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Two-phase Stefan problems with nonlinear boundary conditions described by time-dependent subdifferentials

by

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In the paper, a one-dimensional two-phase Stefan problem with nonlinear conditions imposed on the fixed boundary is considered. The boundary conditions comprehend time-dependent subdifferential operators. Theorems on the local in time existence and uniqueness of solutions are proved. The exposed construction of the local solution exploits methods of the theory of nonlinear evolution equations with time-dependent subdifferential operators in Hilbert space. The Stefan problem under consideration is transformed to the form of such a system of evolution equations.

Introduction

In the previous papers [8, 9] of the author one-phase Stefan problems with boundary conditions described by time-dependent subdifferential operators were studied. The present paper is concerned with two-phase Stefan problems with the same type of nonlinear boundary conditions on the fixed boundary as in the one-phase case.

Our problem is to find a curve $x = l(t)$, $0 < l < 1$, on $[0, T]$ and a function $u = u(t, x)$ on $[0, T]\times[0, 1]$ such that

$$\begin{cases} \varrho_1(u)_t - u_{xx} = 0 \text{ in } Q_l^1(T) = \{(t, x);\ 0 < x < l(t),\ 0 < t < T\}, \\ \varrho_2(u)_t - u_{xx} = 0 \text{ in } Q_l^2(T) = \{(t, x);\ l(t) < x < 1,\ 0 < t < T\}, \\ u(0, x) = u_0(x) \text{ for } 0 \leqslant x \leqslant 1, \\ u_x(t, 0+) \in \partial b_1^t(u(t, 0)) \text{ for } 0 < t < T, \\ -u_x(t, 1-) \in \partial b_2^t(u(t, 1)) \text{ for } 0 < t < T, \\ u(t, l(t)) = 0 \text{ for } 0 \leqslant t \leqslant T \end{cases} \tag{0.1}$$

and

$$\begin{cases} l'(t) = -u_x(t, l(t)-) + u_x(t, l(t)+) & \text{for } 0 < t < T, \\ l(0) = l_0, \end{cases} \tag{0.2}$$

where $0 < T < \infty$ and $0 < l_0 < 1$ are given numbers, u_0 is a given function on $[0, 1]$; $l' = dl/dt$; $\varrho_i = \varrho_i(.)$: $R \to R$, $i = 1, 2$, are given functions which are bi-Lipschitz continuous and increasing on R, and $\varrho_i(0) = 0$; $b_i^t(.)$, $t \in [0, T]$, $i = 1, 2$, are given proper lower semicontinuous and convex functions on R.

The Stefan problem has been discussed by many authors in case of various boundary conditions on the fixed boundary (e.g., [3, 4, 7, 11, 13, 14, 15]). As far as two-phase Stefan problems with flux conditions described by subdifferential operators are concerned, Yotsutani [14, 15] has treated the case when b_i^t, $i = 1$, 2, are independent of time, i.e., $b_i^t(.) = b_i(.)$ and ϱ_i are linear on R; an approximate difference method was employed there for the construction of a local solution, as well as the strong maximum principle for linear heat equations was used in the proof of the uniqueness of solutions. Subsequently, it was shown by Magenes, Verdi and Visintin [11] that the same type of Stefan problems in several space variables can be treated in the framework of the theory of nonlinear semigroups in L^1-spaces (cf., Bénilan [1]).

The system $\{(0.1), (0.2)\}$ is more general than that dealt with in [14, 15] in some respects, such as u being is governed by quasi-linear heat equations of the form $\varrho(u)_t - u_{xx} = 0$ and boundary fluxes $u_x(t, 0+)$, $u_x(t, 1-)$ being controlled by time-dependent subdifferentials $\partial b_1^t(.)$, $\partial b_2^t(.)$, respectively. The purpose of the present paper is to establish a local existence theorem and a uniqueness theorem for $\{(0.1), (0.2)\}$. The construction of a local solution is made by using some results (cf., Kenmochi [5, 6]) in the theory of nonlinear evolution equations involving time-dependent subdifferential operators in Hilbert spaces; in fact, (0.1) is reformulated as a system of nonlinear evolution equations of the form

$$v_i'(t) + \partial\phi_i^t(B_i v_i(t)) \ni 0, \quad 0 < t < T, \quad v_i(0) = v_{i,0}, \quad i = 1, 2, \tag{0.3}$$

where $\partial\phi_i^t$, $i = 1, 2$, are the subdifferentials of proper, lower semicontinuous convex functions ϕ_i^t on $L^2(0, 1)$ and $B_i = \varrho_i^{-1}$ are maximal monotone operators in $L^2(0, 1)$. We construct a local solution to $\{(0.3), (0.2)\}$ rather than to $\{(0.1), (0.2)\}$. The uniqueness of solutions can be shown as a direct consequence of a comparison result for solutions of $\{(0.1), (0.2)\}$.

Notations. For a general (real) Banach space V we denote by $|.|_V$ the norm in V. When V is a Hilbert space, we denote by $(.,.)_V$ the inner product in V.

Let V be a Hilbert space and let ϕ be a proper (i.e. $-\infty < \phi \leq \infty$, $\phi \not\equiv \infty$ on V), l.s.c. (lower semicontinuous) and convex function on V. Then we set

$$D(\phi) = \{z \in V;\ \phi(z) < \infty\}$$

and we refer to Brézis [2] for the definition of subdifferential $\partial\phi$ to ϕ in V and for its general properties.

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1. Two-phase Stefan problems and main results

Given two positive numbers C_1, C_2, we denote by $\Gamma(C_1, C_2)$ the set of all functions $\varrho\colon R \to R$ such that

$$\varrho(0) = 0$$

and

$$C_1(r-r') \leqslant \varrho(r) - \varrho(r') \leqslant C_2(r-r') \text{ for any } r, r' \in R \text{ with } r \leqslant r', \tag{1.1}$$

i.e., $\Gamma(C_1, C_2)$ is a class of strictly increasing and bi-Lipschitz continuous functions on R, vanishing at 0. Given $0 < T < \infty$ and $\alpha_0 \in W^{1,2}(0, T)$, $\alpha_1 \in W^{1,1}(0, T)$, we also denote by $B(\alpha_0, \alpha_1)$ the class of all families $\{b^t(.);\ 0 \leqslant t \leqslant T\}$ of proper, l.s.c. convex functions on R satisfying the following property (1.2):

$$\left\{\begin{array}{l}\text{For any } 0 \leqslant s \leqslant t \leqslant T \text{ and any } r \in D(b^s) \text{ there is } \tilde{r} \in D(b^t) \text{ such that}\\ \qquad |\tilde{r} - r| \leqslant |\alpha_0(t) - \alpha_0(s)|\,(1 + |r| + |b^s(r)|^{1/2})\\ \text{and}\\ \qquad b^t(\tilde{r}) - b^s(r) \leqslant |\alpha_1(t) - \alpha_1(s)|\,(1 + |r|^2 + |b^s(r)|).\end{array}\right. \tag{1.2}$$

DEFINITION 1.1. Let $\varrho_i \in \Gamma(C_1, C_2)$ and $\{b_i^t\} \in B(\alpha_0, \alpha_1)$, $i = 1, 2$, let $u_0 \in W^{1,2}(0, 1)$ and l_0 be a number with $0 < l_0 < 1$. Then $P = P(\varrho_1, \varrho_2; \{b_1^t\}, \{b_2^t\};\ u_0, l_0)$ on $[0, T_0]$, $0 < T_0 \leqslant T$, is the problem of finding

$$x = l(t) \text{ on } [0, T_0]$$

and

$$u = u(t, x) \text{ on } [0, T_0] \times [0, 1]$$

such that

$$l \in W^{1,2}(0, T_0), \quad 0 < l < 1 \text{ on } [0, T_0],$$

$$u \in W^{1,2}(0, T_0;\ L^2(0, 1)) \cap L^\infty(0, T_0;\ W^{1,2}(0, 1)) (\subset C([0, T_0] \times [0, 1])) \tag{1.3}$$

and such that

$$\varrho_1(u)_t - u_{xx} = 0 \quad \text{in } Q_l^1(T_0) = \{(t, x);\ 0 < x < l(t),\ 0 < t < T_0\}, \tag{1.4}$$

$$\varrho_2(u)_t - u_{xx} = 0 \quad \text{in } Q_l^2(T_0) = \{(t, x);\ l(t) < x < 1,\ 0 < t < T_0\}, \tag{1.5}$$

$$u(0, .) = u_0 \text{ on } [0, 1], \tag{1.6}$$

$$u_x(t, 0+) \in \partial b_1^t(u(t, 0)) \quad \text{for a.e. } t \in [0, T_0], \tag{1.7}$$

$$-u_x(t, 1-) \in \partial b_2^t(u(t, 1)) \quad \text{for a.e. } t \in [0, T_0], \tag{1.8}$$

$$u(t, l(t)) = 0 \quad \text{for } t \in [0, T_0], \tag{1.9}$$

$$l'(t) = -u_x(t, l(t)-) + u_x(t, l(t)+) \quad \text{for a.e. } t \in [0, T_0], \tag{1.10}$$

$$l(0) = l_0. \tag{1.11}$$

In the above definition, since

$$u_{xx}(t, .) = \varrho_1(u)_t(t, .) \in L^2(0 < x < l(t)) \text{ for a.e. } t \in [0, T_0],$$

$u_x(t, x)$ is absolutely continuous in $x \in [0, l(t)]$ and $u_x(t, 0+)$ as well as $u_x(t, l(t)-)$ exist for a.e. $t \in [0, T_0]$. Similarly, $u_x(t, 1-)$ and $u_x(t, l(t)+)$ exist for a.e. $t \in [0, T_0]$. Thus (1.7), (1.8) and (1.10) make sense.

The first main result is concerned with the local existence of a solution of P.

THEOREM 1.1. *Let* $\varrho_i \in \Gamma(C_1, C_2)$ *and* $\{b_i^t\} \in B(\alpha_0, \alpha_1)$ *for* $i = 1, 2$, *let* $u_0 \in W^{1,2}(0, 1)$ *and* $0 < l_0 < 1$ *such that*

$$u_0(l_0) = 0, \; u_0(0) \in D(b_1^0) \text{ and } u_0(1) \in D(b_2^0). \tag{1.12}$$

Then there exists a positive number $T_0 \leqslant T$ *such that problem* P *has at least one solution* $\{u, l\}$ *on* $[0, T_0]$, *satisfying*

$$b_i^{(\cdot)}(u(., i-1)) \text{ is bounded on } [0, T_0], \; i = 1, 2.$$

The uniqueness of solution is a direct consequence of a comparison result for solutions of P. Before formulating the result we recall a notion of order for convex functions on R. Given two proper, l.s.c. convex functions $b_1(.)$, $b_2(.)$ on R, we use the symbol "$b_1 \preccurlyeq b_2$ on R" to indicate that

$$b_1(r_1 \wedge r_2) + b_2(r_1 \vee r_2) \leqslant b_1(r_1) + b_2(r_2) \text{ for any } r_1, r_2 \in R, \tag{1.13}$$

where $r_1 \wedge r_2 = \min\{r_1, r_2\}$ and $r_1 \vee r_2 = \max\{r_1, r_2\}$. It is not difficult to see that (1.13) implies

$$(r_1^* - r_2^*)(r_1 - r_2)^+ \geqslant 0 \text{ for any } r_i^* \in \partial b_i(r_i), \; i = 1, 2. \tag{1.14}$$

Clearly, $b_1 \preccurlyeq b_2$ on R if $b_1 = b_2$.

THEOREM 1.2 *Let* $\varrho_i \in \Gamma(C_1, C_2)$, $\{b_i^t\}$, $\{\hat{b}_i^t\} \in B(\alpha_0, \alpha_1)$, $i = 1, 2$, $u_0, \hat{u}_0 \in W^{1,2}(0, 1)$ and l_0, $\hat{l}_0 \in (0, 1)$. *Suppose that*

$$\begin{cases} u_0 \geqslant 0 \text{ on } [0, l_0], \; u_0 \leqslant 0 \text{ on } [l_0, 1], \\ \hat{u}_0 \geqslant 0 \text{ on } [0, \hat{l}_0], \; \hat{u}_0 \leqslant 0 \text{ on } [\hat{l}_0, 1], \end{cases} \tag{1.15}$$

$$\begin{cases} \partial b_1^t(r) \subset (-\infty, 0] & \text{for any } r < 0 \text{ and } t \in [0, T], \\ \partial b_2^t(r) \subset [0, \infty) & \text{for any } r > 0 \text{ and } t \in [0, T], \end{cases} \tag{1.16}$$

$$\begin{cases} \partial\hat{b}_1^t(r) \subset (-\infty, 0] & \text{for any } r < 0 \quad \text{and } t \in [0, T], \\ \partial\hat{b}_2^t(r) \subset [0, \infty) & \text{for any } r > 0 \quad \text{and } t \in [0, T], \end{cases} \tag{1.16$'$}$$

and

$$b_i^t \preccurlyeq \hat{b}_i^t \ \textit{on } R \textit{ for any } t \in [0, T] \textit{ and } i = 1, 2.$$

Also, let $\{u, l\}$ *and* $\{\hat{u}, \hat{l}\}$ *be solutions of* $P = P(\varrho_1, \varrho_2; \{b_1^t\}, \{b_2^t\}; u_0, l_0)$ *and* $\hat{P} = P(\varrho_1, \varrho_2; \{\hat{b}_1^t\}, \{\hat{b}_2^t\}; \hat{u}_0, \hat{l}_0)$ *on* $[0, T_0]$, $0 < T_0 \leqslant T$, *respectively. Then,*

$$\begin{aligned} &\frac{d}{dt}|(v(t,.)-\hat{v}(t,.))^+|_{L^1(0,1)} + \frac{d}{dt}(l(t)-\hat{l}(t))^+ \\ &+(u_x(t, 0+)-\hat{u}_x(t, 0+))\sigma_0([u(t, 0)-\hat{u}(t, 0)]^+) \\ &-(u_x(t, 1-)-\hat{u}_x(t, 1-))\sigma_0([u(t, 1)-\hat{u}(t, 1)]^+) \leqslant 0 \\ &\qquad\qquad\qquad\qquad \text{for a.e. } t \in [0, T_0], \end{aligned} \tag{1.17}$$

where

$$v = \begin{cases} \varrho_1(u) & \text{on } \overline{Q_l^1(T_0)}, \\ \varrho_2(u) & \text{on } \overline{Q_l^2(T_0)}, \end{cases} \qquad \hat{v} = \begin{cases} \varrho_1(\hat{u}) & \text{on } \overline{Q_{\hat{l}}^1(T_0)}, \\ \varrho_2(\hat{u}) & \text{on } \overline{Q_{\hat{l}}^2(T_0)} \end{cases}$$

and σ_0: $R \to R$ *is the function defined by*

$$\sigma_0(r) = \begin{cases} 1 & \text{for } r > 0, \\ 0 & \text{for } r = 0, \\ -1 & \text{for } r < 0. \end{cases}$$

COROLLARY 1.1. *Under the same assumptions and with the same notations as in Theorem 1.2 we have*

$$\begin{aligned} &|(v(t,.)-\hat{v}(t,.))^+|_{L^1(0,1)} + (l(t)-\hat{l}(t))^+ \\ &\leqslant |(v(s,.)-\hat{v}(s,.))^+|_{L^1(0,1)} + (l(s)-\hat{l}(s))^+ \quad \text{for any } 0 \leqslant s \leqslant t \leqslant T_0. \end{aligned} \tag{1.18}$$

In particular, if $u_0 \leqslant \hat{u}_0$ *on* $[0, 1]$ *and* $l_0 \leqslant \hat{l}_0$, *then* $u \leqslant \hat{u}$ *on* $[0, T_0] \times [0, 1]$ *and* $l \leqslant \hat{l}$ *on* $[0, T_0]$.

In fact, by (1.13) (cf., (1.14)) we have

$$(u_x(t, 0+))-\hat{u}_x(t, 0+))\sigma_0([u(t, 0)-\hat{u}(t, 0)]^+) \geqslant 0$$

and

$$-(u_x(t, 1-)-\hat{u}_x(t, 1-))\sigma_0([u(t, 1)-\hat{u}(t, 1)]^+) \geqslant 0$$

for a.e. $t \in [0, T_0]$. Hence we obtain (1.18) from (1.17).

COROLLARY 1.2. *In Theorem* 1.2, *problem P has at most one solution on* $[0, T_0]$.

We now note that expressions (1.7), (1.8) represent various types of boundary conditions. Especially the following two examples are of interest.

EXAMPLE 1.1. Let a_i be positive numbers, and $g_i, h_i \in W^{1,2}(0, T)$ with $g_i < h_i$ on $[0, T]$ for $i = 1, 2$. Set for each $t \in [0, T]$ and $i = 1, 2$,

$$b_i^t(r) = \begin{cases} -a_i(r - g_i(t)) & \text{for } r < g_i(t), \\ 0 & \text{for } g_i(t) \leqslant r \leqslant h_i(t), \\ a_i(r - h_i(t)) & \text{for } r > h_i(t). \end{cases}$$

Then $\{b_i^t\} \in B(\alpha_0, \alpha_1)$ for suitable functions $\alpha_0 \in W^{1,2}(0, T)$ and $\alpha_1 \in W^{1,1}(0, T)$, and (1.7) is written in the following form:

$$u_x(t, 0+) = -a_1 \quad \text{for a.e. } t \in \{t;\ u(t, 0) < g_1(t)\},$$

$$u_x(t, 0+) \in [-a_1, 0] \quad \text{for a.e. } t \in \{t;\ u(t, 0) = g_1(t)\},$$

$$u_x(t, 0+) = 0 \quad \text{for a.e. } t \in \{t;\ g_1(t) < u(t, 0) < h_1(t)\},$$

$$u_x(t, 0+) \in [0, a_1] \quad \text{for a.e. } t \in \{t;\ u(t, 0) = h_1(t)\},$$

$$u_x(t, 0+) = a_1 \quad \text{for a.e. } t \in \{t;\ u(t, 0) > h_1(t)\}.$$

Also, (1.8) is similarly written. Moreover, if $h_1 \geqslant 0$ and $g_2 \leqslant 0$ on $[0, T]$, then condition (1.16) is satisfied.

EXAMPLE 1.2. Let g_i, h_i $(i = 1, 2)$ be as in Example 1.1 and set

$$b_i^t(r) = \begin{cases} 0 & \text{for } g_i(t) \leqslant r \leqslant h_i(t), \\ \infty & \text{otherwise}. \end{cases}$$

These are the limits of the functions considered in the above example as $a_i \to \infty$, $i = 1, 2$. It is not difficult to see that $\{b_i^t\} \in B(\alpha_0, \alpha_1)$ for suitable $\alpha_0 \in W^{1,2}(0, T)$ and $\alpha_1 \in W^{1,1}(0, T)$, and (1.6) is satisfied, provided $h_1 \geqslant 0$ and $g_2 \leqslant 0$ on $[0, T]$.

2. Variational formulation for P

For simplicity we use the following notations:

$$H = L^2(0, 1), \qquad X = W^{1,2}(0, 1)(\subset C([0, 1]))$$

and

$$\Lambda_T = \{l \in C([0, T]);\ 0 < l < 1 \text{ on } [0, T]\}.$$

In order to reformulate problem P as a system of nonlinear evolution equations involving time-dependent subdifferential operators in H, we consider the following functions $\phi_i^t(.) = \phi_i^t(\{b^t\}, l; .)$, $i = 1, 2$, on H associated with $\{b^t\} \in B(\alpha_0, \alpha_1)$, $l \in \Lambda_T$ and $t \in [0, T]$:

$$\phi_i^t(z) = \begin{cases} \frac{1}{2}|z_x|_H^2 + b^t(z(i-1)) & \text{if } z \in K_i(t), \\ \infty & \text{if } z \in H \setminus K_i(t), \end{cases} \tag{2.1}$$

where $K_i(t) = K_i(\{b^t\}, l; t)$, $i = 1, 2$, are the sets defined by

$$K_1(t) = \{z \in X;\ z = 0 \text{ on} [l(t), 1],\ z(0) \in D(b^t)\}$$

and

$$K_2(t) = \{z \in X;\ z = 0 \text{ on } [0, l(t)],\ z(1) \in D(b^t)\}.$$

LEMMA 2.1 (cf., [8; *Lemmas* 1.1, 1.2])

(1) *There is a constant* $R_1 = R_1(\alpha_0, \alpha_1, T)$ *such that* $b^t(r) + R_1|r| + R_1 \geqslant 0$ *and hence*

$$|b^t(r)| \leqslant b^t(r) + 2R_1|r| + 2R_1$$

for any $\{b^t\} \in B(\alpha_0, \alpha_1)$, $t \in [0, T]$ *and* $r \in R$.

(2) *For each* $\{b^t\} \in B(\alpha_0, \alpha_1)$, $l \in \Lambda_T$ *and* $t \in [0, T]$, $\phi_i^t(.) = \phi_i^t(\{b^t\}, l; .)$ *is a proper, l.s.c. convex function on* H *and* $D(\phi_i^t) = K_i(\{b^t\}, l; t)$ *for* $i = 1, 2$.

(3) *There is a non-negative constant* $R_2 = R_2(\alpha_0, \alpha_1, T)$ *such that*

$$|b^t(z(i-1))| \leqslant \phi_i^t(z) + R_2$$

and

$$\tfrac{1}{4}|z_x|_H^2 \leqslant \phi_i^t(z) + R_2$$

for any $\{b^t\} \in B(\alpha_0, \alpha_1)$, $l \in \Lambda_T$, $t \in [0, T]$, $z \in K_i(t)$ *and* $i = 1, 2$, *where* $\phi_i^t = \phi_i^t(\{b^t\}, l; .)$ *and* $K_i(t) = K_i(\{b^t\}, l; t)$.

(4) *For each* $\{b^t\} \in B(\alpha_0, \alpha_1)$, $l \in \Lambda_T$ *and* $t \in [0, T]$, *the subdifferential* $\partial\phi_i^t$ *of* $\phi_i^t(.) = \phi_i^t(\{b^t\}, l; .)$ $(i = 1, 2)$ *in* H *is given as follows:* $z^* \in \partial\phi_1^t(z)$ *(resp.* $z^* \in \partial\phi_2^t(z)$*) if and only if* $z^* \in H$, $z \in X$ *and*

$$\begin{cases} -z_{xx} = z^* \text{ in } (0, l(t)) \text{ (resp., } (l(t), 1)), \\ z = 0 \text{ on } [l(t), 1] \text{ (resp., } [0, l(t)]), \text{ and} \\ z_x(0+) \in \partial b^t(z(0)) \ \big(\text{resp., } -z_x(1-) \in \partial b^t(z(1))\big). \end{cases}$$

Now, let $\varrho \in \Gamma(C_1, C_2)$, $\{b^t\} \in B(\alpha_0, \alpha_1)$, and fix them for the moment. We then consider $\phi_i^t = \phi_i^t(\{b^t\}, l; .)$ for each $l \in \Lambda_T$ and $i = 1, 2$, and the operator B: $D(B) = H \to H$ defined by

$$[Bz](x) = \varrho^{-1}(z(x)) \quad \text{for } z \in H \text{ and } x \in (0, 1). \tag{2.2}$$

Given $l \in \Lambda_T$, $g \in L^2(0, T; H)$ and $v_0 \in H$, we denote by $CP_i(l, g, v_0)$ (or

$CP_i(\varrho, \{b^t\}; l, g, v_0)$ when ϱ and $\{b^t\}$ are necessary to be indicated) the Cauchy problem on $[0, T]$:

$$v'(t)+\partial\phi_i^t(Bv(t))\ni g(t),\ 0<t<T,\ v(0)=v_0.$$

By a solution of $CP_i(l, g, v_0)$ we mean a function v: $[0, T]\to H$ such that

$$\begin{gathered} v\in W^{1,2}(0, T; H)(\subset C([0, T]; H)), \\ v(0)=v_0, \end{gathered} \tag{2.3}$$

$$\begin{gathered} \phi_i^{(\cdot)}(B(.)) \text{ is bounded on } [0, T], \text{ and} \\ g(t)-v'(t)\in\partial\phi_i^t(Bv(t)) \quad \text{for a.e. } t\in[0, T]. \end{gathered} \tag{2.4}$$

LEMMA 2.2. *Let* $\varrho_i\in\Gamma(C_1, C_2)$, $\{b_i^t\}\in B(\alpha_0, \alpha_1)$, $i=1, 2$, $u_0\in X$ *and* $0<l_0<1$. *Assume* (1.12) *holds. Also, let* $l\in\Lambda_T\cap W^{1,2}(0, T)$ *with* $l(0)=l_0$, *and put*

$$v_{1,0}=\begin{cases}\varrho_1(u_0) & \text{on } [0, l_0],\\ 0 & \text{on } (l_0, 1],\end{cases}\qquad v_{2,0}=\begin{cases}0 & \text{on } [0, l_0],\\ \varrho_2(u_0) & \text{on } (l_0, 1].\end{cases} \tag{2.5}$$

Further, let v_i *be a solution of* $CP_i(\varrho_i, \{b_i^t\}; l, 0, v_{i,0})$ *on* $[0, T_0]$, $0<T_0\leqslant T$, *for* $i=1, 2$. *Then,*

$$u=B_1v_1+B_2v_2 \tag{2.6}$$

satisfies (1.3) *and solves* (1.4)–(1.9), *where* B_i, $i=1, 2$, *is the operator in* H *given by* (2.2) *with* $\varrho=\varrho_i$. *Conversely, if a function* u: $[0, T_0]\to H$ *satisfies* (1.3)–(1.9), *then*

$$v_1=\begin{cases}\varrho_1(u) & \text{on } \overline{Q_l^1(T_0)},\\ 0 & \text{on } [0, T_0]\times[0, 1]\setminus\overline{Q_l^1(T_0)}\end{cases}$$

and

$$v_2=\begin{cases}0 & \text{on } [0, T_0]\times[0, 1]\setminus\overline{Q_l^2(T_0)},\\ \varrho_2(u) & \text{on } \overline{Q_l^2(T_0)},\end{cases}$$

are solutions of $CP_1(\varrho_1, \{b_1^t\}; l, 0, v_{1,0})$ *and* $CP_2(\varrho_2, \{b_2^t\}; l, 0, v_{2,0})$ *on* $[0, T_0]$, *respectively.*

Proof. Assuming that v_i is a solution $CP_i(\varrho_i, \{b_i^t\}; l, 0, v_{i,0})$ on $[0, T_0]$ for $i=1, 2$, we see from (2.3) and (2.4) with the help of Lemma 2.1(3) that u given by (2.6) satisfies (1.3). Besides, on account of Lemma 2.1(4), it solves (1.4)–(1.9). Similarly we can show the converse. ■

By Lemma 2.2, problem $P(\varrho_1, \varrho_2; \{b_1^t\}, \{b_2^t\}; u_0, l_0)$ can be reformulated as a quasi-variational problem $QV=QV(\varrho_1, \varrho_2; \{b_1^t\}, \{b_2^t\}; v_{1,0}, v_{2,0}, l_0)$ on $[0, T_0]$, $0<T_0\leqslant T$, as stated below: Find v_i: $[0, T_0]\to H$, $i=1, 2$, and

$l \in C[0, T_0])$ with $0 < l < 1$ on $[0, T_0]$ such that

$(QV1)$ v_i is a solution of $CP_i(\varrho_i, \{b_i^t\}; l, 0, v_{i,0})$ on $[0, T_0]$ for $i = 1, 2$;

$(QV2)$ $l \in W^{1,2}(0, T_0)$, $l(0) = l_0$ and

$$l'(t) = -[B_1 v_1]_x(t, l(t)-) + [B_2 v_2]_x(t, l(t)+) \quad \text{for a.e. } t \in [0, T_0].$$

3. The Cauchy problem CP_i

We begin with the investigation of some properties of the family $\{\phi_i^t(\{b^t\}, l; .)\}$ defined by (2.1) in the previous section, and discuss the solvability of the problem CP_i.

For $0 < L < \infty$ and $0 < \delta < 1$ with $\delta < 1-\delta$ we put

$$\Lambda_{T,\delta} = \{l \in C([0, T]);\ \delta \leqslant l \leqslant 1-\delta \text{ on } [0, T]\}$$

and

$$\Lambda_{T,\delta}(L) = \{l \in \Lambda_{T,\delta} \cap W^{1,2}(0, T);\ |l'|_{L^2(0,T)} \leqslant L\}.$$

Also, let $\Gamma(C_1, C_2)$ and $B(\alpha_0, \alpha_1)$ be as in the first section. In what follows, various constants depend on the numbers C_1, C_2 and functions α_0, α_1 in general, unless otherwise stated.

LEMMA 3.1. *Let $0 < \delta < 1-\delta$. Then there is a constant $R_3 = R_3(\alpha_0, \alpha_1; T, \delta) \geqslant 0$ such that for any $\{b^t\} \in B(\alpha_0, \alpha_1)$ and $l \in \Lambda_{T,\delta}$, $\phi_i^t(.) = \phi_i^t(\{b^t\}, l; .)$ has the following property (*) for each $i = 1, 2$:*

(*) For any $0 \leqslant s \leqslant t \leqslant T$ and $z \in D(\phi_i^s)$ there is $\tilde{z} \in D(\phi_i^t)$ such that

$$|\tilde{z} - z|_H \leqslant R_3 \{|l(t) - l(s)| + |\alpha_0(t) - \alpha_0(s)|\}(1 + |\phi_i^s(z)|^{1/2}). \tag{3.1}$$

and

$$\phi_i^t(\tilde{z}) - \phi_i^s(z) \leqslant R_3 \{|l(t) - l(s)| + |\alpha_0(t) - \alpha_0(s)| + |\alpha_1(t) - \alpha_1(s)|\}(1 + |\phi_i^s(z)|). \tag{3.2}$$

Proof. We give the proof only in case of $i = 1$. Given $z \in D(\phi_1^s)$, we consider the function

$$\tilde{z}(x) = z\left(\frac{l(s)}{l(t)} x\right) + (\tilde{r} - z(0)) z_\delta\left(\frac{l(s)}{l(t)} x\right), \quad 0 \leqslant x \leqslant 1,$$

where

$$z_\delta(x) = \begin{cases} 1 - x/\delta & \text{for } 0 \leqslant x \leqslant \delta, \\ 0 & \text{for } \delta < x \leqslant 1, \end{cases}$$

and $\tilde{r}$ is a number in $D(b_1^t)$, which is chosen so that

$$|\tilde{r} - z(0)| \leqslant |\alpha_0(t) - \alpha_0(s)|\left(1 + |z(0)| + |b_1^s(z(0))|^{1/2}\right) \tag{3.3}$$

and

$$b_1^t(\tilde{r}) - b_1^s(z(0)) \leqslant |\alpha_1(t) - \alpha_1(s)|\left(1 + |z(0)|^2 + |b_1^s(z(0))|\right). \tag{3.4}$$

Obviously, $\tilde{z} \in D(\phi_1^t)$ with $\tilde{z}(0) = \tilde{r}$. Putting

$$w(x) = z\left(\frac{l(s)}{l(t)}x\right) \text{ and } w_\delta(x) = z_\delta\left(\frac{l(s)}{l(t)}x\right),$$

we obtain after some elementary calculations (cf., [8; Lemma 3.1]) that

$$|w-z|_H \leqslant \frac{1}{\delta}|l(t)-l(s)|\,|z_x|_H,$$

$$|w_\delta - z_\delta|_H \leqslant \frac{1}{\delta}|l(t)-l(s)|\,|z_{\delta,x}|_H \leqslant \frac{1}{\delta^2}|l(t)-l(s)|,$$

$$|w_x|_H^2 - |z_x|_H^2 \leqslant \frac{1}{\delta}|l(t)-l(s)|\,|z_x|_H^2, \text{ and}$$

$$|w_{\delta,x}|_H^2 - |z_{\delta,x}|_H^2 \leqslant \frac{1}{\delta}|l(t)-l(s)|\,|z_{\delta,x}|_H^2 \leqslant \frac{1}{\delta^2}|l(t)-l(s)|.$$

From these inequalities together with (3.3), (3.4) and Lemma 2.1(3) it follows that (3.1) and (3.2) hold for some constant $R_3 \geqslant 0$ depending only on α_0, α_1, T, δ. ∎

LEMMA 3.2. *Let $l \in \Lambda_T$, $t \in [0, T]$, $\varrho \in \Gamma(C_1, C_2)$ and $\{b^t\}$, $\}\{\hat{b}^t\} \in B(\alpha_0, \alpha_1)$ such that*

$$b^t \leqslant \hat{b}^t \text{ on } R.$$

Then, for any $z^* \in \partial\phi_i^t(B_z)$ and $v^* \in \partial\hat{\phi}_i^t(Bv)$, i = 1, 2,

$$(z^* - v^*, \sigma_0([z-v]^+))_H \geqslant 0,$$

where $\phi_i^t = \phi_i^t(\{b^t\}, l; .)$, $\hat{\phi}_i^t = \phi_i^t(\{\hat{b}^t\}, l; .)$, $i = 1, 2$, B is the operator in H defined by (2.2) and σ_0 is the same function as in the statement of Theorem 1.2.

Proof. Take a sequence of smooth functions $\sigma_n\colon R \to R$ such that

$$\sigma_n' \geqslant 0 \text{ on } R, \quad -1 \leqslant \sigma_n \leqslant 1 \text{ on } R, \quad \sigma_n(0) = 0$$

and

$$\sigma_n(r) \to \sigma_0(r) \text{ as } n \to \infty \text{ for each } r \in R.$$

Then, with the help of Lemma 2.1(4) for $i = 1$, we see that

$$\begin{aligned}(z^* - v^*, \sigma_n([Bz-Bv]^+))_H &= -\int_0^{l(t)} (Bz-Bv)_{xx}\,\sigma_n([Bz-Bv]^+)\,dx \\ &= \int_{z>v} |(Bz-Bv)_x|^2\,\sigma_n'([Bz-Bv]^+)\,dx \\ &\quad + ([Bz]_x](0+) - [Bv]_x(0+)) \cdot \\ &\quad \cdot \sigma_n(\{[Bz](0) - [Bv](0)\}^+) \geqslant 0 \end{aligned} \tag{3.5}$$

because $[Bz]_x(0+) \in \partial b^t([Bz](0))$, $[Bv]_x(0+) \in \partial\hat{b}^t([Bv](0))$ and $r^* \geqslant \hat{r}^*$ for $r^* \in \partial b^t(r)$, $\hat{r}^* \in \partial\hat{b}^t(\hat{r})$ with $r > \hat{r}$ (cf. (1.14)). Now, letting $n \to \infty$ in (3.5) yields

$$(z^*-v^*, \sigma_0([z-v]^+))_H \geq 0,$$

since $\sigma_0([z-v]^+) = \sigma_0([Bz-Bv]^+)$. ■

COROLLARY 3.1. *Let ϕ_i^t and B be as in Lemma 3.2, and γ be the continuous convex function on H defined by*

$$\gamma(z) = \int_0^1 z^+(x)\,dx \quad \textit{for } z \in H.$$

Then $\partial\phi_i^t \circ B$ is γ-accretive in H, that is, if $z^* \in \partial\phi_i^t(Bz)$ and $v^* \in \partial\phi_i^t(Bv)$, then

$$(z^*-v^*, w)_H \geq 0 \quad \textit{for some } w \in \partial\gamma(z-v),$$

where $\partial\gamma$ is the subdifferential of γ in H.

Proof. It is easy to see that $\sigma_0(z^+) \in \partial\gamma(z)$ for any $z \in H$, so the corollary is a direct consequence of Lemma 3.2. ■

REMARK 3.1. Lemma 3.2 is essentially due to Bénilan [1] and Damlamian [3]. On account of the above lemmas, the abstract results in [5, 6] apply to Cauchy problems $CP_i(\varrho, \{b^t\}; l, g, v_0)$, and we have

PROPOSITION 3.1. For $i = 1, 2$, we have the following statements:

(1) Let $\varrho \in \Gamma(C_1, C_2)$, $\{b^t\} \in B(\alpha_0, \alpha_1)$, $l \in \Lambda_T \cap W^{1,2}(0, T)$ and $\phi_i^t = \phi_i^t(\{b^t\}, l; .)$. For each $g \in L^2(0, T; H)$, $v_0 \in D(\phi_i^0)$, $CP_i(\varrho, \{b^t\}; l, g, v_0)$ has one and only one solution v on $[0, T]$; hence v is bounded in X on $[0, T]$, and $b^{(\cdot)}(Bv(.,i-1))$ is bounded on $[0, T]$, where B is the operator defined by (2.2).

(2) Let $0 < \delta < 1-\delta$ and $0 < L < \infty$, and consider the class $\Lambda_{T,\delta}(L)$. Let k_0 be a positive number. Then there is a constant $M_0 = M_0(C_1, C_2; \alpha_0, \alpha_1; T, \delta, L, k_0) \geq 0$ having the following property: for any $\varrho \in \Gamma(C_1, C_2)$, $\{b^t\} \in B(\alpha_0, \alpha_1)$, $l \in \Lambda_{T,\delta}(L)$, $g \in L^2(0, T; H)$ with $|g|_{L^2(0,T;H)} \leq k_0$ and $v_0 \in D(\phi_i^0)$ with $|\phi_i^0(v_0)| \leq k_0$, the solution v of $CP_i(\varrho, \{b^t\}; l, g, v_0)$ on $[0, T]$ has the following bounds:

$$|v|_{W^{1,2}(0,T;H)} \leq M_0, \quad |v|_{L^\infty(0,T;X)} \leq M_0 \quad \text{and}$$
$$|b^t(Bv(t, i-1))| \leq M_0 \quad \text{for all } t \in [0, T],$$

where B is the operator given by (2.2).

The next proposition is concerned with the comparison between two solutions of CP_i.

PROPOSITION 3.2. Let $\varrho \in \Gamma(C_1, C_2)$, $l \in \Lambda_T \cap W^{1,2}(0, T)$, and $\{b^t\}$, $\{\hat{b}^t\} \in B(\alpha_0, \alpha_1)$ such that

$$b^t \leq \hat{b}^t \text{ on } R \text{ for all } t \in [0, T].$$

Also, let $i = 1$ or 2, and let v and $\hat{v}$ be the solutions of $CP_i(\varrho, \{b^t\}; l, g, v_0)$ and $CP_i(\varrho, \{\hat{b}^t\}; l, \hat{g}, \hat{v}_0)$ on $[0, T]$, respectively, where $g, \hat{g} \in L^2(0, T; H)$ and $v_0, \hat{v}_0 \in X$. Then,

$$|(v(t)-\hat{v}(t))^+|_{L^1(0,1)} \leqslant |(v(s)-\hat{v}(s))^+|_{L^1(0,1)} + \int_s^t |(g(\tau)-\hat{g}(\tau))^+|_{L^1(0,1)}\, d\tau \quad \text{for any } 0 \leqslant s \leqslant t \leqslant T. \tag{3.6}$$

Proof. We have by Lemma 3.2,

$$\begin{aligned}\frac{d}{d\tau}\gamma(v(\tau)-\hat{v}(\tau)) &= \left(v'(\tau)-\hat{v}'(\tau), \sigma_0([v(\tau)-\hat{v}(\tau)]^+)\right)_H \\ &\leqslant \left(g(\tau)-\hat{g}(\tau), \sigma_0([v(\tau)-\hat{v}(\tau)]^+)\right)_H \\ &\leqslant |(g(\tau)-\hat{g}(\tau))^+|_{L^1(0,1)} \quad \text{for a.e. } \tau \in [0, T],\end{aligned}$$

so that we obtain (3.6) by integrating the above inequalities on $[s, t]$. ■

COROLLARY 3.2. *In Proposition* 3.2, *suppose further that* $g \leqslant \hat{g}$ *a.e. on* $[0, T] \times [0, 1]$ *and* $v_0 \leqslant \hat{v}_0$ *on* $[0, 1]$. *Then* $v \leqslant \hat{v}$ *on* $[0, T] \times [0, 1]$.

In the proof of Theorem 1.2 we shall use the following types of approximations $\{b^t_\varepsilon\}$ and $\{b^t_{-\varepsilon}\}$ to $\{b^t\}$:

$$b^t_\varepsilon(r) = b^t(r-\varepsilon)-\varepsilon r, \qquad b^t_{-\varepsilon}(r) = b^t(r+\varepsilon)+\varepsilon r, \tag{3.7}$$

where $0 < \varepsilon \leqslant 1$.

LEMMA 3.3. *Let* $\{b^t\} \in B(\alpha_0, \alpha_1)$, *and* $0 < \varepsilon \leqslant 1$. *Then the families* $\{b^t_\varepsilon\}$, $\{b^t_{-\varepsilon}\}$ *given by* (3.7) *are contaned in* $B(\tilde{\alpha}_0, \tilde{\alpha}_1)$, *where*

$$\tilde{\alpha}_0(t) = (3+4R_1)\int_0^t |\alpha_0'(\tau)|\, d\tau,$$

and

$$\tilde{\alpha}_1(t) = (R_1+1)(R_1+5)\int_0^t \{|\alpha_0'(\tau)+|\alpha_1'(\tau)|\}\, d\tau$$

with the same constants R_1 *as in Lemma* 2.1(1).

Proof. Given $r \in D(b^s_\varepsilon)$ and $0 \leqslant s \leqslant t$, we see that $r-\varepsilon \in D(b^s)$ and hence there is $r' \in D(b^t)$ satisfying

$$|r'-r+\varepsilon| \leqslant |\alpha_0(t)-\alpha_0(s)|\left(1+|r-\varepsilon|+|b^s(r-\varepsilon)|^{1/2}\right) \tag{3.8}$$

By Lemma 2.1(1),

$$|b^s(r-\varepsilon)| \leqslant b^s(r-\varepsilon)+2R_1|r-\varepsilon|+2R_1$$
$$\leqslant b^s_\varepsilon(r)+(2R_1+1)|r|+4R_1 \tag{3.10}$$

and

$$|b^s(r-\varepsilon)|^{1/2} \leqslant |b^s_\varepsilon(r)|^{1/2}+1+(2R_1+1)|r|+4R_1. \tag{3.11}$$

Now, take $\tilde{r}=r'+\varepsilon$. Then it follows from (3.8) and (3.11) that

$$\begin{aligned}|\tilde{r}-r| &\leqslant |r'-r+\varepsilon|\\ &\leqslant |\alpha_0(t)-\alpha_0(s)|\,(3+4R_1+(2+2R_1)|r|+|b^s_\varepsilon(r)|^{1/2})\\ &\leqslant (3+4R_1)\,|\alpha_0(t)-\alpha_0(s)|\,(1+|r|+|b^s_\varepsilon(r)|^{1/2}).\end{aligned}$$

Also, it follows from (3.8)–(3.11) that

$$\begin{aligned}b^t_\varepsilon(\tilde{r})-b^s_\varepsilon(r) &= b^t(r')-\varepsilon(r'+\varepsilon)-b^s(r-\varepsilon)+\varepsilon r\\ &= b^t(r')-b^s(r-\varepsilon)-\varepsilon(r'-r+\varepsilon)\\ &\leqslant |\alpha_1(t)-\alpha_1(s)|\,\{(R_1+1)(R_1+5)+3|r|^2+|b^s_\varepsilon(r)|\}\\ &\quad +|\alpha_0(t)-\alpha_0(s)|\,\{(R_1+1)(R_1+5)+|r|^2+|b^s_\varepsilon(r)|\},\end{aligned}$$

from which we get

$$\begin{aligned}&b^t_\varepsilon(\tilde{r})-b^s_\varepsilon(r)\\ &\qquad \leqslant (R_1+1)(R_1+5)\,\{|\alpha_0(t)-\alpha_0(s)|+|\alpha_1(t)-\alpha_1(s)|\}\,(1+|r|^2+|b^s_\varepsilon(r)|).\end{aligned}$$

Thus $\{b^t_\varepsilon\}\in B(\tilde{\alpha}_0,\tilde{\alpha}_1)$. Similarly we see that $\{b^t_{-\varepsilon}\}\in B(\tilde{\alpha}_0,\tilde{\alpha}_1)$. ■

Besides we have the following lemma which admits an easy proof.

LEMMA 3.4. *Let* $\{b^t_{\pm\varepsilon}\}$ *be given by (3.7), and*

$$I_\varepsilon(r)\,(\textit{resp. } I_{-\varepsilon}(r)) = \begin{cases} -\varepsilon r\,(\textit{resp. } \varepsilon r) \ \textit{if } r\leqq\varepsilon\,(\textit{resp. } r\geqq-\varepsilon)\\ \infty \ \textit{if } r>\varepsilon\,(\textit{resp. } r<-\varepsilon).\end{cases}$$

Then we have:

(1) $b^t_\varepsilon\leqslant b^t\leqslant b^t_{-\varepsilon}$ *on* R *for any* $t\in[0,T]$.

(2) *If* $\partial b^t(r)\subset(-\infty,0]$ (*resp.* $\partial b^t(r)\subset[0,\infty)$) *for any* $r<0$ (*resp.* $r>0$) *and any* $t\in[0,T]$, *then* $I_\varepsilon\leqslant b^t_\varepsilon$ (*resp.* $b^t_{-\varepsilon}\leqslant I_{-\varepsilon}$) *on* R *for any* $t\in[0,T]$.

From Lemmas 3.3 and 3.4 we see that Propositions 3.1 and 3.2 can apply to the Cauchy problems CP_i associated with $\varrho\in\Gamma(C_1,C_2)$, $\{b^t_{\pm\varepsilon}\}$ and $\{I^t_{\pm\varepsilon}\}$.

4. Convergence of solutions of CP_i

First of all we recall the notion of convergence of convex functions, which is due to Mosco [12].

Let V be a Hilbert space and $\{\psi_n\}$ be a sequence of proper, l.s.c. convex functions on V. Then it is said that ψ_n converges to a proper, l.s.c. convex function ψ on V as $n \to \infty$ (denoted by $\psi_n \to \psi$ on V as $n \to \infty$) in the sense of Mosco, if the following two properties (m1), (m2) are fulfilled:

(m1) *If $\{n_k\}$ is a subsequence of $\{n\}$ and $v_k \to v$ weakly in V as $k \to \infty$, then*

$$\liminf_{k\to\infty} \psi_{n_k}(v_k) \geq \psi(v).$$

(m2) *For each $v \in D(\psi)$ there is a sequence $\{v_n\}$ in V such that $v_n \in D(\psi_n)$, $v_n \to v$ in H and $\psi_n(v_n) \to \psi(v)$ as $n \to \infty$.*

For the families $\{b^t_{\pm\varepsilon}\}$ given by (3.7) one can easily check that $b^t_{\pm\varepsilon} \to b^t$ on R as $\varepsilon \to 0$ in the sense of Mosco; more precisely, $b^t_{\pm\varepsilon_k} \to b^t$ on R as $k \to \infty$ in the sense of Mosco for every sequence $\{\varepsilon_k\}$ with $\varepsilon_k \downarrow 0$.

LEMMA 4.1. *Let $i = 1$ or 2, $l \in \Lambda_T$, $\{b^t\} \in B(\alpha_0, \alpha_1)$ and $l_n \in \Lambda_T$, $\{b^t_n\} \in B(\alpha_0, \alpha_1)$ for $n = 1, 2, \dots$. Suppose that*

$$l_n \to l \text{ pointwise on } [0, T]$$

and

$$b^t_n \to b^t \text{ on } R \text{ in the sense of Mosco for each } t \in [0, T]$$

as $n \to \infty$. Denoting by $\phi^t_{i,n}$ (resp. ϕ^t_i) the proper, l.s.c. convex function on H given by (2.1) corresponding to l_n, $\{b^t_n\}$ (resp. l, $\{b^t\}$), we have

$$\phi^t_{i,n} \to \phi^t_i \text{ on } H \text{ in the sense of Mosco for each } t \in [0, T] \text{ as } n \to \infty.$$

Proof. We prove only the case corresponding to $i = 1$. First, let $\{n_k\}$ be any subsequence of $\{n\}$ and $\{z_k\}$ be any weakly convergent sequence in H. Assume that

$$\liminf_{k\to\infty} \phi^t_{1,n_k}(z_k) < \infty$$

and $z_k \to z$ weakly in H as $k \to \infty$. Then it follows from Lemma 2.1(3) that $z_k \to z$ weakly in X and $z_k(0) \to z(0)$ as $k \to \infty$. Since $b^t_n \to b^t$ on R in the sense of Mosco, we see that

$$\liminf_{k\to\infty} \phi^t_{1,n_k}(z_k) = \liminf_{k\to\infty} \{\tfrac{1}{2}|z_{k,x}|^2_H + b^t_{n_k}(z_k(0))\} \geq \tfrac{1}{2}|z_x|^2_H + b^t(z(0)) = \phi^t_1(z).$$

Next, let v be any function in $D(\phi^t_1)$, i.e., $v \in X$, $v = 0$ on $[l(t), 1]$ and $v(0) \in D(b^t)$. We take a sequence $\{r_n\}$ in R so that

$$r_n \to v(0) \text{ and } b^t_n(r_n) \to b^t(v(0)) \text{ (as } n \to \infty),$$

and define a sequence $\{v_n\}$ in X by

$$v_n(x) = v\left(\frac{l(t)}{l_n(t)}x\right) + (r_n - v(0))\, z_0(x),$$

where

$$z_0(x) = \begin{cases} 1 - \dfrac{2x}{l(t)} & \text{for } 0 \leqslant x \leqslant \dfrac{l(t)}{2}, \\ 0 & \text{for } \dfrac{l(t)}{2} < x \leqslant 1. \end{cases}$$

Just as in the proof of Lemma 3.1, we obtain that $v_n \in D(\phi^t_{1,n})$ with $v_n(0) = r_n$ and

$$v_n \to v \text{ in } H \text{ and } |v_{n,x}|^2_H \to |v_x|^2_H.$$

This shows that $\phi^t_{1,n}(v_n) \to \phi^t_1(v)$. Thus, $\phi^t_{1,n} \to \phi^t_1$ on H in the sense of Mosco. ■

We are now in a position to prove a convergence result for Cauchy Problems CP_i.

PROPOSITION 4.1. Let $i = 1$ or 2, $0 < \delta < 1 - \delta$, $0 < L < \infty$, $\varrho \in \Gamma(C_1, C_2)$, $l \in \Lambda_{T,\delta}(L)$, $\{b^t\} \in B(\alpha_0, \alpha_1)$ and $l_n \in \Lambda_{T,\delta}(L)$, $\{b^t_n\} \in B(\alpha_0, \alpha_1)$, $n = 1, 2, \ldots$. Also, let $g \in L^2(0, T; H)$, $v_0 \in D(\phi^0_i)$ and $g_n \in L^2(0, T; H)$, $v_{0,n} \in D(\phi^0_{i,n})$, where $\phi^t_i = \phi^t_i(\{b^t\}, l; .)$ and $\phi^t_{i,n} = \phi^t_i(\{b^t_n\}, l_n; .)$. If

$l_n \to l$ uniformly on $[0, T]$,
$b^t_n \to b^t$ on R in the sense of Mosco for each $t \in [0, T]$,
$g_n \to g$ in $L^2(0, T; H)$,
$\{\phi^0_{i,n}(v_{0,n})\}$ is bounded, and
$v_{0,n} \to v_0$ in H,

then the solution v_n of $CP_{i,n} = CP_i(\varrho, \{b^t_n\}; l_n, g_n, v_{0,n})$ converges to the solution v of $CP_i = CP_i(\varrho, \{b^t\}; l, g, v_0)$ on $[0, T]$ in the following sense:

$$Bv_n \to Bv \text{ in } C([0, T]; H) \text{ and } L^2(0, T; X), \tag{4.1}$$

$$v'_n \to v' \text{ weakly in } L^2(0, T; H), \tag{4.2}$$

$$Bv_n(., i-1) \to Bv(., i-1) \text{ in } L^2(0, T), \tag{4.3}$$

$$\int_0^T b^t_n(Bv_n(t, i-1))\, dt \to \int_0^T b^t(Bv(t, i-1))\, dt, \tag{4.4}$$

where B is the operator in H given by (2.2), and

$$\left.\begin{array}{l} [Bv_n]_x(., 0+) \to [Bv]_x(., 0+) \text{ in } L^2(0, T) \\ {[Bv_n]_x(., l_n(.)-) \to [Bv]_x(., l(.)-) \text{ in } L^2(0, T)} \end{array}\right\} \text{ if } i = 1, \tag{4.5}$$

$$\left.\begin{array}{l}[Bv_n]_x(.,1-)\to[Bv]_x(.,1-) \text{ in } L^2(0,T),\\ [Bv_n]_x(.,l_n(.)+)\to[Bv]_x(.,l(.)+) \text{ in } L^2(0,T)\end{array}\right\} \text{ if } i=2. \tag{4.5)'$$

Proof. We show only the assertion at $i=1$, since the other case can be considered similarly. For simplicity we write ϕ_n^t, ϕ^t for $\phi_{1,n}^t$, ϕ_1^t, respectively. By virtue of a convergence result [5; §2.8] (or [10; Theorem 1.3]) and Lemma 4.1 we have, as $n\to\infty$,

$$v_n\to v \text{ in } C([0,T];H) \text{ and weakly in } W^{1,2}(0,T;H) \tag{4.6}$$

and

$$\int_0^T \phi_n^t(Bv_n(t))\,dt\to\int_0^T \phi^t(Bv(t))\,dt. \tag{4.7}$$

Since $b_n^t\to b^t$ on R in the sense of Mosco, it is inferred from (4.6) and (4.7) with the help of the uniform estimates for solutions in Proposition 3.1(2) that

$$Bv_n\to Bv \text{ in } C([0,T];H) \text{ and weakly in } L^2(0,T;X), \tag{4.8}$$

$$Bv_n(.,0)\to Bv(.,0) \text{ weakly in } L^2(0,T), \tag{4.9}$$

$$|Bv_n|_{L^2(0,T;X)}\to|Bv|_{L^2(0,T;X)} \tag{4.10}$$

and

$$\int_0^T b_n^t(Bv_n(t,0))\,dt\to\int_0^T b^t(Bv(t,0))\,dt.$$

Therefore, by the uniform convexity of $L^2(0,T;X)$, (4.8) and (4.10) imply that $Bv_n\to Bv$ in $L^2(0,T;X)$ and hence $v_n\to v$ weakly in $L^2(0,T;X)$ as well as the convergence of (4.9) is valid in the strong topology. Thus (4.1)–(4.4) hold for $i=1$.

Now we show (4.5). Given $\varepsilon>0$, we take a smooth function $\hat{l}$ in $\Lambda_{T,\delta}$ so that

$$0\leqslant l_n-\hat{l}\leqslant\varepsilon \text{ on } [0,T] \text{ for all large } n.$$

Observe that for a.e. $t\in[0,T]$,

$$\begin{aligned}&|[Bv_n]_x(t,l_n(t)-)-[Bv]_x(t,l(t)-)|^2\\ &\qquad\leqslant 9|[Bv_n]_x(t,\hat{l}(t))-[Bv]_x(t,\hat{l}(t))|^2\\ &\qquad+9|[Bv_n]_x(t,l_n(t)-)-[Bv_n]_x(t,\hat{l}(t))|^2\\ &\qquad+9|[Bv]_x(t,l(t)-)-[Bv]_x(t,\hat{l}(t))|^2.\end{aligned}$$

Also, for a.e. $t\in[0,T]$,

$$|[Bv]_x(t,l(t)-)-[Bv]_x(t,\hat{l}(t))|^2\leqslant\Big\{\int_{\hat{l}(t)}^{l(t)}[Bv]_{xx}(t,x)\,dx\Big\}^2\leqslant$$

$$\leqslant \{\int_{\hat{l}(t)}^{l(t)} \{|v_t(t, x)| + |g(t, x)|\} dx\}^2 \leqslant 4|l(t) - \hat{l}(t)| \{|v'(t)|_H^2 + |g(t)|_H^2\}$$

$$\leqslant 4\varepsilon \{|v'(t)|_H^2 + |g(t)|_H^2\}$$

and similarly

$$|[Bv_n]_x(t, l_n(t)-) - [Bv_n]_x(t, \hat{l}(t))|^2 \leqslant 4\varepsilon \{|v_n'(t)|_H^2 + |g_n(t)|_H^2\}$$

for all large n.

Besides, using the same function z_δ as in the proof of Lemma 3.1, we have

$$\int_0^T |[Bv_n]_x(t, \hat{l}(t)) - [Bv]_x(t, \hat{l}(t))|^2 dt$$

$$= \int_0^T \int_0^{\hat{l}(t)} \frac{\partial}{\partial x} \{(1 - z_\delta(x))([Bv_n]_x(t, x) - [Bv]_x(t, x))\}^2 dx\, dt$$

$$\leqslant \frac{2}{\delta^2} \int_0^T \int_0^\delta |[Bv_n]_x - [Bv]_x|^2 dx\, dt + 2\int_0^T \int_0^{\hat{l}(t)} |[Bv_n]_x - [Bv]_x|\, |[Bv_n]_{xx} - [Bv]_{xx}|\, dx\, dt$$

$$\leqslant \frac{2}{\delta^2} |Bv_n - Bv|_{L^2(0,T;X)}^2 + 2|Bv_n - Bv|_{L^2(0,T;X)} \{|v_n' - v'|_{L^2(0,T;H)} + |g_n - g|_{L^2(0,T;H)}\}.$$

Consequently,

$$\limsup_{n\to\infty} |[Bv_n]_x(\cdot, l_n(t)-) - [Bv]_x(\cdot, l(\cdot)-)|_{L^2(0,T)}^2 \leqslant \text{const. } \varepsilon.$$

Thus $[Bv_n](\cdot, l_n(\cdot)-) \to [Bv]_x(\cdot, l(\cdot)-)$ in $L^2(0, T)$. The other convergence of (4.5) can be concluded in a similar way. ■

5. Proof of Theorem 1.1

Let ϱ_i, $\{b_i^t\}$, $i = 1, 2$, u_0 and l_0 be as in the statement of Theorem 1.1; (1.12) is satisfied as well. Also, let $v_{i,0}$, $i = 1, 2$, be the functions in X given by (2.5).

Now, let L be any positive number and fix it in this section. Next, taking a positive number δ with

$$2\delta < l_0 < 1 - 2\delta,$$

we consider the subclass $\Lambda_{T,\delta}(L, l_0)$ of $\Lambda_{T,\delta}(L)$:

$$\Lambda_{T,\delta}(L, l_0) = \{l \in \Lambda_{T,\delta}(L);\ l(0) = l_0\}.$$

By virtue of Proposition 3.1 we have:

(a) For each $l \in \Lambda_{T,\delta}(L, l_0)$, $CP_i = CP_i(\varrho_i, \{b_i^t\}; l, 0, v_{i,0})$ has a unique solution v_i^l on $[0, T]$ for $i = 1, 2$.

(b) There exists a positive constant $\tilde{M}_0$ such that

$$|v_i^l|_{W^{1,2}(0,T;H)} \leqslant \tilde{M}_0,\ |v_i^l|_{L^\infty(0,T;X)} \leqslant \tilde{M}_0,\ |b_i^l(B_i v_i^l(t, i-1))| \leqslant \tilde{M}_0,\ t \in [0, T],$$

for any $l \in \Lambda_{T,\delta}(L, l_0)$ and $i = 1, 2$,

where B_i is the operator in H given by (2.2) with $\varrho = \varrho_i$.

LEMMA 5.1. *There is a number T' with $0 < T' \leqslant T$ such that for all $l \in \Lambda_{T,\delta}(L, l_0)$*

$$|[B_1 v_1^l]_x(.,\, l(.)-)|_{L^2(0,T')} + |[B_2 v_2^l]_x(.,\, l(.)+)|_{L^2(0,T')} \leqslant L.$$

Proof. With the same convention of notations for $i = 1$ as in the proof of Proposition 4.1 we have for $v \equiv v_1^l$ and $B = B_1$

$$\int_0^t |[Bv]_x(\tau, l(\tau)-)|^2\, d\tau = \int_0^t \int_0^{l(\tau)} \frac{\partial}{\partial x} |(1 - z_\delta(x))\, [Bv]_x(\tau, x)|^2\, dx\, d\tau$$

$$\leqslant \frac{2}{\delta^2} \int_0^t \int_0^\delta |[Bv]_x|^2\, dx\, d\tau + 2 \int_0^t \int_0^{l(\tau)} |[Bv]_x [Bv]_{xx}|\, dx\, d\tau$$

$$\leqslant \frac{2}{\delta^2} \int_0^t |Bv(\tau)|_X^2\, d\tau + 2 \int_0^t |Bv(\tau)|_X\, |v'(\tau)|_H\, d\tau \leqslant \frac{2\tilde{M}_0^2}{\delta^2} t + 2\tilde{M}_0^2 t^{1/2}.$$

We have the same type of inequality for v_2^l as above. Hence the required inequality holds for a certain T' with $0 < T' \leqslant T$. ∎

Now we define an operator P: $\Lambda_{T,\delta}(L, l_0) \to C([0, T])$ by putting

$$[Pl](t) = l_0 - \int_0^t [B_1 v_1^l]_x(\tau, l(\tau)-)\, d\tau + \int_0^t [B_2 v_2^l]_x(\tau, l(\tau)+)\, d\tau$$

for each $l \in \Lambda_{T,\delta}(L, l_0)$ and $t \in [0, T]$.

Moreover, let us consider the operator P^*: $\Lambda_{T,\delta}(L, l_0) \to C([0, T])$ given by

$$[P^* l](t) = \begin{cases} [Pl](t) & \text{for } 0 \leqslant t \leqslant T_0, \\ [Pl](T_0) & \text{for } T_0 < t \leqslant T, \end{cases}$$

for each $l \in \Lambda_{T,\delta}(L, l_0)$, where $T_0 = \min\{T', (\delta/L)^2\}$, T' being as in Lemma 5.1.

LEMMA 5.2.

(1) *P^* maps $\Lambda_{T,\delta}(L, l_0)$ into itself.*

(2) *P^* is continuous in the topology of $C([0, T])$.*

Proof. Since

$$\left|\frac{d}{dt}[Pl](t)\right| \leqslant |[B_1 v_1^l]_x(t, l(t)-)| + |[B_2 v_2^l]_x(t, l(t)+)| \quad \text{for a.e. } t \in [0, T_0],$$

it follows from Lemma 5.1 that

$$\left|\frac{d}{dt}[P^* l]\right|_{L^2(0,T)} = \left|\frac{d}{dt}[Pl]\right|_{L^2(0,T_0)} \leqslant L \quad \text{for any } l \in \Lambda_{T,\delta}(L, l_0).$$

Also,

$$|[Pl](t) - l_0| \leqslant \int_0^t \{|[B_1 v_1^l]_x(\tau, l(\tau)-)| + |[B_2 v_2^l]_x(\tau, l(\tau)+)|\} \, d\tau$$

$$\leqslant T_0^{1/2} \{|[B_1 v_1^l]_x(\cdot, l(\cdot)-)|_{L^2(0,T')} + |[B_2 v_2^l]_x(\cdot, l(\cdot)+)|_{L^2(0,T')}\}$$

$$\leqslant T_0^{1/2} L \leqslant \delta \text{ for any } l \in \Lambda_{T,\delta}(L, l_0) \text{ and } t \in [0, T_0].$$

Hence, P^* maps $\Lambda_{T,\delta}(L, l_0)$ into itself. Thus (1) holds. In order to prove (2), let $l_n \in \Lambda_{T,\delta}(L, l_0)$ and assume $l_n \to l$ in $C([0, T])$. Then it follows from Proposition 4.1 that

$$[B_1 v_1^{l_n}]_x(\cdot, l_n(\cdot)-) \to [B_1 v_1^l]_x(\cdot, l(\cdot)-) \quad \text{in } L^2(0, T)$$

and

$$[B_2 v_2^{l_n}]_x(\cdot, l_n(\cdot)+) \to [B_2 v_2^l]_x(\cdot, l(\cdot)+) \quad \text{in } L^2(0, T),$$

which shows that $P^* l_n \to P^* l$ in $C([0, T])$. Thus we have (2). ■

We note that $\Lambda_{T,\delta}(L, l_0)$ is a compact convex set in $C([0, T])$. Therefore, by the classical fixed point theorem, P^* has a fixed point in $\Lambda_{T,\delta}(L, l_0)$, i.e., there exists $l \in \Lambda_{T,\delta}(L, l_0)$ such that

$$P^* l = l.$$

It is easy to see that $\{v_1^l, v_2^l, l\}$ is a solution of $QV(\varrho_1, \varrho_2; \{b_1^t\}, \{b_2^t\}; v_{1,0}, v_{2,0}, l_0)$ on $[0, T_0]$. Thus the proof of Theorem 1.1 is complete.

6. Proof of Theorem 1.2

We make the same assumptions and use the same notations as in the statement of Theorem 1.2.

For the solution $\{u, l\}$ of $P = P(\varrho_1, \varrho_2; \{b_1^t\}, \{b_2^t\}; u_0, l_0)$ on $[0, T_0]$ we put

$$v_1 \text{ (resp., } v_2) = \begin{cases} \varrho_1(u) \text{ (resp., } 0) & \text{on } \overline{Q_l^1(T_0)}, \\ 0 \text{ (resp., } \varrho_2(u)) & \text{on } \overline{Q_l^2(T_0)} \end{cases},$$

$$v = v_1 + v_2$$

and

$$u_i = B_i v_i, \text{ i.e., } \varrho_i(u_i) = v_i \text{ for } i = 1, 2,$$

where B_i is the operator in H given by (2.2) associated with $\varrho = \varrho_i$ for $i = 1, 2$. Clearly, $u = u_1 + u_2$. As was seen in Lemma 2.2, for each $i = 1, 2$, v_i is the solution of

$$v_i'(t) + \partial\phi_i^t(B_i v_i(t)) \ni 0, \quad 0 < t < T_0, \quad v_i(0) = v_{i,0},$$

where $v_{i,0}$ is the function given by (2.5) and $\phi_i^t = \phi_i^t(\{b_i^t\}, l; .)$. Now, we approximate v_i by the following function $v_{i,\varepsilon}$, $\varepsilon > 0$, which is the solution of

$$v_{i,\varepsilon}'(t) + \partial\phi_{i,\varepsilon}^t(B_i v_{i,\varepsilon}(t)) \ni f_\varepsilon(t), \quad 0 < t < T_0, \quad v_{i,\varepsilon}(0) = v_{i,\varepsilon,0},$$

where $\phi_{i,\varepsilon}^t = \phi_1^t(\{b_{1,\varepsilon}^t\}, l;.)$ with $b_{1,\varepsilon}^t(t) = b_1^t(r-\varepsilon) - \varepsilon r$, $\phi_{2,\varepsilon}^t = \phi_2^t(\{b_{2,-\varepsilon}^t\}, l;.)$ with $b_{2,-\varepsilon}^t(r) = b_2^t(r+\varepsilon) + \varepsilon r$,

$$f_\varepsilon(t, x) = \frac{\partial}{\partial t}\{\varrho_i(\varepsilon(l(t) - x))\} \text{ for } (t, x) \in Q_l^i(T_0),\ \text{i} = 1, 2,$$

$$v_{1,\varepsilon,0}(x) = \varrho_1(u_0(x) + \varepsilon[(l_0 - x) \vee z_1(x)]) \text{ on } [0, l_0], = 0 \text{ on } [l_0, 1],$$

$$\text{with } z_1 \in C^\infty(R),\ z_1 \geqslant 0,\ z_1(0) = 1,\ z_1 = 0 \text{ on } [(l_0 \wedge \hat{l}_0)/2, \infty),$$

and

$$v_{2,\varepsilon,0}(x) = 0 \text{ on } [0, l_0], = \varrho_2(u_0(x) + \varepsilon[(l_0 - x) \wedge z_2(x)]) \text{ on } [l_0, 1],$$

$$\text{with } z_2 \in C^\infty(R),\ z_2 \leqslant 0,\ z_2(1) = -1,\ z_2 = \text{ on } (-\infty, (1 + l_0 \vee \hat{l}_0)/2].$$

Also, we put

$$v_\varepsilon = v_{1,\varepsilon} + v_{2,\varepsilon},\ u_{1,\varepsilon} = B_1 v_{1,\varepsilon},\ u_{2,\varepsilon} = B_2 v_{2,\varepsilon} \text{ and } u_\varepsilon = u_{1,\varepsilon} + u_{2,\varepsilon}.$$

LEMMA 6.1. *We have*:

$$u_{1,\varepsilon}(x) \geqslant \varepsilon(l(t) - x) \text{ for } (t, x) \in \overline{Q_l^1(T_0)}, \tag{6.1}$$

and

$$u_{2,\varepsilon}(x) \leqslant \varepsilon(l(t) - x) \text{ for } (t, x) \in \overline{Q_l^2(T_0)}. \tag{6.2}$$

Proof. It is easy to see that $w_\varepsilon(t, x) = \{\varrho_1(\varepsilon(l(t) - x))\}^+$ satisfies

$$w_\varepsilon'(t) + \partial\psi_\varepsilon^t(B_1 w_\varepsilon(t)) \ni f_\varepsilon(t) \quad \text{for a.e. } t \in [0, T_0],$$

where $\psi_\varepsilon^t = \phi_1^t(\{I_\varepsilon^t\}, l; .)$ with $I_\varepsilon^t = I_\varepsilon$ for any t. Also, by (1.15),

$$v_{1,\varepsilon,0}(x) \geqslant \{\varrho_1(\varepsilon(l_0 - x))\}^+ = w_\varepsilon(0, x).$$

Hence it follows from Proposition 3.2 and Lemma 3.4(2) that

$$v_{1,\varepsilon} \geqslant w_\varepsilon = \varrho_1(\varepsilon(l(t) - x)) \text{ on } \overline{Q_l^1(T_0)},$$

which implies (6.1). Similarly (6.2) is obtained by making use of $I_{-\varepsilon}$. ■

In view of the above lemma, u_ε is strictly positive in $Q_l^1(T_0)$ and strictly negative in $Q_l^2(T_0)$. This fact will be used in the proof of Lemma 6.4.

LEMMA 6.2. *As* $\varepsilon \downarrow 0$, *we have:*

$$u_\varepsilon \to u \text{ in } C([0, T_0]; H) \text{ and } L^2(0, T_0; X),$$
$$v'_\varepsilon \to v' \text{ weakly in } L^2(0, T_0; H),$$

and

$$\left.\begin{array}{l} u_{\varepsilon,x}(.,0+) \to u_x(.,0+) \\ u_{\varepsilon,x}(.,1-) \to u_x(.,1-), \\ u_{\varepsilon,x}(.,l(.)\pm) \to u_x(.,l(.)\pm) \end{array}\right\} \text{ in } L^2(0, T_0). \tag{6.3}$$

The above lemma is a direct consequence of Proposition 4.1.

COROLLARY 6.1. $u \geqslant 0$ *on* $\overline{Q_l^1(T_0)}$ *and* $u \leqslant 0$ *on* $\overline{Q_l^2(T_0)}$.

LEMMA 6.3. *For any* $\eta \in W^{1,2}(0, T_0; H) \cap L^\infty(0, T_0; X)$ *we have:*

$$\begin{aligned} (v'_\varepsilon(t), \eta(t))_H &+ (u_{\varepsilon,x}(t), \eta_x(t)) + l'(t)\eta(t, l(t)) \\ &+ u_{\varepsilon,x}(t, 0)\eta(t, 0) - u_{\varepsilon,x}(t, 1-)\eta(t, 1) \\ = \{l'(t) &+ u_{\varepsilon,x}(t, l(t)-) - u_{\varepsilon,x}(t, l(t)+)\}\eta(t, l(t) + (f_\varepsilon(t), \eta(t))_H \end{aligned} \tag{6.4}$$

for a.e. $t \in [0, T_0]$.

Proof. We observe with the help of Lemma 2.1(4) that

$$\begin{aligned} &(v'_\varepsilon(t), \eta(t))_H = (v'_{1,\varepsilon}(t), \eta(t))_H + (v'_{2,\varepsilon}(t), \eta(t))_H \\ &= \int_0^{l(t)} u_{1,\varepsilon,xx}(t, x)\eta(t, x)\,dx + \int_{l(t)}^1 u_{2,\varepsilon,xx}(t, x)\eta(t, x)\,dx + (f_\varepsilon(t), \eta(t))_H \\ &= -\int_0^{l(t)} u_{1,\varepsilon,x}(t, x)\eta_x(t, x)\,dx + u_{1,\varepsilon,x}(t, l(t)-)\eta(t, l(t)) - u_{1,\varepsilon,x}(t, 0+)\eta(t, 0) \\ &\quad -\int_{l(t)}^1 u_{2,\varepsilon,x}(t, x)\eta_x(t, x)\,dx + u_{2,\varepsilon,x}(t, 1-)\eta(t, 1) - u_{2,\varepsilon,x}(t, l(t)+)\eta(t, l(t)) \\ &\quad + (f_\varepsilon(t), \eta(t))_H \\ &= -(u_{\varepsilon,x}(t), \eta_x(t))_H + \{u_{\varepsilon,x}(t, l(t)-) - u_{\varepsilon,x}(t, l(t)+)\}\eta(t, l(t)) \\ &\quad - u_{\varepsilon,x}(t, 0+)\eta(t, 0) + u_{\varepsilon,x}(t, 1-)\eta(t, 1) + (f_\varepsilon(t), \eta(t))_H. \end{aligned}$$

From this we infer (6.4). ■

Let $\{\hat{u}, \hat{l}\}$ be the solution of $\hat{P} = P(\varrho_1, \varrho_2; \{\hat{b}_1^t\}, \{\hat{b}_2^t\}; \hat{u}_0, \hat{l}_0)$ on $[0, T_0]$, and consider the similar functions $\hat{v}_\varepsilon$, $\hat{u}_\varepsilon$, $\hat{f}_\varepsilon$ corresponding to this solution as v_ε,

u_ε, f_ε. Now, taking the difference between (6.4) and that for $\hat{v}_\varepsilon$, $\hat{u}_\varepsilon$, $\hat{f}_\varepsilon$, $\hat{l}$, we have

$$\begin{aligned}
&(v'_\varepsilon(t)-\hat{v}'_\varepsilon(t),\ \eta(t))_H+(u_{\varepsilon,x}(t)-\hat{u}_{\varepsilon,x}(t),\ \eta_x(t))_H\\
&+l'(t)\eta(t,\ l(t))-\hat{l}'(t)\eta(t,\ \hat{l}(t))\\
&+(u_{\varepsilon,x}(t,\ 0+)-\hat{u}_{\varepsilon,x}(t,\ 0+))\eta(t,\ 0)-(u_{\varepsilon,x}(t,\ 1-)-\hat{u}_{\varepsilon,x}(t,\ -1))\eta(t,\ 1)\\
&=\{l'(t)+u_{\varepsilon,x}(t,\ l(t)-)-u_{\varepsilon,x}(t,\ l(t)+)\}\ \eta(t,\ l(t))\\
&-\{\hat{l}'(t)+\hat{u}_{\varepsilon,x}(t,\ \hat{l}(t)-)-\hat{u}_{\varepsilon,x}(t,\ \hat{l}(t)+)\}\ \eta(t,\ \hat{l}(t))+(f_\varepsilon(t)-\hat{f}_\varepsilon(t),\ \eta(t))_H
\end{aligned}\tag{6.5}$$

for any $\eta\in W^{1,2}(0,\ T_0;\ H)\cap L^\infty(0,\ T_0;\ X)$ and a.e. $t\in[0,\ T_0]$.

Next, substitute the function $\sigma_n([u_\varepsilon-\hat{u}_\varepsilon]^+)$ as η in (6.5), where σ_n is the same function as in the proof of Lemma 3.2, and note that

$$\begin{aligned}
&(u_{\varepsilon,x}(t)-\hat{u}_{\varepsilon,x}(t),\ \sigma_n([u_\varepsilon(t)-\hat{u}_\varepsilon(t)]^+)_x)_H\\
&\qquad=\int_0^1(u_{\varepsilon,x}(t,\ x)-\hat{u}_{\varepsilon,x}(t,\ x))^2\sigma'_n([u_\varepsilon(t,\ x)-\hat{u}_\varepsilon(t,\ x)]^+)\,dx\geqslant 0
\end{aligned}$$

for a.e. $t\in[0,\ T_0]$.

Accordingly, it follows from (6.5) that

$$V_{\varepsilon,n}(t)+L_{\varepsilon,n}(t)+U^{(1)}_{\varepsilon,n}(t)+U^{(2)}_{\varepsilon,n}(t)\leqslant S^{(1)}_\varepsilon(t)+S^{(2)}_\varepsilon(t)+F_\varepsilon(t)\tag{6.6}$$

for a.e. $t\in[0,\ T_0]$,

where

$$\begin{aligned}
V_{\varepsilon,n}(t)&=(v'_\varepsilon(t)-\hat{v}'_\varepsilon(t),\ \sigma_n([u_\varepsilon(t)-\hat{u}_\varepsilon(t)]^+))_H,\\
L_{\varepsilon,n}(t)&=l'(t)\sigma_n([-\hat{u}_\varepsilon(t,\ l(t))]^+)-\hat{l}'(t)\sigma_n([u_\varepsilon(t,\ \hat{l}(t))]^+),\\
U^{(1)}_{\varepsilon,n}(t)&=(u_{\varepsilon,x}(t,\ 0+)-\hat{u}_{\varepsilon,x}(t,\ 0+))\sigma_n([u_\varepsilon(t,\ 0)-\hat{u}_\varepsilon(t,\ 0)]^+),\\
U^{(2)}_{\varepsilon,n}(t)&=-(u_{\varepsilon,x}(t,\ 1-)-\hat{u}_{\varepsilon,x}(t,\ 1-))\sigma_n([u_\varepsilon(t,\ 1)-\hat{u}_\varepsilon(t,\ 1)]^+),\\
S^{(1)}_\varepsilon(t)&=|l'(t)+u_{\varepsilon,x}(t,\ l(t)-)-u_{\varepsilon,x}(t,\ l(t)+)|,\\
S^{(2)}_\varepsilon(t)&=\hat{l}'(t)+\hat{u}_{\varepsilon,x}(t,\ \hat{l}(t)-)-\hat{u}_{\varepsilon,x}(t,\ \hat{l}(t)+)|
\end{aligned}$$

and

$$F_\varepsilon(t)=|f_\varepsilon(t)-\hat{f}_\varepsilon(t)|_H.$$

■

LEMMA 6.4 *For each* $0<\varepsilon\leqslant 1$ *we have*:

$$\lim_{n\to\infty} L_{\varepsilon,n}(t)=\frac{d}{dt}(l(t)-\hat{l}(t))^+\quad \textit{for a.e. } t\in[0,\ T_0].$$

Proof. By Lemma 6.1, u_ε (resp., $\hat{u}_\varepsilon$) is strictly positive on $Q^1_l(T_0)$ (resp., $Q^1_{\hat{l}}(T_0)$)

and strictly negative on $Q_l^2(T_0)$ (resp., $Q_{\hat{l}}^2(T_0)$), so that

$$\sigma_n([u_\varepsilon(t, \hat{l}(t))]^+) \to \sigma_0([u_\varepsilon(t, \hat{l}(t))]^+) = \begin{cases} 1 & \text{if } l(t) > \hat{l}(t), \\ 0 & \text{if } l(t) \leqslant \hat{l}(t) \end{cases}$$

and

$$\sigma_n([-\hat{u}_\varepsilon(t, l(t))]^+) \to \sigma_0([-\hat{u}_\varepsilon(t, l(t))]^+) = \begin{cases} 1 & \text{if } l(t) > \hat{l}(t), \\ 0 & \text{if } l(t) \leqslant \hat{l}(t). \end{cases}$$

Hence, for a.e. $t \in [0, T_0]$,

$$\begin{aligned} \lim_{n\to\infty} L_{\varepsilon,n}(t) &= l'(t)\,\sigma_0([-\hat{u}_\varepsilon(t, \hat{l}(t))]^+) - l'(t)\,\sigma_0([u_\varepsilon(t, \hat{l}(t))]^+) \\ &= \begin{cases} l'(t) - \hat{l}'(t) & \text{if } l(t) > \hat{l}(t), \\ 0 & \text{if } l(t) \leqslant \hat{l}(t) \end{cases} \\ &= \frac{d}{dt}(l(t) - \hat{l}(t))^+. \end{aligned}$$ ■

Noting Lemma 6.4, we deduce from (6.6) by letting $n \to \infty$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}} |(v_\varepsilon(t) - \hat{v}_\varepsilon(t))^+|_{L^1(0,1)} + \frac{d}{dt}(l(t) - \hat{l}(t))^+ + U_\varepsilon^{(1)}(t) + U_\varepsilon^{(2)}(t) \\ &\qquad \leqslant S_\varepsilon^{(1)}(t) + S_\varepsilon^{(2)}(t) + F_\varepsilon(t) \quad \text{for a.e. } t \in [0, T_0], \end{aligned} \tag{6.7}$$

where $U_\varepsilon^{(i)}(t)$ $(i = 1, 2)$ is the function $U_{\varepsilon,n}^{(i)}(t)$ with $\sigma_n(.)$ replaced by $\sigma_0(.)$. Integration of both sides of (6.7) over $[s, t]$ yields

$$\begin{aligned} &|(v_\varepsilon(t) - \hat{v}_\varepsilon(t))^+|_{L^1(0,1)} - |(v_\varepsilon(s) - \hat{v}_\varepsilon(s))^+|_{L^1(0,1)} + (l(t) - \hat{l}(t))^+ \\ &\qquad - (l(s) - \hat{l}(s))^+ + \int_s^t \{U_\varepsilon^{(1)}(\tau) + U_\varepsilon^{(2)}(\tau)\}\, d\tau \\ &\qquad \leqslant \int_s^t \{S_\varepsilon^{(1)}(\tau) + S_\varepsilon^{(2)}(\tau) + F_\varepsilon(\tau)\}\, d\tau \quad \text{for any } [s, t] \subset [0, T_0]. \end{aligned} \tag{6.8}$$

LEMMA 6.5 *There is a sequence $\{\varepsilon_k\}$ with $\varepsilon_k \downarrow 0$ (as $k \to \infty$) such that*

$$\liminf_{k\to\infty} U_{\varepsilon_k}^{(1)}(t) \geqslant (u_x(t, 0+) - \hat{u}_x(t, 0+))\,\sigma_0([u(t, 0) - \hat{u}(t, 0)]^+) \tag{6.9}$$

and

$$\liminf_{k\to\infty} U_{\varepsilon_k}^{(2)}(t) \geqslant -(u_x(t, 1-) - \hat{u}_x(t, 1-))\,\sigma_0([u(t, 1) + \hat{u}(t, 1)]^+) \tag{6.10}$$

for a.e. $t \in [0, T_0]$.

Proof. On account of Lemma 6.2, there is a sequence $\{\varepsilon_k\}$ with $\varepsilon_k \downarrow 0$ such that $u_{\varepsilon_k}(., 0) \to u(., 0)$, $\hat{u}_{\varepsilon_k}(., 0) \to \hat{u}(., 0)$, $u_{\varepsilon_k}(., 1) \to u(., 1)$, $\hat{u}_{\varepsilon_k}(., 1) \to \hat{u}(., 1)$,

$u_{\varepsilon_k,x}(.,0+) \to u_x(.,0+)$, $\hat{u}_{\varepsilon_k,x}(.,0+) \to (\hat{u}_x(.,0+)$, $u_{\varepsilon_k,x}(.,1-) \to u_x(.,1-)$ and $\hat{u}_{\varepsilon_k,x}(.,1-) \to \hat{u}_x(.,1-)$ a.e. on $[0, T_0]$. Also, since $b_{1,\varepsilon}^t \leqslant \hat{b}_{1,\varepsilon}^t$ on R, we see by (1.14) that

$$u_{\varepsilon,x}(t, 0+) \geqslant \hat{u}_{\varepsilon,x}(t, 0+) \quad \text{if } u_\varepsilon(t, 0) > \hat{u}_\varepsilon(t, 0).$$

Hence, by the lower semicontinuity of $\sigma_0(.)$ on $[0, \infty)$, (6.9) holds for a.e. $t \in [0, T_0]$. Similarly, (6.10) holds for a.e. $t \in [0, T_0]$. ■

LEMMA 6.6 *$S_\varepsilon^{(i)} \to 0$ in $L^2(0, T_0)$ for $i = 1, 2$, and $F_\varepsilon \to 0$ in $L^2(0, T_0)$ as $\varepsilon \downarrow 0$.*

Proof. Since $l'(t) = -u_x(t, l(t)-) + u_x(t, l(t)+)$ and $\hat{l}'(t) = -\hat{u}_x(t, \hat{l}(t)-) + \hat{u}_x(t, \hat{l}(t)+)$ for a.e. $t \in [0, T_0]$, (6.3) of Lemma 6.2 implies that $S_\varepsilon^{(i)} \to 0$ in $L^2(0, T_0)$ as $\varepsilon \downarrow 0$ for $i = 1, 2$. Next, we observe from (1.1) that

$$|f_\varepsilon(t, x) \leqslant \varepsilon |l'(t)| \max\{C_1, C_2\}, \quad |\hat{f}_\varepsilon(t, x)| \leqslant \varepsilon |\hat{l}'(t)| \max\{C_1, C_2\}.$$

Hence, $F_\varepsilon \to 0$ in $L^2(0, T_0)$ as $\varepsilon \downarrow 0$. ■

Letting $\varepsilon = \varepsilon_k \downarrow 0$ in (6.8), we see by Lemmas 6.2, 6.5, 6.6 that

$$\begin{aligned} &|(v(t)-\hat{v}(t))^+|_{L^1(0,1)} - |(v(s)-\hat{v}(s))^+|_{L^1(0,1)} + (l(t)-\hat{l}(t))^+ - (l(s)-\hat{l}(s))^+ \\ &\quad + \int_s^t (u_x(\tau, 0+) - \hat{u}_x(\tau, 0+))\, \sigma_0([u(\tau, 0) - \hat{u}(\tau, 0)]^+)\, d\tau \\ &\quad - \int_s^t (u_x(\tau, 1-) - \hat{u}_x(\tau, 1-))\, \sigma_0([u(\tau, 1) - \hat{u}(\tau, 1)]^+)\, d\tau \\ &\leqslant 0 \quad \text{for any } [s, t] \subset [0, T_0], \end{aligned}$$

so that (1.17) holds. Thus the proof of Theorem 1.2 is complete.

REMARK 6.1 As is easily seen in the above proof of inequality (1.17), when u (resp. $\hat{u}$) is strictly positive on $Q_l^1(T_0)$ (resp., $Q_{\hat{l}}^1(T_0)$) and strictly negative on $Q_l^2(T_0)$ (resp., $Q_{\hat{l}}^2(T_0)$), it is not necessary to approximate u (resp., $\hat{u}$) by u_ε (resp., $\hat{u}_\varepsilon$) but one can directly obtain (1.17).

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Dwufazowe zagadnienie Stefana z nieliniowymi warunkami brzegowymi opisywanymi przez subróżniczki zależne od czasu

W pracy rozważane jest jednowymiarowe dwufazowe zagadnienie Stefana z nieliniowymi warunkami zadanymi na nieruchomej części brzegu. Warunki te zawierają operatory subróżniczkowe zmienne w czasie. Udowodnione zostają twierdzenia o lokalnym w czasie istnieniu i jednoznaczności rozwiązania rozważanego zagadnienia. Konstrukcja rozwiązania lokalnego korzysta z metod teorii nieliniowych równań ewolucyjnych ze zmiennymi w czasie operatorami subróżniczkowymi w przestrzeniach Hilberta, rozwiniętej przez autora. Rozważane zagadnienie Stefana zostaje sprowadzone do układu równań ewolucyjnych takiego typu.

Двухфазные проблемы Стефана с нелинейными краевыми условиями содержающими субдифференциалы зависящие от времени

В работе рассуждается одномерная двухфазная проблема Стефана с нелинейными условиями на фиксированном крае области. Эти условия содержают субдифференциальные операторы зависящие от времени. Доказаны теоремы о локальном во времени существовании и единственности решения проблемы. Конструкция локального решения использует методы теории нелинейных эволюционных уравнений с зависящими от времени субдифференциальными операторами в гилбертовом пространстве. Рассуждается проблема Стефана сведена к системе эволюционных уравнений такого рода.