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Impulsive control of a monothonic process with long run average gain function

by

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In the paper a non-Bellman approach is given to impulsive control with long run average gain of a nonincreasing stochastic process, being a model of quality of an abstract object. A theorem describing optimal policies is formulated and two examples are given.

KEY WORDS: point process, impulsive control, long run average gain, quality control.

1. Introduction

Consider a stochastic process x_t, describing quality of an abstract object Q. Assume its state to belong to the interval $[0, 1]$ from "completely efficient" $(x_t = 1)$ to "completely inefficient" $(x_t = 0)$. Quality of a working object is nonincreasing. Therefore the process $x_t(\omega)$ is nonincreasing for any ω. Consider its right-continuous version. For any ω Lebesgue decomposition implies:

$$x_t(\omega) = x_0(\omega) + \int_0^t a_s(\omega)\, ds + \int_0^t b_s(\omega)\, dN_s(\omega) + C_t(\omega) \tag{1}$$

where a_t and b_t are nonpositive and paths of C_t are continuous functions and N_t is a counting process.

Assume $C_t = 0$ as having no physical interpretation and initial state to be "completely efficient" i.e. $x_0 = 1$. Moreover assume that there exists a filtration $\mathcal{G}_t$ with respect to which:

a. a_t is well measurable,

b. b_t is predictable,

c. N_t has a predictable intensity $q_t \geq 0$, i.e.

$N_t - \int_0^t g_s \, ds$ is a $\mathcal{G}_t$-martingale, see [2].

Denote $\mathcal{X}_t = \sigma(x_s, s \leqslant t)$. Consider an information σ-field $\mathcal{F}_t$ satisfying usual conditions and $\mathcal{X}_t \subseteq \mathcal{F}_t \subseteq \mathcal{G}_t$. Controller's intervention will be described as follows: In $\mathcal{F}_t$-stopping times the process x_t is renewed and starts afresh with the same law and independent realization. The gain function to be maximized is long run average gain per unit of time. Formal description will be the following: Let x^t be defined on the probability space $(\tilde{\Omega}, \mathcal{M}, Q)$. Let $(\Omega, \mathcal{F}, P)$ be an infinite tensor product

$$(\Omega, \mathcal{F}, P) = (\tilde{\Omega}^{\mathbf{N}_0}, \mathcal{M}^{\otimes \mathbf{N}_0}, Q^{\otimes \mathbf{N}_0}). \tag{2}$$

Let

$$x_t^i(\omega_1, \omega_2, \ldots) = x_t(\omega_i), \qquad \omega_i \in \Omega. \tag{3}$$

Therefore x_t^i are independent copies of x_t. Assume the control policy to be stationary and denote renewal times by τ_i. Thus

$$\tau_i = \tau_i(\tau) = \sum_{j=1}^{i} \sigma_j, \text{ where } \sigma_j(\omega_1, \omega_2, \ldots) = \tau(\omega_j). \tag{4}$$

If $E\tau > 0$ the renewal process y_t can be constructed by:

$$y_t = y_t(\tau) = \sum_{i=0}^{\infty} x_{t-\tau_i}^{i+1} I\{\tau_i < t \leqslant \tau_{i+1}\}. \tag{5}$$

The long run average gain function is defined by:

$$S(\tau) = \liminf_{t \uparrow \infty} t^{-1} \{\int_0^t f(y_s)\, ds - \sum_n c(y_{\tau_n}) I\{\tau_n \leqslant t\}\}. \tag{6}$$

Define

$$\tau_\varepsilon = \inf\{t : f(x_t) \leqslant \varepsilon\} \tag{7}$$

Assume some regularity conditions for the functions f and c:

1. $f:[0,1] \to \mathbf{R}$ is nondecreasing, $f(0) = 0, f(1) > 0$ and $E\tau_\varepsilon < \infty$ for any $\varepsilon > 0$.
2. $c:[0,1] \to \mathbf{R}_+$ is bounded and differentiable and $c(1) > 0$.

The function f is interpreted as continuous gain per unit of time of working object and c as cost of renewals.

2. Auxiliary results

The formula for gain function (6) can be essentially simplified. Denote

$$J_p(\tau) = \int_0^\tau (f(x_t) - p)\, dt - c(x_\tau) \qquad \text{for } 0 \leqslant p \leqslant f(1) \tag{8}$$

and

$$J^i_p(\tau(\omega_1, \omega_2, \ldots)) = J_p(\tau(\omega_i)) \qquad \text{for } \omega_i \in \tilde{\Omega}. \tag{9}$$

LEMMA 1. *Let τ be any $\mathscr{F}_t$-stopping time. If $0 < E\tau < \infty$ then $S(\tau) = EJ_0(\tau)/E\tau$ and if $E\tau = +\infty$ then $S(\tau) = 0$.*

Proof. Notice that (σ_n) and $(J^n_0(\tau))$ are both sequences of independent and identically distributed random variables. Assume $0 < E\tau < \infty$. Thus $E|J^n_0(\tau)| < \infty$ and by definitions (6) and (9) $\frac{1}{\tau_n} \sum_{i=1}^{n} J^i_0(\tau) \to S(\tau)$.

On the other hand

$$\frac{1}{\tau_n} \sum_{i=1}^{n} J^i_n(\tau) = \left(\frac{\sum_{i=1}^{n} \sigma_i}{n} \right)^{-1} \cdot \left(\frac{\sum_{i=1}^{n} J^i_0(\tau)}{n} \right) \to \frac{EJ_0(\tau)}{E\tau}.$$

Combining we get $S(\tau) = EJ_0(\tau)/E\tau$. Now let $E\tau = +\infty$. Denote $f^z(x) = f(x) \wedge z$ and $f_z(x) = 0$ if $x \leqslant z$ and $f_z(x) = f(x)$ if $z > x$. Notice that $|S(\tau)| \leqslant + \limsup_{t \uparrow \infty} t^{-1} \{\int_0^t f^\varepsilon(y_s)\, ds + \sum_{\tau_n \leqslant t} c(y_{\tau_n})\} \leqslant \limsup_{t \uparrow \infty} t^{-1} \{\sum_{\tau_n \leqslant t} (c(y_{\tau_n})) + \int_{n-1}^{n} f_\varepsilon(y_s)\, ds\} + \varepsilon = \varepsilon$ by theorem 2.1 p. 51 from [1].

Therefore $S(\tau) = 0$. ■

Denote

$$q = \sup_{\tau - \mathscr{F}_t} S(\tau). \tag{10}$$

Notice that q exists and satisfies $0 \leqslant q < f(1)$. The following proposition holds:

PROPOSITION 1. For any $\mathscr{F}_t$-stopping time τ^* with $0 < E\tau^* < \infty$ and any number $p \in (0, f(1))$ the following conditions are equivalent:

(i) $S(\tau^*) = \sup_{\tau - \mathscr{F}_t} S(\tau) = p$

(ii) $EJ_q(\tau^*) = \sup_{\tau - \mathscr{F}_t} EJ_q(\tau)$ and $p = q$.

(iii) $EJ_p(\tau^*) = \sup_{\tau - \mathscr{F}_t} EJ_p(\tau) = 0$.

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) directly by computation following [3].
(ii) $\Rightarrow$ (iii)

$$EJ_p(\tau) = EJ_0(\tau) - pE\tau = E\tau\,(S(\tau) - p) \leqslant 0.$$

It suffices to show that $\sup_{\tau - \mathscr{F}_t} EJ_p(\tau) = 0$.

There is

$$0 \leqslant E \int_0^\tau f(x_t)\, dt \leqslant \varepsilon E\tau + f(1)\, E\tau_\varepsilon$$

for any τ and $\varepsilon > 0$.
For $N(\varepsilon)$ large enough holds

$$f(1)\, E\tau_\varepsilon / N(\varepsilon) \leqslant \varepsilon \tag{11}$$

Let $3\varepsilon < p$, $\delta = \varepsilon / N(\varepsilon)$ and T_δ be a $(\delta \wedge \varepsilon)$-optimal stopping time for the gain $S(\tau)$. Notice that $0 < ET_\delta < \infty$ and therefore $2\varepsilon < p - \varepsilon \leqslant S(T_\delta) = EJ_0(T_\delta)/ET_\delta \leqslant \varepsilon + f(1)\, E\tau_\varepsilon / ET_\delta$ and by (11) $ET_\delta \leqslant N(\varepsilon)$. Hence $EJ_p(T_\delta) = ET_\delta (S(T_\delta) - p) \geqslant -ET_\delta\, \delta \geqslant -\varepsilon$ and $\sup_{\tau - \mathscr{F}_t} EJ_p(\tau) = 0$. ■

3. Solution of the problem

Differentiating by parts we get for every path of the process:

$$c(x_t) = c(x_0) + \int_0^t c'(x_t)\, a_t\, dt + \int_0^t Dc(x_t)\, dN_t \tag{12}$$

where $Dc(x_t) = c(x_{t-} + b_t) - c(x_{t-})$. Since $Dc(x_t)$ is a predictable process for any $\mathscr{G}_t$-stopping time with $E\tau < \infty$ there holds:

$$Ec(x_\tau) = c(x_0) + E \int_0^\tau c'(x_t)\, a_t\, dt + E \int_0^\tau Dc(x_t)\, g_t\, dt. \tag{13}$$

Let h_t be a progressively measurable version of

$$E\left(c'(x_t)\, a_t + Dc(x_t)\, g_t \,|\, \mathscr{F}_t\right)$$

Denote:

$$F_t = f(x_t) - h_t \tag{14}$$

$$\tau(p) = \inf\{t : F_t \leqslant p\} \text{ for } p \geqslant 0 \tag{15}$$

$$K(p) = EJ_p(\tau(p)) \tag{16}$$

The main result of the paper is formulated as follows:

THEOREM 1. *Assume F_t to be nonincreasing.*

A. *If $K(0) \leqslant 0$ then optimal control rule is "do not interfere with the run of the process" ($\tau_{opt} = +\infty$) and hence $q = 0$.*

B. *If $K(0) > 0$ then $q > 0$ and $\tau(q)$ is the optimal stopping rule for the gain S. Moreover q is the only solution of the equation $K(p) = 0$, with the function K nonincreasing, convex and continuous on $(0, +\infty)$ and if $K(0) < \infty$, on $[0, +\infty)$.*

Proof. Notice that $EJ_p(\tau) = E\int_0^{\tau}(F_t - p)\,dt - c(1)$. Since F_t is nonincreasing $\int_0^{\tau(p)(\omega)}(F_t(\omega) - p)\,dt \geqslant \int_0^{t}(F_t(\omega) - p)\,dt$ for any $\omega \in \tilde{\Omega}$, $p \geqslant 0$ and $t \geqslant 0$ by (7). Therefore for any $\mathscr{F}_t$-stopping time $EJ_p(\tau) \leqslant EJ_p(\tau(p)) = K(p)$. Since $c \geqslant 0$ then $EJ_0(\tau(0)) > 0$ implies $\tau(0) \neq 0$ and hence $E(\tau(0) \wedge N) > 0$ for $N \geqslant 1$. Thus for N large enough $q \geqslant S(\tau(0) \wedge N) = EJ_0(\tau(0) \wedge N)/E(\tau(0) \wedge N) > 0$. If $K(0) \leqslant 0$ then $EJ_0(\tau) \leqslant 0$ and $S(\tau) = EJ_0(\tau)/E\tau \leqslant 0$ for any $\mathscr{F}_t$-stopping time with $0 < E\tau < \infty$ and hence $q = 0$ and $\tau_{\text{opt}} = +\infty$. Since $c(1) > 0$ then $\limsup_{p \uparrow \infty} K(p) < 0$. Notice that by Tonelli theorem

$$\int_0^p (F_t - p)\,dt = \int_0^{\infty}\int_p^{\infty} I\{F_t \geqslant r\}\,dr\,dt = \int_p^{\infty}\int_0^{\infty} I\{F_t \geqslant t\}\,dt\,dr = \int_p^{\infty}\tau(r)\,dr \tag{17}$$

Since $\tau(r) \geqslant \tau(p)$ if $r \leqslant p$ holds

$$E\tau(r) \geqslant E\tau(p) \text{ for } p \geqslant r. \tag{18}$$

Prove that $E\tau(p) < \infty$ for any $p > 0$. Suppose that for some $p > 0$ there holds $E\tau(2p) = +\infty$. By assumption 1 $E\tau_p < \infty$. Hence $-h_t \geqslant p$ for $\tau_p \wedge \tau(2p) \leqslant t \leqslant \tau(2p)$. The statements above imply that

$$E\int_{\tau_p \wedge \tau(2p)}^{\tau(2p)} h_t\,dt = -\infty$$

Therefore for any k and N large enough there is:

$$Ec(x_{\tau(2p) \wedge N}) - Ec(x_{\tau_p \wedge \tau(2p) \wedge N}) = E\int_{\tau_p \wedge \tau(2p) \wedge N}^{\tau(2p) \wedge N} h_t\,dt \leqslant -k$$

a contradiction since c is bounded. We have proved that $E\tau(p) < \infty$ for any $p > 0$. Thus $K(p) < \infty$, $p > 0$. Therefore

$$K(p) = E\int_0^{\tau(p)}(F_t - p)\,dt - c(1) = E\int_p^{\infty}\tau(r)\,dr - c(1) = \int_p^{\infty}E\tau(r)\,dr - c(1) \tag{19}$$

By (18) the function K is nonincreasing, convex and continuous on $(0, +\infty)$. Since $K(0) > 0$ and $\lim_{r \uparrow \infty} K(r) < 0$ there exists the only solution of the equation $K(p) = 0$, namely q. Since $EJ_q(0) = -c(1) < 0$ and $EJ_q(\tau(q)) = 0$ there is $\tau(q) \neq 0$ and $0 < E\tau(q) < +\infty$. By proposition 1 condition (iii) $q = \sup_{\tau - \mathscr{F}_t} S(\tau) = S(\tau(q))$. ∎

The following examples describe practical significance of the theorem:

EXAMPLE 1.

The filtration process.

Let N_t be a point process with the intensity $\mu_1 + I_{\{t > T\}}(\mu_0 - \mu_1)$, where T is an exponentially distributed random variable with parameter 1 and

$\mu_0 > \mu_1$. Let $x_t = P(T \geqslant t | N_s, s \leqslant t)$ be the process of sequential estimation of T, which satisfies a stochastic equation: $dx_t = a(x_t)\, dt + b(x_{t-})\, dN_t$ for some determined functions a and b. Let f and c be linear functions: $f(x_t) = fx_t$ and $c(x_t) = c_0(1-x_t)+c_1$ with $f, c_0, c_1 > 0$. If $\mu_0 - \mu_1 \leqslant 1 < f/(c_0+c_1)$ the assumption B of Theorem 1 is satisfied by (3). Such a problem is strictly connected with partially observed quality control problem described in [3].

EXAMPLE 2.
Constant renewal costs. If $c(x) = \text{const} > 0$ application of Theorem 1 is immediate.

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Sterowanie impulsowe procesu monotonicznego ze średnim zyskiem na jednostkę czasu

W artykule podano bezpośrednie podejście do sterowania impulsowego nierosnącym procesem ze średnim zyskiem na jednostkę czasu. Proces taki jest modelem jakości abstrakcyjnego obiektu technicznego. W pracy sformułowane jest twierdzenie opisujące optymalną strategię sterowania i podane są dwa przykłady.

Импульсное управление монотонным процессом со средним выигрышем за единицу времени

В статье приводится непосредственный подход к импульсному управлению не возрастающим процессом со средним выигришем за единицу времени. Такой процесс является моделью качества абстракного технического процесса. В работе формулируется теорема, описывающая оптимальную стратегию управления и даются два примера.