Control and Cybernetics

VOL. 16 (1987) No. 2

Classes of rational polyhedra

by

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This work analyzes some properties of the lattice polyhedra, in view of possible applications to the optimal choice of the cutting plane in discrete programming.

1. Introduction

The optimal choice of the cutting plane involves a characterization of convex figures containing a minimal number of points of the lattice. An old theorem by Pick connects the area of every integer vertex polygon to the number of internal and boundary integer points of the polygon, and therefore it can be useful as a starting point for a new approach. The theorem by Pick was extended from 2-dimensional to 3-dimensional cases by J. E. Reeve thirty years ago, no attempt for further generalizations to n-dimensional polyhedra is known until now.

Here, Pick's and Reeve's results are used to find classifications and similitudes among 2 and 3-dimensional polyhedra having integer or, more generally, rational vertices, in order to examine the applicative possibilities that generalizations of the above theorems might have.

2. Fundamental notions

2.1. An initial transformation

We shall limit our study to nonnegative coordinate polyhedra with at least a vertex on every axis; it is easy to obtain such polyhedra using the transformation

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* Sponsored by the GNAFA of the National Council of Research. The author is also member of the Dept. of Mathematics, Statistics, Informatics and Applications, University of Bergamo, Italy.

$$\bar{x} = x - a, \quad \text{where} \quad a_i = \min_{x \in \text{polyhedron}} x_i \quad (i = 1, \ldots, n) \tag{1}$$

We can divide all polyhedra into equivalence classes, treating all elements obtainable by the transformation into the same polyhedron as being in the same class. All polyhedra of each class can be ordered according to a (for instance by increasing norm and, for equal norm, lexicographically). In particular, all rational polyhedra of the same class can be numbered by combining such ordering with the classic one of rational numbers.

The quotient set defined by the above transformation is composed of three subsets:

A = polyhedra whose vertex coordinates are all integer;

B = polyhedra whose vertex coordinates are all rational, and at least one not integer;

C = complementary set.

We shall consider only A and B.

2.2. A contraction for integer polyhedra

Let $P \in A$ and m be the maximum common divisor of the coordinates of the vertices of P. The contraction $x^1 = x/m$ transforms P into the polyhedron $P^1 \in A$, similar to P. P^1 is "transformed into the maximum network", in the sense that no network with a basis greater than 1 exists, so that all vertices of P^1 belong to it. Therefore we can divide A into equivalence classes; we shall choose as representative for every class the element P^1 having the maximum common divisor of the vertex coordinates equal to one.

2.3. A dilatation for rational polyhedra

Let $P \in B$, and d be the minimum common denominator of the coordinates (at minimal terms) of the vertices of P. The dilatation $x^1 = dx$ transforms P into the polyhedron $P^1 \in A$ similar to P. P^1 is "transformed into the maximum network", according to section 2.2. In particular, B is so divided into equivalence classes; P^1 can be taken (improperly, but pragmatically) as representative of every class.

2.4. A numeration

A numeration of the polyhedra of A and B will be given here only for the representatives of the equivalence classes introduced in the above sections; inside each class the integer m, or d, will provide the sub-numeration identifying the polyhedron. Besides, the numeration will be effected merely for convex polyhedra, since knowledge of their vertices is enough for their identification.

Consider a generic permutation of the v vertices of a convex representative polyhedron, and write the following ordered sequence of nv elements: the n coordinates of the first vertex; the n coordinates of the second one, and so on until the last vertex. The minimal sequence, in lexicographic order, for all possible vertex permutations, is chosen as the representative sequence of the polyhedron. We order polyhedra according to the maximum number belonging to the representative sequence; when equal, lexicographically. Such ordering induces a numeration, which can be generated by a simple algorithm. Number 1 is assigned to the simplex having a vertex in the origin and all other vertices in the unitary points of axes; the first hypercube has a vertex with all coordinates equal 1, all other vertices being those of the first simplex.

3. Some relations among similar polyhedra

Consider a network R with coordinates $(x_1/r, \ldots, x_n/r)$ where $x_1, \ldots, x_n$ are integers and r positive integer. Let P and P' be two generic similar polyhedra, all vertices belonging to R. We call $V_{i,h}$ the hypervolume of the i-th h-dimensional face of P; $p_{i,h,r}$ the number of points of R internal to the i-th face (a numbering for the faces can be that of representative sequences, as introduced in the preceding section).

In particular, $h = 1$ corresponds to an open edge; $h = n$ to the whole open polyhedron. We call $p_{h,r}$ the number of points of R belonging to all faces of P having a dimension not greater than h. We call d the minimum common divisor of the coordinates (at minimal terms) of $\bar{P}$, which is the transform of P in the terms of transformation of section 2. We adopt corresponding symbols for P' and for P^1, which is the common polyhedron obtained from the collineation given through definition of dilatation on P and P'. We call v, e and f respectively the numbers of the vertices, edges and faces of the polyhedron. The similitude of P and P' yields, for all i, h, r:

$$V'_{i,h} = \left(\frac{d}{d'}\right)^h V_{i,h} \tag{2}$$

Observe that all points of R over every straight line are mutually equidistant, with minimal distance c depending only on the trend of the line. Such c is then equal for similar polyhedra (for a more detailed analysis see [1, theorems 2 and 3]). Therefore, the measures of the i-th edge of P and P' are:

$$V_{i,1} = c\,(p_{i,1,r}+1); \qquad V'_{i,1} = c\,(p'_{i,1,r}+1). \tag{3}$$

It follows, for every i-th edge of the polyhedron, that:

$$(p'_{i,1,r}+1) = \frac{d}{d'}\,(p_{i,1,r}+1) \tag{4}$$

Observe that the number of points of R over all edges of P' is:

$$p'_{1,r} = v + \sum_{i=1}^{e} p'_{i,1,r} \tag{5}$$

This new connection between P and P' follows:

$$p'_{1,r} = v - e + \frac{d}{d'}\,(p_{1,r} - (v-e)) \tag{6}$$

In particular, when $P = P'$, the above relations hold among polygon and their representatives. In the same way, using $d = 1/m$, the connections among integer polyhedra of section 2.2 can be deduced.

4. Two particular cases

4.1. $n = 2$

An interesting particularization concerns the case of $n = 2$, i.e. of rational coordinate polygona. We can make the connection between the transform and representative of these polyhedra deeper, and in general between polygona of the same class.

Firstly, we observe that (6) becomes, since $v = e$:

$$p'_{1,r} = \frac{d}{d'}\,p_{1,r} \tag{7}$$

i.e. the numbers of points of R over the boundaries of P and P' are in similitude ratio d/d' (*). We can thus generalize the theorem of Pick [2, p. 378] for the network with basis $1/r$:

$$V_{1,2} = \frac{1}{r^2}\left(\frac{p_{1,r}}{2} + p_{1,2,r} - 1\right) \tag{8}$$

Such connection between the number of internal and boundary lattice

(*) this result and the previous one were obtained by G. Stocco in [3] for similar polygons in the 2-dimensional lattice of integers, and have given the present author the idea for the generalization here contained.

points, and the area of a rational polygon leads to another relation for similar polygons:

$$p'_{1,2,r} = r^2 \left(\frac{d}{d'}\right)^2 V_{1,2} - \frac{d}{d'} \frac{p_{1,r}}{2} + 1 \tag{9}$$

which expresses the number of internal points of P' as a function of the similitude ratio, of the area of P, and of the number of points of R which belong to the boundary of P.

4.2. $n = 3$

It is known that, when $n = 3$, $f + v = e + 2$. Thus, the formula (6) of section 3, related to edges, can be expressed for similar 3-dimensional polyhedra as follows:

$$p'_{1,r} = 2 - f + \frac{d}{d'}(p_{1,r} - (2 - f)) \tag{10}$$

Recall the notations:

$p_{2,r}$ is the number of points of R which belong to the boundary of P;
$p_{3,r}$ is the number of points of R which belong to P;
$p_{1,3,r}$ is the number of points of R which are internal to P.
Of course, $r = 1$ refers to the network of integers. The theorem of Pick has been generalized by J. E. Reeve to 3-dimensional figures. A theorem of [2, p. 382] expresses the volume $V_{1,3}$ of every convex polyhedron as follows:

$$V_{1,3} = (2(p_{3,r} - rp_{3,1}) - (p_{2,r} - rp_{2,1}))/2(r-1)r(r+1) \tag{11}$$

Besides, the following relation holds:

$$p_{2,r} = r^2 p_{2,1} + 2(1 - r^2) \tag{12}$$

This expression of the volume, as shown in [2], always requires information on a sublattice as well as on the integer lattice. In fact, the volumes of polyhedra are not uniquely determined only by the internal and boundary integer points. For instance, a tetrahedron having vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, c)$, where c is a positive integer, has always 0 internal and 4 boundary integer points, but its volume depends on c.

The volume $V_{1,3}$ can also be expressed as a function of the number of the points of the network internal to P as follows:

$$V_{1,3} = (2(p_{1,3,r} - rp_{1,3,1}) + (p_{2,r} - rp_{2,1}))/2(r-1)r(r+1) \tag{13}$$

Therefore, relation (9) becomes, for $n = 3$:

$$p'_{1,3,r} = rp'_{1,3,1} + (1-r)\left(\frac{r}{2}\, p'_{2,1} - (r+1)\left(\left(\frac{d}{d'}\right)^3 V_{1,3}\, r + 1\right)\right)$$

5. Remaining problems

We can observe that all relations given in 3. and 4. are valid for every value of r, until the set of points with rational coordinates is exhausted.

As seen above, the theorems by Pick and Reeve can be used to find new relations between similar polyhedra; such relations could be applied to the optimal choice of the cutting plane using the transformed representatives. Therefore, a trial of generalization of the above theorems for $n > 3$ appears justified for applicative developments.

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Received, January 1985

Klasy wielościanów rzeczywistych

W artykule analizowane są właściwości wielościanów rzeczywistych pod kątem możliwości zastosowań do optymalnego wyboru płaszczyzny odcinającej w programowaniu dyskretnym.

Классы действительных многогранников

В статье анализуруются свойства действительных многогранников с точки зрения применений к оптимальному выбору секущей плоскости в дискретном программировании.