_k)\,Q^{1+r}(p_k)\,A(p_k)}+$$
$$+\sum_{k=1}^{m/(1+r)}\frac{1}{r!}\left[\frac{L(s)\,P(s)\,(s-q_k)^{r+1}}{s^{a(1+b)}M(s)\,Q^{r+1}(s)}\right]^{(r)}_{s=q_k} \tag{55}$$

It is evident from (55) that we also need to calculate the sum represented by the second term in (55). To do this we choose, similarly as in the preceding case, the function

$$\Phi_2(s)=\frac{L(s)\,A'(s)\,P(s)}{s^{a(1+b)}M(s)\,A(s)\,Q^{r+1}(s)} \tag{56}$$

Following the same way and using the partial fraction method we find that:

$$\Phi_2(s)=a\sum_{k=0}^{b}\frac{1}{k!}\left[\frac{L(s)\,A'(s)\,P(s)}{M(s)\,A(s)\,Q^{1+r}(s)}\right]^{(k)}_{s=0}\cdot\frac{1}{s^{1+b-k}}+$$
$$+\sum_{k=1}^{2n-a(1+b)}\frac{L(p_k)\,A'(p_k)\,P(p_k)}{p_k^{a(1+b)}M'(p_k)\,A(p_k)\,Q^{1+r}(p_k)}\,\frac{1}{s-p_k}+$$
$$+\sum_{k=1}^{m/(1+r)}\sum_{l=0}^{r}\frac{1}{l!}\left[\frac{L(s)\,A'(s)\,P(s)\,(s-q_k)^{1+r}}{s^{a(1+b)}M(s)\,A(s)\,Q^{1+r}(s)}\right]^{(l)}_{s=q_k}\frac{1}{(s-q_k)^{1+r-l}}+$$
$$+\sum_{k=1}^{n-a(1+b)}\frac{L(r_k)\,A'(r_k)\,P(r_k)}{r_k^{a(1+b)}M(r_k)\,A'(r_k)\,Q^{1+r}(r_k)}\,\frac{1}{s-r_k} \tag{57}$$

where r_k, $k = 1, 2, \dots, n-a(1+b)$ are the roots of the equation

$$\begin{aligned} A(r_k) &= 0 \\ A'(r_k) &\neq 0 \end{aligned} \tag{58}$$

After similar algebraic manipulations we obtain that

$$\begin{aligned} \varphi_2(o) = 0 = a\frac{1}{b!}\left[\frac{L(s)A'(s)P(s)}{M(s)A(s)Q^{1+r}(s)}\right]^{(b)}_{s=0} + \\ + \sum_{k=1}^{2n-a(1+b)} \frac{L(p_k)A'(p_k)P(p_k)}{p_k^{a(1+b)}M'(p_k)A(p_k)Q^{1+r}(p_k)} + \\ + \sum_{k=1}^{m/(1+r)} \frac{1}{r!}\left[\frac{L(s)A'(s)P(s)(s-q_k)^{1+r}}{s^{a(1+b)}M(s)A(s)Q^{1+r}(s)}\right]^{(r)}_{s=q_k} + \\ + \sum_{k=1}^{n-a(1+b)} \frac{B(r_k)P(r_k)}{r_k^{a(1+b)}A'(r_k)Q^{1+r}(r_k)} \end{aligned} \tag{59}$$

We have substituted

$$\begin{aligned} L(r_k) &= B(r_k)C(r_k) \\ M(r_k) &= A'(r_k)C(r_k) \end{aligned} \tag{60}$$

(see (18) and (58)).

$$\begin{aligned} S_2 = \sum_{j=1}^{\infty} \Phi_2(s_j) = a\frac{1}{b!}\left\{\frac{L(s)P(s)}{M(s)Q^{1+r}(s)}\left[\frac{A'(s)}{A(s)} - \frac{V'(s)}{V(s)}\right]\right\}^{(b)}_{s=0} + \\ + \sum_{k=1}^{m/(1+r)} \frac{1}{r!}\left\{\frac{L(s)P(s)(s-q_k)^{1+r}}{s^{a(1+b)}M(s)Q^{1+r}(s)}\left[\frac{A'(s)}{A(s)} - \frac{V'(s)}{V(s)}\right]\right\}^{(r)}_{s=q_k} + \\ + \sum_{k=1}^{n-a(1+b)} \frac{B(r_k)P(r_k)}{r_k^{a(1+b)}A'(r_k)Q^{1+r}(r_k)} \end{aligned} \tag{61}$$

Taking into account that

$$\begin{aligned} \frac{A'(s)}{A(s)} - \frac{V'(s)}{V(s)} = \frac{A'(s)V(s) - V'(s)A(s)}{A(s)V(s)} = \\ = \frac{A'(A + C\exp(-hs)) - A[A' + (C' - hC)\exp(-hs)]}{AV} = \\ = \left.\frac{(hAC + A'C - AC')\exp(-hs)}{AV}\right|_s = \frac{M(s)\exp(-hs)}{A(s)V(s)} \end{aligned} \tag{62}$$

we obtain finally

$$S_2 = a\frac{1}{b!}\left[\frac{L(s)\,P(s)\exp(-hs)}{A(s)\,V(s)\,Q^{1+r}(s)}\right]^{(b)}_{s=0} +$$
$$+\sum_{k=1}^{m/(1+r)}\frac{1}{r!}\left[\frac{L(s)\,P(s)\,(s-q_k)^{1+r}}{s^{a(1+b)}A(s)\,V(s)\,Q^{1+r}(s)}\right]^{(r)}_{s=q_k} +$$
$$+\sum_{k=1}^{n-a(1+b)}\frac{B(r_k)\,P(r_k)}{r_k^{a(1+b)}A'(r_k)\,Q^{1+r}(r_k)} \tag{63}$$

If

$$B(s)\equiv 0 \tag{64}$$

then we have

$$L(s) = -A(s)\,D(s) \tag{65}$$

and

$$U(s) = D(s)\exp(-hs) \tag{66}$$

The sum S_2 takes the form:

$$S_2 = -a\left[\frac{1}{b!}\,\frac{U(s)}{V(s)}\,\frac{P(s)}{Q^{1+r}(s)}\right]^{(b)}_{s=0} - \sum_{k=1}^{m/(1+r)}\frac{1}{r!}\left[\frac{U(s)P(s)(s-q_k)^{1+r}}{s^{a(1+b)}V(s)\,Q^{1+r}(s)}\right]^{(r)}_{s=q_k} \tag{67}$$

If

$$B(s)\not\equiv 0 \tag{68a}$$

we introduce in the similar way the function

$$\Phi_3(s) = \frac{B(s)\,P(s)}{s^{a(1+b)}A(s)\,Q^{1+r}(s)} =$$
$$= a\sum_{k=0}^{b}\frac{1}{k!}\left[\frac{B(s)\,P(s)}{A(s)\,Q^{1+r}(s)}\right]^{(k)}_{s=0}\cdot\frac{1}{s^{1+b-k}} +$$
$$+\sum_{k=1}^{n-a(1+b)}\frac{B(r_k)\,P(r_k)}{r_k^{a(1+b)}A'(r_k)\,Q^{1+r}(r_k)}\cdot\frac{1}{s-r_k} +$$
$$+\sum_{k=1}^{m/(1+r)}\sum_{l=0}^{r}\frac{1}{l!}\left[\frac{B(s)\,P(s)\,(s-q_k)^{1+r}}{s^{a(1+b)}A(s)\,Q^{1+r}(s)}\right]^{(l)}_{s=q_k}\frac{1}{(s-q_k)^{1+r}} \tag{68}$$

From the relation $\varphi_3(o)=0$ we obtain in the same way the relation:

$$\sum_{k=1}^{n-a(1+b)}\frac{B(r_k)\,P(r_k)}{r_k^{a(1+b)}A'(r_k)\,Q^{1+r}(r_k)} = -a\frac{1}{b!}\left[\frac{B(s)\,P(s)}{A(s)\,Q^{1+r}(s)}\right]^{(b)}_{s=0} +$$
$$-\sum_{k=1}^{m/(1+r)}\frac{1}{r!}\left[\frac{P(s)\,(s-q_k)^{r+1}}{s^{a(1+b)}A(s)\,Q^{r+1}(s)}\right]^{(r)}_{s=q_k} \tag{69}$$

The sum S_2 from (63), using substitution (69) takes the form:

$$S_2 = a\,\frac{1}{b!}\left\{\frac{P(s)}{A(s)\,Q^{1+r}(s)}\left[\frac{L(s)\exp(-hs)}{V(s)} - B(s)\right]\right\}^{(b)}_{s=0} +$$

$$+ \sum_{k=1}^{n-a(1+b)} \frac{1}{r!}\left\{\frac{P(s)\,(s-q_k)^{r+1}}{s^{a(1+b)}A(s)\,Q^{r+1}(s)}\left[\frac{L(s)\exp(-hs)}{V(s)} - B(s)\right]\right\}^{(r)}_{s=q_k} \tag{70}$$

But

$$\frac{L(s)\exp(-hs)}{V(s)} - B(s) =$$

$$= \frac{[B(s)\,C(s) - A(s)\,D(s)]\exp(-hs) - B(s)\,[A(s) + C(s)\exp(-hs)]}{V(s)} =$$

$$= -\frac{A(s)\,U(s)}{V(s)} \tag{71}$$

Finally, the general formula (30) takes the form:

$$\sum_{j=1}^{\infty} \frac{L(s_j)\,P(s_j)}{s_j^{a(1+b)} M(s_j)\,Q^{1+r}(s_j)} = -\left\{a\,\frac{1}{b!}\left[\frac{U(s)}{V(s)}\,\frac{P(s)}{Q^{1+r}(s)}\right]^{(b)}_{s=0} +\right.$$

$$\left. + \sum_{k=1}^{n-a(1+b)} \frac{1}{r!}\left[\frac{U(s)}{s^{a(1+b)}V(s)}\cdot\frac{P(s)\,(s-q_k)^{r+1}}{Q^{r+1}(s)}\right]^{(r)}_{s=q_k}\right\} \tag{72}$$

In the particular case which is very often in use, when there exists a single root equal to zero the formula (72) takes the form: for $b=0$

$$\sum_{j=1}^{\infty} \frac{L(s_j)\,P(s_j)}{s_j^{a} M(s_j)\,Q^{1+r}(s_j)} =$$

$$= -\left\{a\,\frac{U(o)\,P(o)}{V(o)\,Q^{1+r}(o)} + \sum_{k=1}^{n-a} \frac{1}{r!}\left[\frac{U(s)}{s^{a}V(s)}\cdot\frac{P(s)\,(s-q_k)^{r+1}}{Q^{r+1}(s)}\right]^{(r)}_{s=q_k}\right\} \tag{73}$$

We can use the derived general formulas (72) and (73) for the evaluation of the integral J_{2r}. To achieve this goal we assume the existence of a single root equal to zero $b=0$ (in the absence of the root equal to zero we put $a=0$). For the calculation of the inner sum S_i (see relation (29)) we put in the general formula (73)

$$\begin{aligned} P(s_j) &\equiv 1 \\ Q(s_j) &= s_j - q_1 \\ q_1 &= -s_i \end{aligned} \tag{74}$$

and

$$-S_i(s_i) = a\,\frac{U(o)\,P(o)}{V(o)\,Q^{1+r}(o)} + \frac{1}{r!}\left[\frac{U(s)}{s^a V(s)}\cdot\frac{P(s)\,(s+s_i)^{1+r}}{Q^{1+r}(s)}\right]^{(r)}_{s=-s_i} =$$

$$= a\,\frac{U(o)}{V(o)\,s_i^{1+r}} + \frac{1}{r!}\left[\frac{U(s)}{s^a V(s)}\right]^{(r)}_{s=-s_i} \tag{75}$$

The integral determined by (28) is:

$$J_{2,r} = (-1)^r \cdot r! \sum_{i=1}^{\infty} \frac{L(s_i)}{s_i^a M(s_i)} \left\{ a\,\frac{U(o)}{V(o)\,s_i^{1+r}} + \frac{1}{r!}\left[\frac{U(s)}{s^a V(s)}\right]^{(r)}_{s=-s_i} \right\} \tag{76}$$

$$J_{2,r} = (-1)^r\, r!\, a\,\frac{U(o)}{V(o)} \sum_{i=1}^{\infty} \frac{L(s_i)}{s_i^{a+1+r} M(s_i)} +$$

$$+ (-1)^r \sum_{i=1}^{\infty} \frac{L(s_i)}{s_i^a M(s_i)} \left[\frac{U(s)}{s^a V(s)}\right]^{(r)}_{s=-s_i} \tag{77}$$

Shortly

$$J_{2,r} = (-1)^r\, r!\, a\,\frac{U(o)}{V(o)}\, S_o + (-1)^r S_r \tag{78}$$

where

$$S_o = \sum_{i=1}^{\infty} \frac{L(s_i)}{s_i^{a+1+r} M(s_i)} \tag{79}$$

$$S_r = \sum_{i=1}^{\infty} \frac{L(s_i)}{s_i^a M(s_i)} \left[\frac{U(s)}{s^a V(s)}\right]^{(r)}_{s=-s_i} \tag{80}$$

Now we compare formula (79) with formula (72). Putting in the general formula (72) $b = 1+r$, $P(s) \equiv 1$, $Q(s) \equiv 1$ we calculate the sum S_o from the relation:

$$S_o = -a\,\frac{1}{(1+r)!}\left[\frac{U(s)}{V(s)}\right]^{(r+1)}_{s=0}$$

and respectively

$$J_{2,r} = (-1)^{1+r}\,\frac{1}{1+r}\cdot a\,\frac{U(o)}{V(o)}\left[\frac{U(s)}{V(s)}\right]^{(1+r)}_{s=0} + (-1)^r S_r \tag{82}$$

The sum S_r determined by (80) may be written in the form:

$$S_r = \sum_{i=1}^{\infty} \frac{L(s_i)}{s_i^a M(s_i)} \left[\frac{U(s)}{s^a V(s)}\right]^{(r)}_{s=-s_i} = \sum_{i=1}^{\infty} \frac{L(s_i)}{s_i^a M(s_i)} \cdot \frac{P(s_i)}{s_i^{a(1+r)} F^{1+r}(s_i)} =$$

$$= \sum_{i=1}^{\infty} \frac{L(s_i)\,P(s_i)}{s_i^{a(2+r)} M(s_i)\,F^{1+r}(s_i)}, \tag{83}$$

where

$$\left[\frac{U(s)}{s^a V(s)}\right]^{(r)}_{s=-s_i} = \frac{P(s_i)}{s_i^{a(1+r)} F^{1+r}(s_i)} \tag{84}$$

Comparison of the relation (84) with (72) gives:

$$F(s_i) = Q(s_i) \tag{85}$$

where we define from (3) by the elimination of the exponential term from (13)

$$F(s) = A(s)\,V(-s) = A(s)\,A(-s) - C(s)\,C(-s) \tag{86}$$

and we calculate the roots q_k from the equation

$$F(q_k) = Q(q_k) = 0 \tag{87}$$

Now it is necessary to attain the explicit form of $P(s)$. Starting from the relation (84) we obtain:

$$\left[\frac{U(s)}{s^a V(s)}\right]^{(r)} = \frac{1}{[s^a V(s)]^{1+r}} \times$$

$$\times \begin{vmatrix} (s^a V), & (s^a V)', & (s^a V)'', & \ldots\ldots, & (s^a V)^{(r)} \\ 0\ , & (s^a V), & 2(s^a V)', & \ldots\ldots & \\ 0\ , & 0\ , & (s^a V)\ , & \ldots\ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0\ , & 0\ , & \ldots\ldots, & (s^a V), & \ldots\ldots \\ U\ , & U'\ , & U''\ , & \ldots\ldots, & U^{(r)} \end{vmatrix} \tag{88}$$

Taking into account the relation (86) we find

$$\begin{vmatrix} (s^a V)_{-s_i} A(s_i), & (s^a V)'_{-s_i} A(s_i), & \ldots, & (s^a V)^{(r)}_{-s_i} A(s_i) \\ \vdots & & & \\ \vdots & & & \\ (U)_{-s_i} A(s_i)\ , & (U')_{-s_i} A(s_i)\ , & \ldots, & (U^{(r)})_{-s_i} A(s_i) \end{vmatrix} \times$$

$$\times \frac{1}{(-s_i)^{a(1+r)} F^{1+r}(s_i)} =$$

$$= \frac{(-1)^{a(1+r)} \begin{vmatrix} (s^a V)_{-s_i} A(s_i)\ldots \\ \vdots \\ (U)_{-s_i} A(s_i)\ldots\ldots \end{vmatrix}}{s_i^{a(1+r)} \cdot F^{1+r}(s_i)} = \frac{P(s_i)}{s_i^{a(1+r)} Q^{1+r}(s_i)} \tag{89}$$

From (89) we obtain finally

$$P(s_i) = (-1)^{a(1+r)} \times$$

$$\times \begin{vmatrix} (s^a V)_{-s_i} A(s_i), & (s^a V)'_{-s_i} A(s_i), & \ldots, & (s^a V)^{(r)}_{-s_i} A(s_i) \\ 0 \quad , & (s^a V)_{-s_i} A(s_i), & \ldots, & \binom{r}{1} (s^a V)^{(r-1)}_{-s_i} A(s_i) \\ 0 \quad , & 0 \quad , & \ldots, & \binom{r}{2} (s^a V)^{(r-2)}_{s_i} A(s_i) \\ \cdots & \cdots & \cdots & \cdots \\ & & & \binom{r}{r-1} (s^a V)'_{-s_i} A(s_i) \\ (U)_{-si} A(s_i), & (U')_{-si} A(s_i), & \ldots, & (U^{(r)})_{-s_i} A(s_i) \end{vmatrix} \tag{90}$$

Direct application of the general formula (72) gives:

$$b = 1+r \tag{91}$$

$$S_r = -\left\{ a \frac{1}{(1+r)!} \left[\frac{U(s) P(s)}{V(s) F^{1+r}(s)} \right]^{(1+r)}_{s=0} + \right.$$
$$\left. + \sum_{k=1}^{n-a(1+b)} \frac{1}{r!} \left[\frac{U(s)}{s^{a(2+r)} V(s)} \cdot \frac{P(s)(s-f_k)^{r+1}}{F^{1+r}(s)} \right]^{(r)}_{s=f_k=A} \right\} \tag{92}$$

Finally, substitution of the relation (92) to (82) gives:

$$J_{2,r} = (-1)^{r+1} \left\{ a \frac{1}{1+r} \frac{U(o)}{V(o)} \left[\frac{U(s)}{V(s)} \right]^{(r+1)}_{s=0} + \frac{1}{r!} \left[\frac{U(s) P(s)}{V(s) F^{1+r}(s)} \right]^{(1+r)}_{s=0} + \right.$$
$$\left. + \sum_{k=1}^{n-a(2+r)} \frac{1}{r!} \frac{U(s)}{s^{a(2+r)} V(s)} \cdot \left[\frac{P(s)(s-f_k)^{1+r}}{F^{1+r}(s)} \right]^{(r)}_{s=f_k} \right\} \tag{93}$$

REMARK 1. If $a = 0$ this means absence of a single zero root, the first term in (93) disappears.

REMARK 2. It is worth to note that the extension of this formula to the case which instead of the error $e(t)$ has its derivative $e^{(k)}(t)$, or aggregate $\sum_{k=0}^{n-1} e^{(k)}(t)$ is straight-forward. To do this it is sufficient to replace the error transform $E(s)$ by the transform of its appropriate derivative:

$$\begin{aligned} & e(t) \fallingdotseq E(s) \\ & e'(t) \fallingdotseq sE(s) - e(o) \\ & \vdots \\ & e^{(k)}(t) \fallingdotseq s^k E(s) - \sum_{i=1}^{k} e^{(k-1)}(o) \cdot s^{k-i} \end{aligned} \tag{94}$$

4. Some simple practical examples

Let us consider the process whose transmittance is described by the relation:

$$G_o(s) = \frac{k_o \exp(-hs)}{Ts} \tag{E.1}$$

The transmittance of the controller is

$$G_R(s) = K_R \tag{E.2}$$

and the transform of input is equal to $\dfrac{W_o}{s}$ (E.3)

Error's transform is equal to

$$E(s) = \frac{1}{1+G_o G_R} \frac{W_o}{s} =$$

$$= \frac{1}{1+\dfrac{K_o K_R}{sT}\exp(-hs)} \frac{W_o}{s} = \frac{W_o}{s+\alpha \exp(-hs)} \tag{E.4}$$

where

$$\alpha = \frac{K_o K_R}{T} \tag{E.5}$$

First of all we consider the case for $\underline{r=0}$:

From (1) we have

$$J_{2,0} = \int_o^\infty e^2(t)\,dt \tag{E.5}$$

From (2) we identify that

$$U(s) = W_o \tag{E.6}$$

and in particular from (6a) we have

$$B(s) = W_o \tag{E.7}$$

$$D(s) = 0 \tag{E.8}$$

$$v = 0 \tag{E.9}$$

From (3) comparing with (E.4) we see that

$$V(s) = s + \alpha \exp(-hs) \tag{E.10}$$

and respectively

$$A(s) = s \tag{E.11}$$

$$C(s) = \alpha \tag{E.12}$$

From (18) we have also

$$L(s) = W_o\,\alpha \tag{E.13}$$

$$M(s) = h\alpha s + \alpha = \alpha\,(1+hs) \tag{E.14}$$

From (28) we have

$$J_{2,0} = (-1)^1 \cdot 0! \sum_{i=1}^{\infty} \frac{L(s_i)}{M(s_i)} S_i(s_i) \tag{E.15}$$

where from (29) the inner sum is

$$S_i(s_i) = \sum_{j=1}^{\infty} \frac{L(s_j)}{M(s_j)(s_j+s_i)^1} \tag{E.16}$$

From (72) we have $a = 0$ and

$$\sum_{j=1}^{\infty} \frac{L(s_j)\,P(s_j)}{M(s_j)\,Q^{1+r}(s_j)} = -\sum_{k=1}^{n} \frac{1}{0!}\left[\frac{U(s)}{V(s)}\,\frac{P(s)(s-q_k)}{Q(s)}\right]_{s=q_k}^{(0)} \tag{E.17}$$

Putting $s_j = s$ and comparing (E.16) with (E.17) gives:

$$\begin{aligned} P(s) &= 1 \\ Q(s) &= s - q_1 \\ q_1 &= -s_i \\ n &= 1 \end{aligned} \tag{E.18}$$

The inner sum is equal to

$$S_i(s_i) = -\left[\frac{U(s)}{V(s)}\right]_{s=q_1} \tag{E.19}$$

Substitution of (E.6) and (E.10), (E.18) into (E.19) gives

$$S_i(s_i) = -\frac{W_o}{-s_i + \alpha\exp(hs_i)} \tag{E.20}$$

From the characteristic equation (13) we can calculate the exponential term

$$s_i + \alpha\exp(-hs_i) = 0 \tag{E.21}$$

We have

$$s_i\exp(hs_i) = -\alpha \tag{E.22}$$

and finally

$$S_i(s_i) = -\frac{W_o\,s_i}{-s_i^2-\alpha^2} = W_o\,\frac{s_i}{s_i^2+\alpha^2} \tag{E.22}$$

Calculation of the integral according to (E.15) yields

$$J_{2,0} = \sum_{i=1}^{\infty} \frac{L(s_i)}{M(s_i)} \cdot W_o \frac{s_i}{s_i^2+\alpha^2} \tag{E.23}$$

Application of formula (93) requires identification of $P(s)$ and $Q(s)$. In this case we have

$$J_{2,0} = -\sum_{i=1}^{\infty} \frac{L(s_i)\,P(s_i)}{M(s_i)\,Q(s_i)} = +\sum_{k=1}^{n} \left[\frac{U(s)}{V(s)} \cdot \frac{P(s)(s-q_k)}{Q(s)} \right]_{s=q_k} \tag{E.24}$$

Comparing with (E.23) gives

$$Q(s_i) = s_i^2+\alpha^2$$

$$Q(s) = s^2+\alpha^2 = (s-q_1)(s-q_2) \tag{E.25}$$

$$n = 2$$

$$q_1 = j\alpha, \qquad q_2 = -j\alpha \tag{E.26}$$

$$P(s) = W_o\, s \tag{E.27}$$

Finally we obtain

$$J_{2,0} = \sum_{k=1}^{2} \frac{W_o^2\, s}{[s+\alpha \exp(-hs)]\,(s-q_{k+1})} \bigg|_{s=q_k} =$$

$$= W_o^2 \frac{q_1}{q_1+\alpha \exp(-hq_1)} \cdot \frac{1}{q_1-q_2} + W_o^2 \frac{q_2}{q_2+\alpha \exp(-hq_2)} \cdot \frac{1}{q_2-q_1},$$

but $q_2 = -q_1$

$$J_{2,0} = \frac{W_o^2}{2} \left[\frac{1}{q_1+\alpha \exp(-hq_1)} + \frac{1}{-q_1+\alpha \exp(hq_1)} \right] =$$

$$= \frac{W_o^2\, \alpha}{2} \frac{\exp(q_1 h)+\exp(-q_1 h)}{(\alpha^2-q_1^2)+q_1 \alpha\,[\exp(hq_1)-\exp(-hq_1)]}$$

and

$$J_{2,0} = \frac{W_o^2}{2\alpha} \frac{\cos \alpha h}{1-\sin \alpha h} \tag{E.28}$$

Now we calculate, for this standard example, the weighted integral for $r = 1$:

$$J_{2,1} = (-1)^2\, 1! \sum_{i=1}^{\infty} \frac{L(s_i)}{M(s_i)} S_i(s_i) \tag{E.29}$$

Here we have

$$S_i(s_i) = -\frac{1}{1!} \left[\frac{\Psi(s)}{V(s)} \right]'_{-s_i} \tag{E.30}$$

Using (88) we obtain

$$-S_i(s_i)=\left.\frac{\begin{vmatrix}V(s), & V'(s)\\ U(s), & U'(s)\end{vmatrix}}{V^2(s)}\right|_{s=-s_i}=\frac{\begin{vmatrix}-s_i+\alpha\exp(hs_i), & 1-\alpha h\exp(hs_i)\\ W_o\quad , & 0\end{vmatrix}}{[-s_i+\alpha\exp(hs_i)]^2} \tag{E.31}$$

Elimination of the exponential term, similarly to the preceding case yields:

$$-S_i(s_i)=\frac{s_i\begin{vmatrix}-s_i^2-\alpha^2, & s_i+\alpha^2 h\\ W_o\quad , & 0\end{vmatrix}}{(-s_i^2-\alpha^2)^2}=-\frac{s_i W_o(s_i+\alpha^2 h)}{(s_i^2+\alpha^2)^2}$$

Finally we obtain

$$S_i(s_i)=\frac{W_o s_i(s_i+\alpha^2 h)}{(s_i^2+\alpha^2)^2}=\frac{P(s_i)}{Q^2(s_i)} \tag{E.32}$$

and

$$P(s_i)=W_o s_i(s_i+\alpha^2 h) \tag{E.33}$$

$$Q(s_i)=s_i^2+\alpha^2 \tag{E.34}$$

Similarly to the preceding case

$$J_{2,1}=-\sum_{k=1}^{2}\left[\frac{U(s)P(s)(s-q_k)^2}{V(s)Q^2(s)}\right]'_{s=q_k} \tag{E.35}$$

We remember that

$$U(s)=W_o$$

so

$$\left[\frac{P(s)}{V(s)Q_k(s)}\right]_{s=q_k}=\left.\frac{\big(P'(s)V(s)-P(s)V'(s)\big)Q_k(s)-2P(s)V(s)Q_k'(s)}{V^2(s)Q_k^3(s)}\right|_{s=q_k} \tag{E.36}$$

where

$$Q_k(s)=\frac{Q(s)}{s-q_k} \tag{E.37}$$

$$\left.\begin{aligned}Q_1(s)&=s-q_2=s+q_1\\ Q_2(s)&=s-q_1=s+q_2\\ Q_k'(s)&=1\\ q_1&=j\alpha\\ q_2&=-j\alpha\\ P(s)&=s^2+\alpha^2 hs\end{aligned}\right\} \tag{E.38a}$$

$$\left.\begin{aligned} P'(s) &= 2s+\alpha^2 h \\ V(s) &= s+\alpha \exp(-hs) \\ V'(s) &= 1-\alpha h \exp(-hs) \end{aligned}\right\} \tag{E.38b}$$

Finally from (E.35), after substitution of (E.36) and (E.38) into (E.35) we obtain

$$J_{2,1} = \frac{W_o^2}{4\alpha^2} \frac{1+\alpha^2 h^2-\alpha h \cos \alpha h}{1-\sin \alpha h} \tag{E.39}$$

Case of $\underline{r=2}$:

$$\left[\frac{U(s)}{V(s)}\right]''_{-s_i} = \frac{1}{V^3} \begin{vmatrix} V & V' & V'' \\ 0 & V & 2V' \\ 1 & 0 & 0 \end{vmatrix}_{-s_i} = \frac{2(V')^2-VV''}{V^3}\bigg|_{-s_i} =$$

$$= -s_i \frac{(2-\alpha^2 h^2) s_i^2+4\alpha^2 hs_i+\alpha^4 h^2}{(s_i^2+\alpha^2)^3} \tag{40}$$

$$R(s) = -[(2-\alpha^2 h^2) s^3+4\alpha^2 hs^2+\alpha^4 h^2] \tag{41}$$

$$J_{2,2} = -\frac{1}{2}\sum_{k=1}^{2}\left[\frac{-[(2-\alpha^2 h^2) s^3+4\alpha^2 hs^2+\alpha^4 h^2]}{[s+\alpha \exp(-sh)](s+f_k)^3}\right]^{(2)}_{f_k} W_o^2 =$$

$$= -\frac{1}{2}\sum_{i=1}^{2}\left[\frac{-R(s)}{F(s)}\right]^{2}_{f_k} W_o^2 \tag{42}$$

$$\left[\frac{-R(s)}{F(s)}\right]''_{f_k} = \frac{1}{F^3(s)} \begin{vmatrix} F(s), & F'(s), & F''(s) \\ 0, & F(s), & 2F'(s) \\ -R(s), & -R'(s), & -R''(s) \end{vmatrix}_{f_k} \tag{43}$$

$$F(s) = [s+\alpha \exp(-hs)](s+f_1)^3 \tag{44}$$

$$F'(s) = [1-\alpha h \exp(-hs)](s+f_1)^3+3[s+\alpha \exp(-hs)](s+f_1)^2 \tag{45}$$

$$F''(s) = \alpha h^2 \exp(-hs)(s+f_1)^3+G[1-\alpha h \exp(-hs)](s+f_1)^2+$$
$$+G[s+\alpha \exp(-hs)](s+f_1) \tag{46}$$

$$-R(s) = (2-\alpha^2 h^2) s^3+4\alpha^2 hs^2+\alpha^4 h^2 s \tag{47}$$

$$-R'(s) = 3(2-\alpha^2 h^2) s^2+8\alpha^2 hs+\alpha^4 h^2 \tag{48}$$

$$-R''(s) = G(2-\alpha^2 h^2) s+8\alpha^2 h \tag{49}$$

In particular for $s=f_1$ we have

$$F(f_1) = 8\alpha^4 [1-j \exp(-j\alpha h)] \tag{50}$$

$$F'(f_1) = 4\alpha^3 \{-5j+(2j\alpha^4-3)[\exp(-j\alpha h)]\} \tag{51}$$

$$F''(f_1) = 4\alpha^2 \{-9+(G\alpha h+3j-2j\alpha^2 h^2)\exp(-j\alpha h)]\} \tag{52}$$

Table of final results and algorithm

		Remarks
1. Error's transform	$E(s) = \frac{U(s)}{s^{a(1+b)} V(s)}$, $\quad U(s) = B(s) + D(s)\exp(-hs)$, $\quad V(s) = A(s) + C(s)\exp(-hs)$	
2. Error as function of time	$e(t) = \sum_{i=1}^{\infty} \frac{L(s_i)}{s_i^{a(1+b)} M(s_i)} \exp(s_i t)$ $\quad L(s) = B(s)C(s) - A(s)D(s)$, $\quad M(s) = hA(s)C(s) + A'(s)C(s) - A(s)C(s)$	
3. Decomposition formula (general)	$\sum_{j=1}^{\infty} \frac{L(s_j)P(s_j)}{s_j^{a(1+b)} M(s_j) Q^{1+r}(s_j)} = -\left\{ a \frac{1}{b!} \left[\frac{U(s)}{V(s)} \cdot \frac{P(s)}{Q^{1+r}(s)} \right]^{(b)}_{s=0} + \frac{1}{r!} \sum_{k=1}^{m} \left[\frac{U(s)}{s^{a(1+b)} V(s)} \cdot \frac{P(s)}{Q_k^{1+r}(s)} \right]^{(r)}_{s=q_k} \right\}$	$Q_k = \frac{Q(s)}{s-q_k}$, $Q(q_k) = 0$, $k = 1, \ldots, m$
4. Decomposition formula for single zero-pole	$b = 0 \left[a = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \right]$ $\quad \sum_{j=1}^{\infty} \frac{L(s_j)P(s_j)}{s^a M(s_j) Q^{1+r}(s_j)} = -\left\{ a \frac{U(o)}{V(o)} \frac{P(o)}{Q^{1+r}(o)} + \frac{1}{r!} \sum_{k=1}^{m} \left[\frac{U(s)}{s^a V(s)} \frac{P(s)}{Q_k^{1+r}(s)} \right]^{(r)}_{s=q_k} \right\}$	
5. Inner sum	$S_i(s_i) = \sum_{j=1}^{\infty} \frac{L(s_j)}{s_j^a M(s_j)(s_j+s_i)^{1+r}} = -\left\{ a \frac{U(o)}{V(o) s_i^{1+r}} + \frac{1}{r!} \left[\frac{U(s)}{s^a V(s)} \right]^{(r)}_{s=-s_i} \right\}$	
6. First integral formula	$J_{2,r} = (-1)^{1+r} \cdot r! \sum_{i=1}^{\infty} \frac{L(s_i)}{s_i^a M(s_i)} S_i(s_i) = (-1)^r a \cdot r! \frac{U(o)}{V(o)} \sum_{i=1}^{\infty} \frac{L(s_i)}{s_i^{a+1+r} M(s_i)} + (-1)^r \sum_{i=1}^{\infty} \frac{L(s_i)}{s_i^a M(s_i)} \left[\frac{U(s)}{s^a V(s)} \right]^{(r)}_{s=-s_i}$	
7. Second integral formula	$J_{2,r} = (-1)^{1+r} \left\{ a \cdot \frac{1}{1+r} \left\{ \frac{U(o)}{V(o)} \left[\frac{U(s)}{V(s)} \right]^{(1+r)}_{s=0} + \frac{1}{r!} \left[\frac{U(s)}{V(s)} \cdot \frac{R(s)}{F^{1+r}(s)} \right]^{(m)} \right\} + \frac{1}{r!} \sum_{k=1}^{m} \left[\frac{U(s)}{s^{a(1+b)} V(s)} \cdot \frac{R(s)}{F_k^{1+r}(s)} \right]^{(r)}_{s=f_k} \right\}$	
8. Third integral formula special case $a = 0$	$J_{2,r} = (-1)^{1+r} \cdot \frac{1}{r!} \sum_{k=1}^{m} \left[\frac{U(s)}{V(s)} \frac{R(s)}{F_k^{1+r}(s)} \right]^{(r)}_{s=f_k}$	$\left[\frac{U(s)}{V(s)} \right]^{(r)}_{s=-s_i} = \frac{R(s_i)}{F^{1+r}(s_i)}$, $F_k(s) = \frac{F(s)}{s-f_k}$

$$-R(f_1) = 2\alpha^3 (j\alpha^2 h^2 - j - 2\alpha h) \tag{53}$$

$$-R'(f_1) = 2\alpha^2 (2\alpha^2 h^2 - 3 + j4\alpha h) \tag{54}$$

$$-R''(f_1) = 2\alpha\,[4\alpha h + j3\,(2 - \alpha^2 h^2)] \tag{55}$$

$$\left[-\frac{R(s)}{F(s)}\right]''_{f_1} = \frac{-j(1+\alpha^2 h^2) - [(1+\alpha^2 h^2)(3+\alpha^2 h^2) - j\alpha h(5+\alpha^2 h^2)] \times \exp(-j\alpha h) + [\alpha h(5+\alpha^2 h^2) - j\alpha^2 h^2(1+\alpha^2 h^2)]\exp(-2j\alpha h)}{4\alpha^3 [1 - j\exp(-j\alpha h)]^3} \tag{56}$$

Similarly

$$\left[-\frac{R(s)}{F(s)}\right]''_{f_2} = \frac{j(1+\alpha^2 h^2) - [(1+\alpha^2 h^2)(3+\alpha^2 h^2) + j\alpha h(5+\alpha^2 h^2)] \times \exp(j\alpha h) + [\alpha h(5+\alpha^2 h^2) + j\alpha^2 h^2(1+\alpha^2 h^2)]\exp(2j\alpha h)}{4\alpha^3 [1 + j\exp(j\alpha h)]^3} \tag{57}$$

$$J_{22} = \frac{1}{2}\left\{\left[\frac{R(s)}{F(s)}\right]''_{f_1} + \left[\frac{R(s)}{F(s)}\right]''_{f_2}\right\} = \frac{(1+\alpha^2 h^2)[2 + h^2\alpha^2 - \sin\alpha h]\cos\alpha h - \alpha h(5+\alpha^2 h^2)(1-\sin\alpha h)}{8\alpha^3 (1-\sin\alpha h)^2} \tag{58}$$

Below we give optimal values of the parameter of the controller for

$r = 0$

From $\dfrac{dJ_{2,0}}{d\alpha} = 0$

we obtain $\alpha_{\text{opt}}\, h = 0.73844$

$r = 1$

From $\dfrac{dJ_{2,1}}{d\alpha} = 0 \qquad \alpha_{\text{opt}}\, h = 0.90658$

and finally for

$r = 2$

From $\dfrac{dJ_{2,2}}{d\alpha} = 0 \qquad \alpha_{\text{opt}}\, h = 1.1701$

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Metoda uogólnionego rozwinięcia Heaviside'a

W pracy przedstawiono ogólną metodę wyznaczania całki wagowego dynamicznego błędu kwadratowego w układach sterowania z opóźnieniem. Wyniki analityczne otrzymane w pracy umożliwiają wyznaczenie optymalnych wartości parametrów urządzenia sterującego, a także określenie wpływu funkcji wagowych. Dla zilustrowania rozważań przedstawiono prosty przykład.

Метод обобщенного разложения Хевисайда

В работе представлен общий метод вычисления интеграла весовой динамической квадратической ошибки в системах управления с запаздыванием. Полученные в работе аналитические результаты позволяют вычислять оптимальные значения параметров управляющего устройства, а также определять влияние весовых функций. В качестве иллюстрации рассмотрения представлен простой пример.