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# Imprecise costs in mathematical programming problems

by

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Linear Programming problems in which some vagueness is supposed about the costs taking part in the objective function are considered in this paper. The main aim is to describe such problems and to give a resolution method being suitable for that kind of problems. As auxiliary problem for solving the former one, a multiobjective linear parametric programming problem is found. This auxiliary problem is obtained by means of the concept of preferential cone.

## 1. Introduction

When in order to pose some problem, a decision-maker is asked about the value of a parameter $c$ taking part in the problem, it is usual to obtain answers as: $c$ is big enough, $c$ is greater than $c_0$, $c$ is about $c_1$, etc., that obviously shows some vagueness about the true value of $c$.

To mathematically model the problem, perhaps a crisp version of it may be advantageous because of necessary tools for solving it will be conventional ones. However it seems reasonable to think that if the former problem has a vague formulation (either partial or total) this vagueness be considered in the mathematical version to pose.

Since the seminal paper of R. E. Bellman and L. A. Zadeh [1], fuzzy decision making problems received an outstanding attention. As is known Fuzzy Mathematical Programming problems are a particular case of those ones. In this way, problems with a fuzzy constraint set have been widely studied ([3], [11], [15], [16], [12], [13], [7], ...). But the case in which the costs taking part in the objective function are fuzzy is less developed.

Often decision-maker gives approximate estimates about the true values of the costs rather than the exact values of these. Moreover those estimates can be given with some vagueness as was said at the start. In this case to consider the costs as fuzzy ones may be a very reasonable way to model the former problem.

The main aim of this paper is to solve the Fuzzy Linear Programming (FLP) problem in which the costs are fuzzy ones with trapezoid membership functions. To do it, taking into account some results shown in [2] a Multiobjective Parametric Linear Programming (MPLP) problem will be derived. The parametric solution of this last problem will serve to give a fuzzy solution to the former fuzzy problem.

## 2. Formulation of the problem

Suppose a Mathematical Programming problem with a constraint set in the form

$$X = \{x \in R^n / Ax \leqq b,\ x \geqq 0\}$$

with $A$ and $b$ being matrices $m \times n$ and $m \times l$ respectively.

If decision-maker has a vague information about the costs taking part in the objective function, then it is natural to think he shall prefer to fix, for each cost, an interval containing the more possible values of those, i.e., for decision-maker it will be easier to say that $c_j$ is in interval $[r_j, R_j]$ than to determine the exact value of $c_j$.

Thus it is also reasonable to think that for decision-maker it will be not equal if the value of the cost be near to either the left or the right or the middle of the interval, that is, it is natural to suppose decision-maker can define in each interval taking part in the formulation of the problem a membership function explaining those options,

$$\mu_j : R \to [0, 1], \quad j \in N = \{1, 2, ..., n\}$$

These functions giving the fuzzy mathematical nature of the problem may be graphically described as in Fig. 1, and analytically can be expressed as,

$$\begin{aligned} \mu_j(x) = 0 \quad & \text{if} \quad R_j \geqq x \leqq r_j \\ h_j(x) \quad & \text{if} \quad r_j \leqq x \leqq \underline{c}_j \\ 1 \quad & \text{if} \quad \underline{c}_j \leqq x \leqq \overline{c}_j \\ g_j(x) \quad & \text{if} \quad \overline{c}_j \leqq x \leqq R_j, \quad j \in N \end{aligned} \tag{1}$$

$h_j(\cdot)$ and $g_j(\cdot)$, $j \in N$, being strictly increasing and decreasing continuous functions, respectively.

Hence, the FLP problem stated by decision-maker can be written as,

$$\text{Max}\ \{c_1 x_1 + c_2 x_2 + \dots + c_n x_n / x \in X\} \tag{2}$$

where $c_j$, $j \in N$, are vaguely stated estimates of the coefficients which are defined by means of membership functions like (1). Let it note that eventually some of these functions may be classical characteristic ones because of corresponding cost being non-fuzzy.

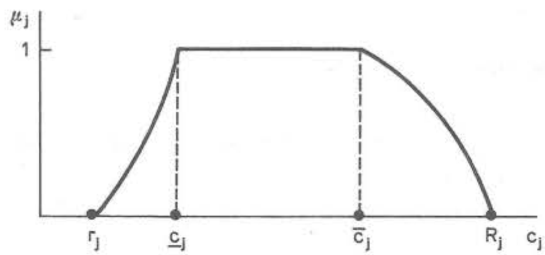


Fig. 1

Now, by taking

$$\forall c \in R^n, \quad c = (c_1, \dots, c_n), \quad \mu(c) = \operatorname*{Inf}_j \mu_j(c_j), \quad j \in N$$

$\mu(\cdot)$ defines a fuzzy objective which induces a fuzzy preorder in $X$ as was shown in [13]. In consequence a fuzzy solution to (2) can be found from the solution of the MPLP problem,

$$\begin{aligned} &\text{Max: } c_1 x_1 + \dots + c_n x_n \\ &\text{s.t.:} \\ &Ax \leqq b \\ &\mu(c) \geqq 1-\alpha \\ &x \geqq 0, \quad c \in R^n, \quad \alpha \in [0, 1] \end{aligned} \tag{3}$$

Taking into account that

$$\mu(c) \geqq 1-\alpha \Leftrightarrow \operatorname*{Inf}_j \mu_j(c_j) \geqq 1-\alpha \Leftrightarrow \mu_j(c_j) \geqq 1-\alpha, \quad j \in N, \quad \alpha \in [0, 1]$$

then, from (1) it is obtained, that

$$\mu_j(c_j) \geqq 1-\alpha \Leftrightarrow h_j^{-1}(1-\alpha) \leqq c_j \leqq g_j^{-1}(1-\alpha), \quad j \in N$$

and denoting $\varphi_j \equiv h_j^{-1}$, $\psi_j \equiv g_j^{-1}$, $j \in N$, problem (3) can be written as,

$$\text{Max}\ \{cx/x \in X,\ \varphi(1-\alpha) \leqq c \leqq \psi(1-\alpha),\ \alpha \in [0, 1]\} \tag{4}$$

where $\varphi(\cdot) = [\varphi_1(\cdot), \dots, \varphi_n(\cdot)]$ and $\psi(\cdot) = [\psi_1(\cdot), \dots, \psi_n(\cdot)]$.

Furthermore if $\Gamma(1-\alpha)$, $\alpha \in [0, 1]$, denotes the set of vectors $c \in R^n$ whose components $c_j$ are in the interval $[\varphi_j(1-\alpha), \psi_j(1-\alpha)]$, $j \in N$, (4) can be finally rewritten as,

$$\text{Max } \{cx/x \in X,\ c \in \Gamma(1-\alpha),\ \alpha \in [0, 1]\} \tag{5}$$

which for each $\alpha \in [0, 1]$ is a Linear Multiobjective problem having in objectives costs that can assume values in respective intervals.

Next section is devoted to development of a possible method of solution to (5) which provides a fuzzy solution to the former problem (2).

## 3. A solution method

Suppose problem (2) and its parametric representation (5). Consider fixed $\alpha \in [0, 1]$. Such a problem will be denoted $M_\alpha$. It is evident that solution of $M_\alpha$ is given by the $x^* \in X$ such that,

$$\forall c \in \Gamma(1-\gamma),\ \forall x \in X \Rightarrow cx^* \geqq cx$$

As the unicity of that $x^* \in X$ is not guaranteed, let $S_{1-\alpha}$ denote the set of those points being solutions of $M_\alpha$,

$$S_{1-\alpha} = \{x^* \in X / \forall c \in \Gamma(1-\alpha),\ x \in X \Rightarrow cx^* \geqq cx\}$$

Let it note that it is possible to obtain $S_{1-\alpha} = \emptyset$ for some $\alpha \in [0, 1]$, but when $\alpha$ runs the unit interval, there is at least one value, e.g. $\alpha = 0$, for which $S_{1-\alpha} \neq \emptyset$, except that the problem be unfeasible.

Thus, in accordance with the Decomposition Theorem for fuzzy sets, it can be defined,

$$S = \bigcup_\alpha \alpha \cdot S_{1-\alpha}$$

which is a fuzzy set giving the fuzzy solution to the former problem (cf. e.g. [10], [12]).

Hence, to round off this approach, it remains to give a method obtaining for each $\alpha \in [0, 1]$ the solution $S_{1-\alpha}$. In order to do it, $M_\alpha$ (always with fixed $\alpha$) shall be rewritten in a more explicit way,

$$\left.\begin{array}{l} \text{Max: } c_1 x_1 + \ldots + c_n x_n \\ \text{s.t.:} \\ \qquad Ax \leqq b \\ \varphi_j(1-\alpha) \leqq c_j \leqq \psi_j(1-\alpha) \\ \qquad x \geqq 0 \end{array}\right\} M_\alpha$$

Thus $M_\alpha$ appears as a Multiobjective Linear Programming problem having one objective by each vector $c \in \Gamma(1-\alpha)$.

Clearly, for any $\alpha \in [0, 1]$, $\Gamma(1-\alpha) \subset R^n$ is a convex set with extremal points defined by those $c \in \Gamma(1-\alpha)$ such that its components are either in the lower or upper bound: $\varphi_j(1-\alpha)$ or $\psi_j(1-\alpha)$, respectively. This characterization was studied in [2], although in a different context (for Linear Multiple Objective Programming problems with interval costs).

Following [2], if

$$V(c) = \{x \in R^n / cx \geqq 0\} \qquad c \in \Gamma(1-\alpha)$$

it is clear that $V(c)$ is a cone.

Likewise,

$$V\,|\Gamma(1-\alpha)| = \bigcup_c V(c) \qquad c \in \Gamma(1-\alpha)$$

is a cone (the preferential cone of $M_\alpha$).

Let $E(1-\alpha) \subset \Gamma(1-\alpha)$ be the subset constituted by vectors whose $j^{\text{th}}$ component is equal to either the upper or the lower bound of $c_j$, $\psi_j(1-\alpha)$ or $\varphi_j(1-\alpha)$, $j \in N$, that is,

$$c = (c_1, \ldots, c_n) \in E(1-\alpha) \Leftrightarrow c_j = \varphi_j(1-\alpha) \text{ or } \psi_j(1-\alpha), \qquad \forall j \in N$$

Then it is evident that the maximum number of elements in $E(1-\alpha)$ is $2^n$. On the other hand, it is verified, [2], that

$$V[\Gamma(1-\alpha)] = \bigcup_c V(c) \qquad c \in E(1-\alpha)$$

what shows the finite subfamily of problems with costs $c \in E(1-\alpha)$, i.e., vectors of costs whose components are only in the upper or lower bound, it may be used to solve $M_\alpha$ rather than the complete (infinite) family involved by the said $M_\alpha$.

In accordance with that, $M_\alpha$ can be solved by means of the problem

$$\begin{aligned} &\text{Max: } c_1 x_1 + \ldots + c_n x_n \\ &\text{s.t.:} \\ &Ax \leqq b \\ &\quad x \geqq 0, \quad c \in E(1-\alpha) \end{aligned} \tag{6}$$

or more explicitly,

$$\begin{aligned} &\text{Max: } (c^1 x, c^2 x, \ldots, c^{2^n} x)' \\ &\text{s.t.:} \\ &Ax \leqq b \\ &x \geqq 0,\ c^k \in E(1-\alpha), \quad k = 1, 2, \ldots, 2^n \end{aligned} \tag{7}$$

which is a conventional Multiobjective Linear Programming problem.

Thus the resolution of all problems in the family $\{M_\alpha,\ \alpha \in [0, 1]\}$ can be carried out using (7) and considering $\alpha \in [0, 1]$ as a parameter, i.e., the fuzzy solution to (2) will be obtained from the solution of the MPLP problem,

$$\begin{aligned} &\text{Max: } (c^1 x, c^2 x, \ldots, c^{2^n} x)' \\ &\text{s.t.:} \\ &Ax \leqq b \\ &x \geqq 0, \quad c^k \in E(1-\alpha) \\ &\alpha \in [0, 1], \quad k = 1, 2, \ldots, 2^n \end{aligned} \tag{8}$$

REMARK. Notice that several approaches can be considered to solve a Multiobjective Linear Programming problem. Since the aim of this paper is to give a mathematical formulation for the former fuzzy problem, this point will be not treated here, it will be the theme of a forthcoming paper.

Finally notice that for (8) there can be defined, in accordance with the Decomposition Theorem for fuzzy sets,

$$V = \bigcup_\alpha \alpha \cdot V\,[\Gamma(1-\alpha)]$$

which, because $V\,[\Gamma(1-\alpha)]$ is a cone, is a fuzzy cone according to the corresponding definition given in [10] and that may be called the Preferential Fuzzy Cone corresponding to (2).

Now all of this is clarified with a concrete problem.

## 4. Numerical example

Consider the following FLP problem with fuzzy objective,

$$\begin{aligned} &\text{Max: } z = c_1 x_1 + 2x_2 \\ &\text{s.t.:} \\ &\qquad 2x_1 - x_2 \leqq 6 \\ &\qquad x_1 + 5x_2 \leqq 10 \\ &\qquad x_1, x_2 \geqq 0 \end{aligned}$$

in which the fuzziness is supposed present in $c_1 \subset R$ only, being measured by the membership function,

$$\mu_1(x)\begin{cases} = x-2 & \text{if} \quad 2 \leqq x \leqq 3 \\ = (5-x)/2 & \text{if} \quad 3 \leqq x \leqq 5 \\ = 0 & \text{elsewhere} \end{cases}$$

Thus $M_\alpha$ will be,

$$\begin{aligned} &\text{Max: } c_1 x_1 + 2x_2 \\ &\text{s.t.:} \\ &\qquad 2x_1 - x_2 \leqq 6 \\ &\qquad x_1 + 5x_2 \leqq 10 \\ &\qquad 3-\alpha \leqq c_1 \leqq 3+2\alpha \\ &\qquad x_1, x_2 \geqq 0 \end{aligned}$$

Then, in this case,

$$E(1-\alpha) = \{(3-\alpha, 2), (3+2\alpha, 2)\} \qquad \alpha \in [0, 1]$$

and in accordance with (7) one has to solve,

$$\begin{aligned} &\text{Max: } [(3-\alpha)x_1 + 2x_2, (3+2\alpha)x_1 + 2x_2] \\ &\text{s.t.:} \\ &\qquad 2x_1 - x_2 \leqq 6 \\ &\qquad x_1 + 5x_2 \leqq 10 \\ &\qquad x_1, x_2 \geqq 0, \quad \alpha \in [0, 1] \end{aligned}$$

which is a MPLP problem.

Solving it by means of the Multicriterion Simplex Method, [14], one can obtain

$$\begin{aligned} x_1^* = 3.64, \quad x_2^* = 1.27 \\ z^*(\alpha) = [13.45 - 3.64\alpha, \ 13.45 + 7.27\alpha] \end{aligned} \tag{9}$$

which is the fuzzy solution to the problem. It is clear that this solution has a membership degree equal to one (The supremum of the respective degrees for which it is a solution).

REMARK. Note that in this particular example, for any $\alpha \in [0, 1]$ (9) is an optimal solution in the classical sense for both objectives. However, this is not a very frequent case and thus, when the membership degree of the optimal solution is $\alpha^* < 1$, one can to look for nondominated solutions in the interval $(\alpha^*, 1]$ by means of a nondominance test as is shown in [14].

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## References

[1] BELLMAN R. E., ZADEH L. A. Decision Making in a Fuzzy Environment. *Man. Sci.* **17** (B), (1970) 4, 141–164.

[2] BITRAN G. R. Linear Multiple Objective Problems with Interval Coefficients. *Man. Sci.* **26** (1980) 7, 694–706.

[3] CHANAS S. Parametric Programming in Fuzzy Linear Programming. *Fuzzy Sets and Systems*, **11** (1983), 243–251.

[4] DELGADO M., VERDEGAY J. L., VILA M. A. Mathematical Programming Problems with Fuzzy Costs. First IFSA Congress. Palma de Mallorca (Spain). July (1985).

[5] GAL T. Postoptimal Analyses, Parametric Programming and Related Topics. McGraw Hill, 1979.

[6] HAMACHER H., LEBERLING H., ZIMMERMANN H. J. Sensitivity Analysis in Fuzzy Linear Programming. *Fuzzy Sets and Systems*, **1** (1978), 269–281.

[7] NEGOITA C. V., RALESCU D. Applications of Fuzzy Sets to Systems Analysis. Birkhauser Verlag, 1975.

[8] ORLOVSKY S. A. On Programming with Fuzzy Constraint Sets. *Kybernetes*, **6** (1977), 197–201.

[9] STEUER R. E. Algorithms for Linear Programming Problems with Interval Objective Function Coefficients. *Math. of Oper. Res.* **6** (1981) 3, 333–348.

[10] TAKEDA E., NISHIDA T. Multiple Criteria Decision Problems with Fuzzy Domination Structures. *Fuzzy Sets and Systems*, **3** (1980), 123–136.

[11] TANAKA H., OKUDA T., ASAI K. On Fuzzy Mathematical Programming. *J. of Cybernetics*, **3** (1974) 4, 37–46.

[12] VERDEGAY J. L. Fuzzy Mathematical Programming. In M. M. Gupta and E. Sanchez (Eds): Fuzzy Information and Decision Processes. North Holland 1982, 231–237.

[13] VERDEGAY J. L. A Dual Approach to Solve the Fuzzy Linear Programming Problem. *Fuzzy Sets and Systems*, **14** (1984), 131–141.

[14] ZELENY M. Multiple Criteria Decision Making. McGraw Hill 1982.

[15] ZIMMERMANN H. J. Description and Optimization of Fuzzy Systems. *Int. J. General Systems*, **2** (1975), 209–215.

[16] ZIMMERMANN H. J. Fuzzy Programming and Linear Programming with Several Objective Functions. *Fuzzy Sets and Systems*, **1** (1977), 445–455.

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## Niedokładne koszty w zadaniach programowania matematycznego

Przedmiotem pracy są zadania programowania liniowego, w których funkcja celu zawiera niepewne co do wartości współczynniki kosztów. Celem artykułu jest analiza takich zadań i podanie metody rozwiązywania odpowiadającej tego typu zadaniom. Otrzymano zadanie wielokryterialnego parametrycznego programowania liniowego jako zadania pomocniczego do rozwiązywania zadania pierwotnego. To zadanie zostało otrzymane przy użyciu pojęcia stożka preferencji.

## Неточные затраты в задачах математического программирования

Предметом работы являются задачи линейного программирования, в которых функция цели содержит сомнительные, в отношении значений, коэффициенты затрат. Целью статьи является анализ таких задач и представление метода решения, удобного для этого типа задач. Получена задача многокритериального параметрического линейного программирования в качестве вспомогательной для решения первичной задачи. Вспомогательная задача была получена при использовании понятия конуса предпочтений.