Control
and Cybernetics
VOL. 16 (1987) No. 2

Bang-bang controllers for an optimal cooling problem

by

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The existence of optimal bang-bang boundary controllers for a free boundary control problem of parabolic type is studied. The problem is of one-phase Stefan type.

1. Introduction

Consider the controlled one-phase Stefan problem in n dimensions which physically models the melting of a body of ice $\Omega \subseteq R^3$ maintained in contact with a region of water. The boundary Γ of Ω is composed of two disjoint sets Γ_1 and Γ_2. The temperature on the boundary Γ_1 is $\eta(x, t)$ while the temperature on Γ_2 is zero. Initially the water (liquid) occupies the domain Ω_0 (see Figure 1 below). If $\theta = \theta(x, t)$ is the

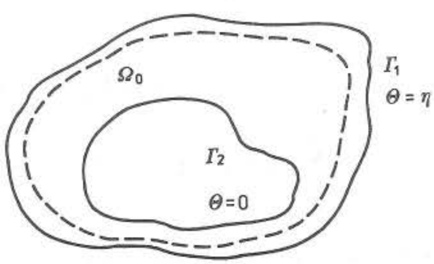

Fig. 1

*) The work of this author was partially supported by a grant from the AFOSR.

temperature at point x and at time t and if $t=\sigma(x)$ is the equation of the water-ice interface then the temperature distribution θ satisfies the classical Stefan problem

$$\begin{aligned}
&\theta_t-\Delta\theta=0 && \text{in } \{(x,t)\in\Omega\times(0,T);\ \ \sigma(x)<t<T\}\\
&\theta=0 && \text{in } \{(x,t)\in\Omega\times(0,T);\ \ \sigma(x)\geqq t\}\\
&\nabla_x\theta\cdot\nabla\sigma(x)=-\varrho && \text{in } \{(x,t);\ t=\sigma(x)\} \qquad (1.1)\\
&\theta=\eta && \text{in } \Gamma_1\times(0,T);\ \ \theta=0 \ \text{ in } \ \Gamma_2\times(0,T)\\
&\theta(x,0)=\theta_0(x) && \text{if } \ x\in\Omega_0;\ \ \theta(x,0)=0 \ \text{ if } \ x\in\Omega-\Omega_0.
\end{aligned}$$

where ϱ is a positive constant.

We will assume that the temperature η on $\Gamma_1\times(0,T)$ is controlled by the system

$$\eta(x,t)=\sum_{i=1}^{m} g_i(x)\,u_i(t) \qquad x\in\Gamma_1, \qquad t\in(0,T) \tag{1.2}$$

$$du_i(t)/dt+f_i(u_i(t))=v_i(t) \qquad \text{a.e.} \qquad t\in[0,T]; \qquad i=1,2,\dots,m,$$

$$u_i(0)=u_i^0 \tag{1.3}$$

where $g_i\geqq 0$ are given functions on Γ_1 and $f_i: R\to R$ are Lipschitz and continuously differentiable. The vector function $v(t)=(v_i(t),\dots,v_m(t))\in \in L^\infty(0,T;R^m)$ represents the supply of fuel provided by a system of heaters which control the temperatures $(u_1,\dots,u_m)$. If the functions g_i have disjoint supports in Γ_i this corresponds to the physical situation in which the temperature on Γ_i is determined by m heated regions with densities $g_1,\dots,g_m$. The case when $g_i(x)=\delta(x-x_i)$, where δ is the Dirac delta function, is of special physical interest since it represents point heating. This case is approximated as the limit of C^1 functions g_i that we are considering here.

We assume that the control functions $v=v(t)$ are subject to the following constraints

$$0\leqq v_i(t)\leqq N_i, \qquad \text{a.e. in} \quad [0,T], \qquad i=1,2,\dots,m \tag{1.4}$$

$$\sum_{i=1}^{m} a_i\int_0^T v_i(t)\,dt=M \tag{1.5}$$

where N_i, a_i are nonnegative constants such that $\sum_{i=1}^{m} a_i>0$ and $T\sum_{i=1}^{m} a_i N_i\geqslant \geqslant M$. The class of all control functions $v\in L^\infty(0,T;R^m)$ satisfying the constraints (1.4)–(1.5) will be denoted by U.

Our goal here is to consider several optimization problems associated with (1.1)–(1.5) which are approaches to the following controllability problem. Given a surface $S \subseteq \Omega \times (0, T)$ we seek a $v \in U$ such that S is "as close as possible" to the free surface $S_v = \{(x, t); t = \sigma(x)\}$ which is the solution of system (1.1)–(1.5) corresponding to the control v.

The first step in solving this problem involves transforming the Stefan problem (1.1) into a parabolic variational inequality using the well-known device of Baiocchi and Duvaut. We may then apply the necessary conditions for optimal control of variational inequalities to characterize the optimal control.

The main emphasis of this paper is to explicitly determine the optimal control. We will characterize the optimal control as a bang-bang control.

Optimal control of free boundary problems have been studied elsewhere by a variety of methods. Some examples include Saguez [6], Barbu [1, 2], Friedman [4] and Bermudez and Saguez [3]. There is a growing literature on optimal control of variational inequalities. We refer to [1, 2, 5, 6].

2. Optimal control for parabolic variational inequalities

Let θ be the solution of the system (1.1). Define the function

$$H(x, t) = \int_0^t \theta(x, s)\, \chi(x, s)\, ds \qquad \forall (x, t) \in \Phi = \Omega \times (0, T)$$

where $\chi(x, t) = 1$ if $\sigma(x) \leqq t$ and $\chi(x, t) = 0$ if $\sigma(x) > t$. Then, H is the solution of the variational inequality

$$\begin{aligned} & H_t - \Delta H = f \quad \text{on the set} \quad \{(x, t); \quad H(x, t) > 0\} \\ & H \geqq 0, \quad H_t - \Delta H \geqq f \quad \text{on} \quad \Phi \\ & H = \sum_{i=1}^{m} g_i \int_0^t u_i(s)\, ds \quad \text{in} \quad \Sigma_1 = \Gamma_1 \times (0, T) \\ & H = 0 \quad \text{in} \quad \Sigma_2 = \Gamma_2 \times (0, T) \\ & H(x, 0) = 0 \quad \text{in} \quad \Omega \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} f(x) &= \theta_0(x) \quad \text{if} \quad x \in \Omega_0 \\ &= -\varrho \quad \text{if} \quad x \in \Omega - \Omega_0 . \end{aligned}$$

Furthermore, we have that

$$\{(x, t) \in \Phi; \quad \sigma(x) > t\} = \{(x, t) \in \Phi; \quad H(x, t) > 0\}.$$

In the general case, the controls v are chosen so as to minimize the payoff

$$\text{(P)} \qquad \int_{\Phi} g\left(H(x,t),t\right) dxdt + \int_{\Omega} g_0\left(H(x,T)\right) dx$$

subject to the constraints (2.1), (1.3)–(1.5).

We assume that

$$\theta_0 \in L^\infty(\Omega), \quad g_i \in W^{2-1/q}(\Gamma_1), \quad \text{with} \quad q > \max\left(\frac{n+2}{2}, 2\right). \tag{2.2}$$

Then problem (2.1) has a unique solution $H \in W_q^{2,1}(\Phi) \cap C(\bar{\Phi})$, where, as usual, $W_q^{2,1}(\Phi) = \{y \in L^2(\Phi) | \partial y^{r+s}/\partial t^r \partial x^s \in L^q(\Phi) \text{ for } 2r+s \leqq 2\}$. See, for example, Barbu [1, p. 162]. More precisely, we have that

$$H_\varepsilon \to H \quad \text{weakly in} \quad W_q^{2,1}(\Phi) \quad \text{as} \quad \varepsilon \to 0, \tag{2.3}$$

where $H_\varepsilon \in W_q^{2,1}(\Phi)$ is the solution to the approximating problem

$$\begin{aligned} &\partial H_\varepsilon/\partial t - \Delta H_\varepsilon + \beta^\varepsilon(H_\varepsilon) = f \quad \text{in} \quad \Phi \\ &H_\varepsilon = Bu \quad \text{in} \quad \Sigma_1; \quad H_\varepsilon = 0 \quad \text{in} \quad \Sigma_2 \\ &H_\varepsilon(x,0) = 0 \quad \text{if} \quad x \in \Omega \end{aligned} \tag{2.4}$$

where $Bu = \sum_{i=1}^{m} g_i(x) \int_0^t u_i(s)\, ds$ and β^ε is a smooth approximation of the multivalued function β given by $\beta(r) = 0$ for $r > 0$, $\beta(0) = (-\infty, 0]$. $\beta(r) = \emptyset$ for $r < 0$.

The following estimate also holds (see [1, p. 163]):

$$\| H_\varepsilon \|_{W_q^{2,1}(\Phi)} + \| \beta^\varepsilon(H_\varepsilon) \|_{L^q(\Phi)} \leqq C\left[1 + \| Bu \|_{W^{2-1/q,1-1/2q}(\Sigma_1)}\right] \tag{2.5}$$

For the functions $g: R \times [0,T] \to R$ and $g_0: R \to R$ we will make the assumption

(A) $g(y,t)$ and $g_0(y)$ are continuously differentiable in y, g is measurable in t and

$$|\partial g(y,t)/\partial y| \leqq \alpha_R(t) \quad \text{a.e.} \quad t \in [0,T], \quad |y| \leqq R, \quad \alpha_R \in L^2(0,T), \tag{2.6}$$

$$g(y,t) \geqq -C_1(1+|y|), \quad g_0(y) \geqq -C(1+|y|) \quad \forall y \in R, \quad 0 \leqq t \leqq T. \tag{2.7}$$

By standard arguments it follows that problem (P) admits at least one solution. Accordingly, let (H^*, u^*, v^*) denote an optimal triple for problem (P). Note that H^* and u^* are state variables and v^* is the optimal control.

Consider now the approximating problem for each $\varepsilon > 0$:

(P_ε) Minimize the payoff

$$\int_{\Phi} g\left(H_{\varepsilon}(x,t),t\right)dxdt+\int_{\Omega} g_0\left(H_{\varepsilon}(x,T)\right)dx+\frac{1}{2}\int_0^T \|v(t)-v^*(t)\|_m^2\,dt$$

over all (H_ε, u, v) subject to (2.4), (1.3)–(1.5).

Let $(H_\varepsilon, u_\varepsilon, v_\varepsilon)$ denote an optimal triple for problem (P_ε). Then by arguments similar to those in [1, p. 240] we obtain.

LEMMA 1. *As* $\varepsilon \to 0$ *we have*

$$v_\varepsilon \to v^* \quad \text{strongly in} \quad \left(L^2(0,T)\right)^m, \tag{2.8}$$

$$H_\varepsilon \to H^* \quad \text{strongly in} \quad W_q^{2,1}(\Phi), \quad u_\varepsilon \to u^* \quad \text{strongly in} \quad C(0,T). \tag{2.9}$$

We may now determine the necessary conditions which $(H_\varepsilon, u_\varepsilon, v_\varepsilon)$ must satisfy. Using equations (2.4), (1.3)–(1.5) we obtain the following.

For each $\varepsilon > 0$ there exist functions p_ε and q_ε with $p_\varepsilon \in L^2\left(0,T;H_0^1(\Omega)\right)\cap$ $\cap\, C\left([0,T];L^2(\Omega)\right)$, $\partial p_\varepsilon/\partial t \in L^2\left(0,T;H^{-1}(\Omega)\right)$ which together with H_ε, u_ε, and v_ε satisfy the system

$$\begin{aligned}
&\partial H_\varepsilon/\partial t - \Delta H_\varepsilon + \beta^\varepsilon(H_\varepsilon) = f \quad \text{in} \quad \Phi\\
&H_\varepsilon = Bu_\varepsilon \quad \text{in} \quad \Sigma_1; \quad H_\varepsilon = 0 \quad \text{in} \quad \Sigma_2\\
&H_\varepsilon(x,0) = 0, \quad x\in\Omega
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&\partial p_\varepsilon/\partial t + \Delta p_\varepsilon - \dot\beta^\varepsilon(H_\varepsilon)\,p_\varepsilon = \partial g(H_\varepsilon,t)/\partial y \quad \text{in} \quad \Phi\\
&p_\varepsilon = 0 \quad \text{in} \quad \Sigma_1\cup\Sigma_2\\
&p_\varepsilon(x,T) = -g_0'\left(H_\varepsilon(x,T)\right), \quad \forall x\in\Omega.
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&d(u_\varepsilon)_i/dt + f_i\left((u_\varepsilon)_i\right) = (v_\varepsilon)_i \quad \text{a.e. in} \quad [0,T], \quad i=1,2,\dots,m\\
&(u_\varepsilon)_i(0) = u_i^0
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
&d(q_i^\varepsilon)/dt - f_i'\left((u_\varepsilon)_i\right)q_i^\varepsilon = \int_t^T ds \int_{\Gamma_1} g_i(x)\left(\partial p_\varepsilon(x,s)/\partial \nu\right)dx \quad 0\leqq t<T\\
&q_i^\varepsilon(T) = 0, \quad i=1,2,\dots,m,
\end{aligned} \tag{2.13}$$

$$\int_0^T \int_{\Gamma_1} \partial p_\varepsilon/\partial\nu\, Bv\,dxdt + h'(v_\varepsilon, w) + \int_0^T \left(w(t),\ v_\varepsilon(t)-v^*(t)\right)dt \geqq 0 \tag{2.14}$$

for all $w\in\left(L^2(0,T)\right)^m$, where h' is the directional derivative of the indicator function h given by

$$h(v) = 0 \quad \text{if} \quad v\in U, \quad h(v) = +\infty \quad \text{if} \quad v\notin U \tag{2.15}$$

and where

$$dv^i/dt + f_i'\left((u_\varepsilon)_i\right)v_i = w_i, \quad \text{a.e. in} \quad (0,T]; \quad v_i(0) = 0. \tag{2.16}$$

By (2.13), (2.14) and (2.16) we get after some manipulation that

$$-\int_0^T \left(q^\varepsilon(t), w(t)\right) dt + \int_0^T \left(w(t), v_\varepsilon(t) - v^*(t)\right) dt + h'(v_\varepsilon, w) \geqq 0$$

for every $w \in \left(L^2(0, T)\right)^m$. Therefore,

$$q^\varepsilon(t) + v^*(t) - v_\varepsilon(t) \in \partial h(v_\varepsilon)(t) \quad \text{a.e.} \quad t \in [0, T], \tag{2.17}$$

where $\partial h: L^2(0, T; R^m) \to L^2(0, T; R^m)$ is the subdifferential of h (see, for example [1, p. 101]).

We see that

$$\partial h(v) = \partial h_0(v) + \partial h_1(v) \qquad \forall v \in \left(L^2(0, T)\right)^m$$

where

$$\begin{aligned} \partial h_0(v) = \{w = (w_i, \dots, w_m); \quad & w_i = 0 \quad \text{in} \quad \{t;\ 0 < v_i(t) < N_i\}, \\ & w_i \leqq 0 \quad \text{in} \quad \{t;\ v_i(t) = 0\}, \\ & w_i \geqq 0 \quad \text{in} \quad \{t;\ v_i(t) = N_i\}\}. \end{aligned} \tag{2.18}$$

$$\partial h_i(v) = \{w = (\lambda a_1, \dots, \lambda a_m); \quad \lambda \in R\}, \quad \forall v \in \left(L^2(0, T)\right)^m. \tag{2.19}$$

Then, by (2.17) we infer that there is $\lambda_\varepsilon \in R$ such that

$$(v_\varepsilon)_i(t) = \begin{cases} 0 & \text{if} \quad q_i^\varepsilon(t) + \left(v_i^* - (v_\varepsilon)_i\right)(t) < \lambda_\varepsilon a_i \\ N_i & \text{if} \quad q_i^\varepsilon(t) + \left(v_i^* - (v_\varepsilon)_i\right)(t) > \lambda_\varepsilon a_i \end{cases} \tag{2.20}$$

Now, by multiplying (2.1) by p_ε and $\operatorname{sgn}(p_\varepsilon)$, and integrating over Φ we obtain the estimate

$$\|p_\varepsilon(t)\|^2_{L^2(\Omega)} + \int_0^T \|p_\varepsilon(t)\|^2_{H_0^1(\Omega)} dt + \int_\Phi |\dot\beta^\varepsilon(H_\varepsilon) p_\varepsilon| \, dxdt \leqq C.$$

Then arguing as in [1, p. 242] we conclude that there exists a function $p \in BV\left([0, T]; H^{-s}(\Omega)\right)$, $s > n/2$, such that for a subsequence, still indexed by ε, as $\varepsilon \to 0$

$$p_\varepsilon \to p \quad \text{strongly in} \quad L^2(\Phi), \quad \text{weakly in} \quad L^2\left(0, T; H_0^1(\Omega)\right)$$
$$\text{and weak star in} \quad L^\infty\left(0, T; L^2(\Omega)\right), \tag{2.21}$$

$$p_\varepsilon(t) \to p(t) \quad \text{strongly in} \quad H^{-s}(\Omega) \quad \text{for every} \quad t \in [0, T], \tag{2.22}$$

$$\dot\beta^\varepsilon(H_\varepsilon) p_\varepsilon \to \mu \quad \text{weak star in} \quad \left(L^\infty(\Phi)\right)^*. \tag{2.23}$$

Now let χ be any function in $W_q^{2-1/q,\, 1-1/2q}(\Sigma_1)$ and let $\varphi \in W_q^{2,1}(\Phi)$ be the solution of the problem

$$\partial\varphi/\partial t - \Delta\varphi = 0 \quad \text{in} \quad \Phi$$

$$\varphi = \chi \quad \text{in} \quad \Sigma_1; \quad \varphi = 0 \quad \text{in} \quad \Sigma_2$$
$$\varphi(x, 0) = 0.$$

Multiply (2.11) by φ and integrate by parts using Green's formula. We obtain

$$\left| \int_0^T \int_{\Gamma_1} \partial p_\varepsilon / \partial v \chi dx dt \right| \leqq M \| \varphi \|_{C(\bar{\Phi})} \leqq C \| \chi \|_{W_q^{2-1/q, 1-1/2q}(\Sigma_1)}$$

Hence, $\{\partial p_\varepsilon / \partial v\}_\varepsilon$ is bounded in the dual of $W_q^{2-1/q, 1-1/2q}(\Sigma_1)$. Consequently we may assume that as $\varepsilon \to 0$

$$\partial p_\varepsilon / \partial v \to \partial p / \partial v \quad \text{weakly in} \quad (W^{2-1/q, 1-1/2q}(\Sigma_1))^*.$$

Then, letting $\varepsilon \to 0$ in equations (2.10)–(2.13) and (2.20) we see that

$$\begin{cases} \partial H^* / \partial t - \Delta H^* = f & \text{in} \quad \{H^* > 0\} \\ H^* = Bu^* \quad \text{in} \quad \Sigma_1, \; H^* = 0 & \text{in} \quad \Sigma_2 \\ H^*(x, 0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.24}$$

$$\begin{cases} \partial p / \partial t + \Delta p - \mu = \partial g / \partial y & \text{in} \quad \Phi \\ p = 0 \quad \text{in} \quad \Sigma \\ p(x, T) = -g_0'(H^*(x, T)) & \text{a.e.} \quad x \quad \text{in} \quad \Omega. \end{cases} \tag{2.25}$$

$$du_i^* / dt + f_i(u_i^*) = v_i^* \quad \text{a.e.} \quad t \in (0, T], \quad u_i^*(0) = u_i^0. \tag{2.26}$$

Since $\{\int_{\Gamma_i} \partial p_\varepsilon / \partial v g_i \, dx\}_\varepsilon$ is bounded in $W_q^{1-1/2q}(0, T)$ we get that as $\varepsilon \to 0$

$$\int_t^T ds \int_{\Gamma_i} g_i \, \partial p_\varepsilon / \partial v dx \to \int_t^T ds \int_{\Gamma_i} g_i \, \partial p / \partial v dx \equiv \psi_i \quad \text{weakly in} \quad L^q(0, T)$$

The functions $\psi_i(t)$ satisfy

$$\int_0^T \psi_i(t) \varphi(t) \, dt = \langle \partial p / \partial v, g_i \int_0^t \varphi(s) \, ds \rangle, \quad \forall \varphi \in L^2(0, T),$$

where $\langle \, , \, \rangle$ is the pairing between $W_q^{2-1/q, 1-1/2q}(\Sigma_1)$ and its dual.

Letting $\varepsilon \to 0$ in (2.13) and (2.20) we get

$$dq_i^* / dt - f_i'(u_i^*) q_i^* = \int_t^T ds \int_{\Gamma_1} g_i(x) \, \partial p / \partial v \, dx \quad \text{a.e.} \quad t \in [0, T)$$

$$q_i^*(T) = 0. \tag{2.27}$$

$$v_i^* = \begin{cases} 0 & \text{if} \quad q_i^*(t) < \lambda a_i \\ N_i & \text{if} \quad q_i^*(t) > \lambda a_i \end{cases}, \quad i = 1, 2, \ldots, m. \tag{2.28}$$

Finally, arguing as in [1] we see that $\mu = 0$ in the set $\{H^* > 0\}$ and $p = 0$ in the set $\{H^* = 0\}$. Therefore, p satisfies the system

$$\partial p/\partial t + \Delta p = \partial g\,(H^*, t)/\partial y \quad \text{in} \quad \{H^* > 0\} \tag{2.30}$$

$$p = 0 \quad \text{in} \quad \{H^* = 0\} \tag{2.31}$$

$$p\,(x, T) = -g_0'\,(H^*\,(x, T)) \quad \text{a.e.} \quad x \in \Omega. \tag{2.32}$$

Summarizing, we have proved

THEOREM 1. *Let (H^*, u^*, v^*) be any optimal triple for problem (P). Then there are functions p, q^* and a constant $\lambda \in R$ with*

$$p \in L^2\,(0, T;\ H_0^1\,(\Omega)) \cap BV\,([0, T];\ H^{-s}\,(\Omega)) \cap L^\infty\,(0, T;\ L^2\,(\Omega)), \qquad s > n/2,$$

$$\partial p/\partial \nu \in \big(W_q^{2-1/q,\,1-1/2q}\,(\Sigma_1)\big)^*, \qquad q^* \in L^2\,(0, T;\ R^m),$$

such that equations (2.24), (2.31), (2.32), (2.27), *and* (2.28) *are satisfied.*

Next, we will assume that

$$f_i' \leqq 0, \qquad f_i'' \geqq 0, \qquad i = 1, 2, \ldots, m \tag{2.33}$$

$$\partial g\,(y, t)/\partial y > 0 \quad \text{if} \quad y > 0; \quad g_0'\,(y) \geqq 0 \quad \text{if} \quad y \geqq 0 \tag{2.34}$$

$$\theta_0 \in C\,(\bar{\Omega}_0) \quad \text{and} \quad \theta_0\,(x) > 0 \quad \forall x \in V\,(\Gamma_1), \tag{2.35}$$

where $V\,(\Gamma_1)$ is a neighborhood of Γ_1.

THEOREM 2. *Under the assumptions* (2.33)–(2.35) *every optimal control $v^* = (v_1^*, \ldots, v_m^*)$ is a bang-bang control. That is, there are $0 \leqq t_i \leqq T$, $i = 1, 2, \ldots, m$, such that*

$$v_i^*\,(t) = \begin{cases} 0 & \text{if} \quad 0 \leqq t \leqq t_i \\ N_i & \text{if} \quad t_i < t \leqq T \end{cases} \qquad i = 1, 2, \ldots, m. \tag{2.36}$$

In particular, if $m = 1$ then problem (P) *has a unique optimal control given by*

$$v^*\,(t) = \begin{cases} 0 & \text{if} \quad 0 \leqq t \leqq t_1 = T - M/(aN) \\ N & \text{if} \quad t_1 < t \leqq T \end{cases} \tag{2.37}$$

where $N = N_1$ and $a = a_1$.

Proof. Let $\varphi \in C_0^\infty\,(R^n)$ with $0 \leqq \varphi \leqq 1$ in R^n, $\varphi = 1$ in a neighborhood $V_1 \subseteq V\,(\Gamma_1)$ of Γ_1 and $\varphi = 0$ in $\Omega_0 - V\,(\Gamma_1)$. Set $\tilde{p} = p\varphi$ and note that by (2.30) we have

$$\partial \tilde{p}/\partial t + \Delta \tilde{p} = \varphi \partial g/\partial y - p \Delta \varphi - \nabla p\, \nabla \varphi \quad \text{in} \quad \Omega_1 \times (0, T)$$

$$\tilde{p} = 0 \quad \text{on} \quad \partial \Omega_1 \tag{2.38}$$

$$\tilde{p}(x, T) = -g_0'(H^*(x, T))\, \varphi(x) \qquad \forall x \in \Omega_1$$

where $\Omega_1 = V(\Gamma_1)$ (see Figure 2).

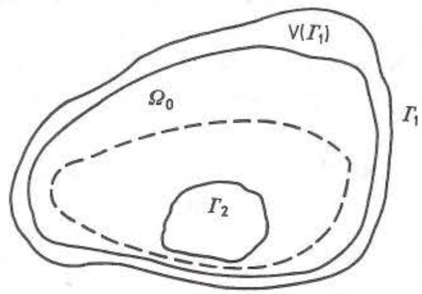

Fig. 2

To see that (2.38) is true we note that $H^* > 0$ in Ω_1. Indeed, by (2.1) we have that

$$\partial H^*/\partial t - \Delta H^* \geqq \theta_0 \quad \text{in} \quad \{H^* = 0\}.$$

Combining this with (2.35) implies that $H^* > 0$ in Ω_1 as claimed.

By (2.38) we see that $\tilde{p} \in L^2(0, T-\delta;\ H_0^1(\Omega_1) \cap H^2(\Omega_1))$ and $\partial\tilde{p}/\partial t \in L^2(0, T-\delta;\ L^2(\Omega_1))$ for every $\delta > 0$. Also,

$$\tilde{p} = p \quad \text{in} \quad \overline{V(\Gamma_1)} \times (0, T).$$

In particular, we conclude that $\partial\tilde{p}/\partial\nu \in L^2(0, T-\delta; H^{1/2}(\Gamma_1))$ for every $\delta > 0$. Moreover, since

$$\partial\tilde{p}/\partial t + \Delta\tilde{p} = \partial g(H^*, t)/\partial y \quad \text{in} \quad V(\Gamma_1) \times (0, T)$$

$$\tilde{p}(x, T) \leqq 0 \qquad \text{in} \quad V(\Gamma_1)$$

and $\partial g(H^*, t)/\partial y \equiv 0$, by the strong maximum principle we conclude that $\tilde{p} < 0$ in $V(\Gamma_1) \times (0, T)$ and $\partial\tilde{p}/\partial\nu > 0$ in $\Gamma_i \times (0, T)$. Since $\partial\tilde{p}/\partial\nu = \varphi\partial p/\partial\nu + p\partial\varphi/\partial\nu = \varphi\partial p/\partial\nu$ in $\Gamma_1 \times (0, T)$, and $\varphi = 1$ on Γ_1, we obtain that

$$\partial p/\partial\nu > 0 \quad \text{in} \quad \Gamma_1 \times (0, T). \tag{2.39}$$

Next, we see from (2.27) that $q_i^* \leqq 0$ in $[0, T]$ and $dq_i^*/dt \equiv \mu_i$ satisfies the equation

$$d\mu_i/dt = f_i'(u_i^*)\,\mu_i + (u_i^*)'\, f_i''(u_i^*)\, q_i^* - \int_{\Gamma_i} g_i\, \partial p/\partial\nu\, dx \quad \text{a.e. in} \quad [0, T),$$

$$\mu_i(T) = 0 \qquad i = 1, 2, \ldots, m.$$

Then, by (2.33) we get

$$\mu_i(t) \geqq \exp\left[\int_t^T -f_i'(u_i^*)\, ds\right] \int_t^T ds \left[\exp \int_s^T f_i(u_i^*)\, d\tau \int_{\Gamma_i} g_i\, \partial p/\partial\nu\, dx\right]$$

for a.e. $t \in (0, T)$, $i = 1, 2, \ldots, m$.

Together with (2.39) the last inequality tells us that $\mu_i(t) > 0$ for a.e. $t \in (0, T)$ and for all $i = 1, 2, \ldots, m$. Therefore, the functions q^* are strictly increasing on $[0, T]$. By (2.28) we conclude that every v_i^* has at most one switch point t_i and so (2.36) must be true.

Now consider the special case $m = 1$. Then, by (2.36) we see that $(T - t_i)\, aN = M$ as claimed. This completes the proof. ■

REMARK 1. From the proof of Theorem 2 it is clear that the conclusions still hold if assumption (2.34) is weakened to

$$\partial g(H^*, t)/\partial y \geqq 0, \quad g_0'(H^*(T)) \geqq 0, \quad t \in [0, T], \tag{2.40}$$

$$\partial g(H^*, t)/\partial y + g_0'(H^*) \equiv 0 \tag{2.41}$$

for every optimal H^*.

3. Optimal control of the one-phase Stefan problem

To begin with we consider the following problem. Given a set $E \subseteq \Phi$ find $v \equiv U$ such that $E \subseteq \{(x, t) \in \Omega \times (0, T);\ \sigma(x) \geqq t\}$. The least squares approach leads us to consider the optimal control problem:

$$\min \left[\int_{\Phi} H^2(x, t)\, \chi_E(x, t)\, dx dt; \quad v \in U \right] \tag{3.1}$$

where H is the solution of (2.1) and χ_E is the characteristic function of E.

Under the assumptions (2.33)–(2.35) we use Theorem 2 to obtain

COROLLARY 1. *Every optimal control v^* of problem (3.1) is of the form (2.37) if $H^* \chi_E \equiv 0$.*

Now let $S = \{(x, t) \in \Phi;\ t = \xi(x)\}$ be a given C^1-surface and let $H^0 \in C^2(\bar{\Phi})$ be such that $H^0(x, t) = 0$ for $0 \leqq t \leqq \sigma(x)$. Consider the optimization problem:

$$\text{Minimize} \int_{\Phi} (H(x, t) - H^0(x, t))^2\, dx dt \tag{3.2}$$

over all (H, u, v) subject to (1.3)–(1.5) and (2.1).

This is a least squares approach to the controllability problem mentioned in section 1, i.e. to the problem of finding $v \in U$ such that $S = \{(x, t);\ t = \sigma(x)\}$.

We will assume that

$$\begin{cases} \partial H^0/\partial t - \Delta H^0 \leqq f & \text{in} \quad \Phi \\ H^0 \leqq 0 & \text{in} \quad \Sigma \equiv \Sigma_1 \cup \Sigma_2 \\ H^0(x, 0) \leqq 0 & \text{in} \quad \Omega. \end{cases} \tag{3.3}$$

Then by (2.10) we see that

$$(H_\varepsilon - H^0)_t - \Delta (H_\varepsilon - H^0) + \beta^\varepsilon (H_\varepsilon) \geqq 0 \quad \text{in} \quad \Phi$$
$$(H^\varepsilon - H^0) \geqq 0 \quad \text{in} \quad \Sigma \tag{3.4}$$
$$(H^\varepsilon - H^0)(x, 0) \geqq 0 \quad \text{in} \quad \Omega.$$

Multiplying (3.4) by $(H_\varepsilon - H^0)^-$ and integrating over Φ we get that $(H_\varepsilon - H^0)^- = 0$ in Φ. By Lemma 1 we then infer that

$$H^* \geqq H^0 \quad \text{in} \quad \Phi \tag{3.5}$$

for every solution H^* to problem (2.1). Then, by Theorem 2, (see also Remark 1), we have

COROLLARY 2. *Let* (H^*, u^*, v^*) *be any optimal triple for problem* (3.2). *Then, if* $H^* \equiv H^0$ *and assumptions* (3.3) *hold, the optimal control* v^* *has the form* (2.37).

Consider, finally, the optimal control problem

$$\text{Maximize} \int_\Phi \theta(x, t)\, dxdt \tag{3.6}$$

over all (θ, u, v) satisfying (1.1)–(1.5).

In terms of the control system (2.1), the problem (3.6) can be expressed as

$$\text{Maximize} \int_\Omega H(x, T)\, dx \tag{3.7}$$

over all (H, u, v) subject to (1.3)–(1.5) and (2.1).

Then, we have the

COROLLARY 3. *Every optimal control* v^* *of problem* (3.7) *is of the form* (2.37).

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Received, August 1986

Sterowania typu „bang-bang" dla pewnej klasy optymalnych procesów chłodzenia

W pracy rozważany jest problem istnienia optymalnych sterowań brzegowych typu „bang-bang" w przypadku pewnej klasy sterowanych zagadnień parabolicznych ze swobodną granicą. Rozważane zagadnienia należą do klasy jednofazowych zadań Stefana.

Управления типа "банг-банг" для некоторого класса оптимальных процессов охлаждения

Рассматриваются проблемы существования оптимальных краевых управлений типа „банг-банг" для некоторого класса управляемых параболических задач со свободной границей. Рассмотрены проблемы типа однофазных задач Стефана.