Control
and Cybernetics
Vol. 16 (1987) No. 3–4

On the approximation of the boundary control in two-phase Stefan-type problems

by

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The paper is concerned with boundary control of two-phase Stefan problems. A construction of optimal solutions, based on exploiting regularization techniques, is presented. Results of some numerical experiments are discussed.

Introduction

Consider the boundary control problem

$$\text{(P)} \qquad \text{Minimize } \left\{\pi(u) = \int_0^T \left[\tfrac{1}{2}|y-d|^2_{L^2(\Omega)} + \tfrac{1}{2}|u|^2_{L^2(\partial\Omega)}\right] dt\right\}$$

over all $u \in L^2(\Sigma)$ and $y = y(u) \in L^2(0, T; H^1(\Omega))$ subject to

$$\begin{cases} \dfrac{\partial}{\partial t} v(t, x) - \Delta y(t, x) = f(t, x) & \text{a.e. in } Q, \\ v(t, x) \in \beta\big(y(t, x)\big) & \text{a.e. in } Q, \end{cases} \tag{1.1}$$

$$\frac{\partial y(t, x)}{\partial n} = u(t, x) \quad \text{a.e. on } \Sigma, \tag{1.2}$$

$$v(0, x) = v_0(x) \quad \text{a.e. on } \Omega. \tag{1.3}$$

In the above problem, $\Omega \subset R^n$, $n \geqslant 1$, is a bounded domain with smooth boundary Γ and $Q =]0, T[\times \Omega$ is a cylinder with the lateral boundary Σ. We assume that $v_0 \in L^2(\Omega)$, $d \in L^2(Q)$ and that β is a strongly maximal monotone graph in $R \times R$, bounded on bounded sets.

The system (1.1) contains the two-phase Stefan problem as a special case. Let namely β be given by

$$\beta(r) = \begin{cases} r - r_0, & r > r_0, \\ [-\delta, 0], & r = r_0 \\ \varkappa(r - r_0) - \delta, & r < r_0 \end{cases} \tag{1.4}$$

where $\varkappa$, $\delta > 0$, then we obtain a two-phase Stefan problem.

This paper can be considered as supplementary to the previous works [10, 11, 12] of the authors. The material is organized as follows. In chapter 2, we briefly discuss the existence questions and a regularization of Problem (P). As usual, the regularization process consists of replacing (P) by a family of smooth problems and tending to the limit with the approximate controls (see Theorem 2.3 and Theorem 2.4). The main purpose of this paper is to show that the standard steepest descent type algorithm (see Algorithm 3.2) may be used for finding iteratively the solution for the regularized control problem. The main result of chapter 3 is Theorem 3.3 where the descent property of the Algorithm 3.2 is proved. Due to the lack of convexity, the emphasis will be on the descent property, not on the convergence properties of the algorithm. To obtain the numerical solution of the state equation and the adjoint system, finite elements in space and finite differences in time are used. In chapter 4, a numerical example is given. Numerical tests show that the descent property of the algorithm is provided.

Concerning the recent literature of this field we first of all refer to the profound habilitation thesis of Pawłow [18] (and references therein) as well as to the special issue of this journal [16]. For the numerical approximation of Stefan-type problems we quote furthermore [6, 7, 17, 18, 20, 22, 23–25], for control problems of the above type and their approximation see [8, 11, 13, 18, 20, 21] and for industrial applications [9, 14, 18, 20]. For an analysis of used regularization technique see [1, 2, 18].

2. Existence and regularization

We will briefly outline the existence of an $L^2(\Sigma)$ optimal control for problem (P). Next, the approximation properties of the regularized controls are given. For more details, we quote [21].

Denote $V = H^1(\Omega)$, $H = L^2(\Omega)$ with scalar product $(\cdot, \cdot)$ and norm $\|\cdot\|$; V^* is the dual of V. System (1.1)–(1.3) can be written in abstract form as

$$\frac{dv}{dt}+Ay=f,\; v(t)\in B(y(t)) \quad \text{a.e. } [0,T], \tag{2.1}$$

$$v(0)=v_0. \tag{2.2}$$

The function $f\in L^2(0,T;V^*)$ is given by

$$\int_0^T (f(t),\psi(t))\,dt=\int_0^T\int_\Gamma u\cdot\psi\,d\Gamma\,dt,\quad \forall\psi\in L^2(0,T;V). \tag{2.3}$$

Operator $A\colon V\to V^*$ is defined by

$$(Ay,z)=\int_\Omega \operatorname{grad} y\cdot\operatorname{grad} z\,dx,\qquad \forall y,\, z\in V. \tag{2.4}$$

and operator $B\colon H\to H$ is the realization of β in $L^2(\Omega)$.
The existence of a solution for problem (2.1), (2.2) is studied, for example, in paper [3], where A and B may be both nonlinear. From [21], we have

THEOREM 2.1 *Let $u_n\to u$ weakly in $L^2(\Sigma)$. Then $y_n\to y$ weakly in $L^2(0,T;V)$, where y_n, y are the solutions of* (2.1), (2.2) *corresponding to u_n, u.*
From this result, one obtains at once

THEOREM 2.2 *There is an optimal pair $[u^*,y^*]$ in $L^2(\Sigma)\times L^2(0,T;V)$ for problem* (P).
Consider the regularized problem

(P_ε) Minimize $\{\pi^\varepsilon(u)=\int_0^T[\frac{1}{2}|y-d|_H^2+\frac{1}{2}|u|_H^2]\,dt\}$

subject to

$$\frac{\partial\beta^\varepsilon(y(t,x))}{\partial t}-\Delta y(t,x)=f(t,x)\quad \text{a.e. in } Q, \tag{2.5}$$

$$\frac{\partial}{\partial n}y(t,x)=u(t,x)\quad \text{a.e. on } \Sigma, \tag{2.6}$$

$$y(0,x)=y_0(x)\quad \text{a.e. on } \Omega, \tag{2.7}$$

where we define

$$\beta^\varepsilon(y)=y+\int_{-\infty}^{\infty}\gamma_\varepsilon(y-\varepsilon^2\theta)\varrho(\theta)\,d\theta, \tag{2.8}$$

and γ_ε is the Yosida approximation of the maximal monotone graph $\gamma(y)=\beta(y)-y$ (we assume for convenience that $\varkappa\geqslant 1$ in (1.4)), and ϱ is a

Friedrichs mollifier, such that $\varrho \in C_0^\infty(R)$, supp $\varrho \subset (-1, 1)$, $\varrho(-\theta) = \varrho(\theta)$ and $\int_{-\infty}^{\infty} \varrho(\theta)\,d\theta = 1$.

Obviously, the problem (P_ε) has on optimal pair $[y_\varepsilon, u_\varepsilon] \in L^2(Q) \times L^2(\Sigma)$. Furthermore we have, see ([21])

THEOREM 2.3 *The subsequences converge as follows*

$$u_\varepsilon \to u^* \text{ strongly in } L^2(\Sigma), \tag{2.9}$$

$$y_\varepsilon \to y^* \text{ strongly in } L^2(Q). \tag{2.10}$$

The corresponding convergence result for the cost functional is, see ([14]):

THEOREM 2.4 *The sequence $\pi(u_\varepsilon) \to \pi(u^*)$, the optimal value of problem* (P), *when $\varepsilon \to 0$, and therefore $\{u_\varepsilon\}$, is a minimizing sequence for* (P).

3. The descent property

In order to obtain a suboptimal control for (P), by Theorem 2.4, one may solve problem (P_ε). Due to the good differentiability properties in (P_ε), a gradient algorithm can be utilized to find u_ε efficiently.
We denote by θ_ε: $L^2(\Sigma) \to L^2(Q)$ the mapping $u \to y$ given by (2.5)–(2.7).

THEOREM 3.1 *For all $u \in L^2(\Sigma)$ there exists a linear operator $\nabla\theta_\varepsilon(u)$: $L^2(\Sigma) \to L^2(Q)$ defined by:*

$$\nabla\theta_\varepsilon(u)\,v = \operatorname*{weak-lim}_{\lambda \to 0} \frac{\theta_\varepsilon(u+\lambda v) - \theta_\varepsilon(u)}{\lambda} \tag{3.1}$$

for all $v \in L^2(\Sigma)$. Moreover

$$\nabla\theta_\varepsilon(u)\,v = \frac{\partial z}{\partial t}, \tag{3.2}$$

where z is the solution of the problem

$$\nabla\beta^\varepsilon(\theta_\varepsilon(u))\frac{\partial z}{\partial t} + Az = h \quad \text{a.e. } [0, T], \tag{3.3}$$

$$z(0) = 0. \tag{3.4}$$

In equation (3.3), *$h \in W^{1,2}(0, T; V^*)$ satisfies*

$$h(t) = \int_0^t g_1(\xi)\,d\xi + v_0, \tag{3.5}$$

$$\int_0^T (g_1(t), \psi(t))\,dt = \int_0^T \int_\Gamma v\psi\, d\Gamma\, dt, \quad \psi \in L^2(0, T; V). \tag{3.6}$$

Proof. Denote by B^ε the realization in H of β^ε and $y_\lambda = \theta_\varepsilon(u+\lambda v)$, $y = \theta_\varepsilon(u)$. Then, by the definition of solution we get

$$B^\varepsilon\left(\frac{dw_\lambda}{dt}\right) + Aw_\lambda = g + \lambda h \quad \text{a.e. } [0, T], \tag{3.7}$$

$$B^\varepsilon\left(\frac{dw}{dt}\right) + Aw = g \quad \text{a.e. } [0, T], \tag{3.8}$$

$$w_\lambda(0) = w(0) = 0. \tag{3.9}$$

Here $g(t) = \int_0^t f(\xi)\,d\xi + v_0$, f given by (2.3) and $w_\lambda(t) = \int_0^t y_\lambda(\xi)\,d\xi$, $w(t) = \int_0^t y(\xi)\,d\xi$

Subtract the above two relations and multiply by $\frac{dw_\lambda}{dt} - \frac{dw}{dt}$, to get

$$\int_0^t \left|\frac{dw_\lambda}{dt} - \frac{dw}{dt}\right|_H^2 ds + \frac{1}{2}\big(A(w_\lambda(t) - w(t)), w_\lambda(t) - w(t)\big) \leqslant \lambda \int_0^t \left(h, \frac{dw_\lambda}{dt} - \frac{dw}{dt}\right) ds.$$

Then, $\frac{dw_\lambda}{dt} \to \frac{dw}{dt}$ and $w_\lambda \to w$ strongly in $L^2(0, T; H)$, $C(0, T; H)$ respectively.

We set $z_\lambda = \frac{w_\lambda - w}{\lambda}$, that is

$$\int_0^t \left|\frac{dz_\lambda}{dt}\right|_H^2 ds + \frac{1}{2}(Az_\lambda(t), z_\lambda(t)) \leqslant \int_0^t \left(h, \frac{dz_\lambda}{dt}\right) ds.$$

Integrating by parts in the right hand side we obtain $\{z_\lambda\}$, $\left\{\frac{dz_\lambda}{dt}\right\}$ bounded in $L^\infty(0, T; V)$, $L^2(0, T; H)$. Since B^ε is Lipschitz, the Lebesgue theorem shows that

$$\frac{B^\varepsilon\left(\frac{dw_\lambda}{dt}\right) - B^\varepsilon\left(\frac{dw}{dt}\right)}{\lambda} = \frac{B^\varepsilon\left(\frac{dw_\lambda}{dt}\right) - B^\varepsilon\left(\frac{dw}{dt}\right)}{\frac{dw_\lambda}{dt} - \frac{dw}{dt}} \cdot \frac{dz_\lambda}{dt}$$

is weakly convergent in $L^2(0, T; H)$ to $\nabla B^\varepsilon\left(\frac{dw}{dt}\right) \cdot \frac{dz}{dt}$, where z is such that $z_\lambda \to z$ strongly in $C(0, T; H)$.

We can pass to the limit and obtain (3.2)–(3.4) to finish the proof. ■

Now, we can define the adjoint system for the control problem (P_ε):

$$\nabla\beta^\varepsilon(y_\varepsilon)\frac{\partial p_\varepsilon}{dt}-Ap_\varepsilon=y_\varepsilon-d \quad \text{a.e. in } [0,T], \tag{3.10}$$

$$p_\varepsilon(T)=0. \tag{3.11}$$

The gradient algorithm for solving problem (P_ε) is obvious (for brevity we omit the subindex ε):

Algorithm 3.2

Step 1. Choose any u_0 and set $n:=0$.
Step 2. Compute y_n by solving (2.5)–(2.7).
Step 3. Test if the pair $[y_n, u_n]$ is satisfactory; if YES then STOP; otherwise GO TO step 4.
Step 4. Compute p_n by (3.10)–(3.11).
Step 5. Compute u_{n+1} by equation

$$u_{n+1}=u_n-\varrho_n(u_n-p_{n|\Sigma}),\ \text{where } \varrho_n \text{ is an appropriate real parameter.} \tag{3.12}$$

Step 6. Set $n:=n+1$ and GO TO step 2.

The convergence test involved in step 3 is based on the difference $|u_n-p_{n|\Sigma}|$ which is to be smaller than a given parameter. In step 5, the parameter ϱ_n can for example be selected by utilizing a line search.
It is known that without convexity assumptions, the above gradient algorithm may be convergent only to a stationary point of the functional (see [5]). Since the state equation is nonlinear, the cost functional is no more convex and our result underlines the descent property of (3.12). In finite dimensional case the situation is evident.

THEOREM 3.3.
(i) *Let ε be fixed. The sequence $\pi_\varepsilon(u_n)$ is convergent, when $n\to\infty$.*
(ii) *Let $\tilde{u}_\varepsilon$ be the approximate value of $\tilde{u}_\varepsilon$ as computed by Algorithm 3.2. The sequence $\pi_\varepsilon(\tilde{u}_\varepsilon)$ is bounded with respect to ε and every cluster point $\tilde{\pi}$ satisfies*

$$\tilde{\pi}\leqslant\pi(u_0) \tag{3.13}$$

where u_0 is the first iteration.

Proof.
(i) The sequence $\{\pi_\varepsilon(u_n)\}$ decreases and it is bounded by $\pi_\varepsilon(u_0)$ and $\pi_\varepsilon(u_\varepsilon)$.
(ii) We have

$$\pi_\varepsilon(u_\varepsilon)\leqslant\pi_\varepsilon(\tilde{u}_\varepsilon)\leqslant\pi_\varepsilon(u_0) \tag{3.14}$$

and, by an easy consequence of Theorem 2.4, $\pi_\varepsilon(u_\varepsilon)\to\pi(u^*)$.

We will show that $\pi_\varepsilon(u_0) \to \pi(u_0)$, too. This is equivalent to

$$y_\varepsilon = \theta_\varepsilon(u_0) \to y \text{ strongly in } L^2(Q),$$

where y is the solution of (1.1)–(1.3) corresponding to u_0. Let $w_\varepsilon(t) = \int_0^t y_\varepsilon(\xi)\,d\xi$. Then

$$B^\varepsilon\left(\frac{dw_\varepsilon}{dt}\right) + Aw_\varepsilon = q_0 \quad \text{a.e. } [0, T], \tag{3.15}$$

$$w_\varepsilon(0) = 0,$$

with $g_0(t) = \int_0^t f_0(\xi)\,d\xi + v_0$ and

$$\int_0^T (f_0(t), \psi(t))\,dt = \int_0^T \int_\Gamma u_0 \cdot \psi\,d\Gamma\,dt, \ \forall \psi \in L^2(0, T; V).$$

Multiply (3.15) by $\left\{\frac{dw_\varepsilon}{dt}\right\}$. Then we obtain $\{w_\varepsilon\}$, $\left\{\frac{dw_\varepsilon}{dt}\right\}$ bounded in $L^\infty(0, T; V)$ and $L^2(0, T; H)$, respectively.

Since B is supposed to be bounded on bounded sets, we get $\left\{B^\varepsilon\left(\frac{dw_\varepsilon}{dt}\right)\right\}$ bounded in $L^2(0, T; H)$.

Next, subtract two equations (3.15) and multiply by $\frac{dw_\varepsilon}{dt} - \frac{dw_\sigma}{dt}$. By (2.8) we get:

$$\int_0^t \left|\frac{dw_\varepsilon}{dt} - \frac{dw_\sigma}{dt}\right|_H^2 + \int_0^t \int_\Omega \left(\gamma^\varepsilon\left(\frac{dw_\varepsilon}{dt}\right) - \gamma^\sigma\left(\frac{dw_\sigma}{dt}\right), \frac{dw_\varepsilon}{dt} - \frac{dw_\sigma}{dt}\right)$$
$$+\frac{1}{2}|\nabla w_\varepsilon(t) - \nabla w_\sigma(t)|_H^2 = 0. \tag{3.16}$$

Here $\gamma^\varepsilon(y) = \beta^\varepsilon(y) - y$, i.e. the second term in (2.8) and $\left\{\gamma^\varepsilon\left(\frac{dw_\varepsilon}{dt}\right)\right\}$ is bounded in $L^2(Q)$.

Taking into account the properties of the Yosida approximation:

$$\gamma_\varepsilon(y) \in \gamma((I + \varepsilon\gamma)^{-1}(y)), \ \varepsilon\gamma_\varepsilon(y) = y - (I + \varepsilon\gamma)^{-1}(y)$$

and the above boundedness, one can infer from (3.16) that $\{w_\varepsilon\}$, $\left\{\frac{dw_\varepsilon}{dt}\right\}$ are Cauchy sequences in $L^2(0, T; V)$ and $L^2(Q)$, respectively. Now, it is possible to pass to the limit in (3.15) and to finish the proof.

REMARK 3.4. The practical meaning of Theorem 3.3 is that in a given problem one should take the control u_0 already used in practice as the first iteration. Then, the algorithm improves the performance given by it.

4. A numerical example

The regularized state problem (2.5)–(2.8) and the adjoint state problem are discretized by applying a finite difference method in time and a finite element method in space. The adjoint state in the discrete case as well as optimality conditions are derived in [11, 18]. For the convergence of approximations see [11, 18]. To illustrate the use of Algorithm 3.2, the following numerical example is considered:

$$\Omega =]0, 1[\times]0, 1[,$$

$$T = 1.$$

Let

$$\beta(y) = \begin{cases} y, & \text{if } y < 0, \\ [0, 2], & \text{if } y = 0, \\ 4y+2, & \text{if } y > 0, \end{cases} \tag{4.1}$$

$$f(t, x_1, x_2) = \begin{cases} 8(2e^{-2t}-1), & \text{if } x_1^2+x_2^2 > e^{-2t}, \\ 2(e^{-2t}-2), & \text{if } x_1^2+x_2^2 \leqslant e^{-2t} \end{cases} \tag{4.2}$$

$$v_0 = \beta(y_0) \tag{4.3}$$

and

$$y_0 = \begin{cases} x_1^2+x_2^2-1, & \text{if } x_1^2+x_2^2 < 1, \\ 2(x_1^2+x_2^2-1), & \text{if } x_1^2+x_2^2 \geqslant 1. \end{cases} \tag{4.4}$$

For the boundary control

$$u(t, x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = 0, \text{ or } x_2 = 0, \\ 4 & \text{on the remaining of } \partial\Omega. \end{cases} \tag{4.6}$$

The exact solution y, of (1.1)–(1.3) with given data (4.1)–(4.4) is

$$y(t, x_1, x_2) = \begin{cases} 2(x_1^2+x_2^2-e^{-2t}), & \text{if } x_1^2+x_2^2 > e^{-t}, \\ x_1^2+x_2^2-e^{-t}, & \text{if } x_1^2+x_2^2 \leqslant e^{-t}. \end{cases}$$

Consider the cost functional

$$\pi_\lambda(u) = \frac{1}{2}\int_0^1 [|y|_{L^2(\Omega)} + \frac{\lambda}{2}|u|^2_{L^2(\partial\Omega)}]\,dt \qquad \text{with } \lambda = 0.1.$$

We shall now test the efficiency of different variants of Algorithm 3.2. The nonlinear programming methods tested are:

– steepest descent Algorithm 3.2,

- a conjugate gradient method with an automatic restart ([19], ZXCGR of IMSL Subroutine Library),
- a bundle algorithm due to C. Lemarechal (BCG).

We have chosen $\Delta t = 1/16$ (time step) and 64 triangular linear elements in discretization of state and adjoint problem.
For simplicity, we have replaced β^ε by a piecewise linear function such as

$$\beta_\varepsilon(y) = \begin{cases} y, & \text{if } y < 0 \\ \dfrac{2+y_\varepsilon}{\varepsilon}, & \text{if } y \in [0, \varepsilon] \\ 4y+2, & \text{if } y > \varepsilon \end{cases}$$

for $\varepsilon = 1/16$ (with appropriate modification for $y = 0$ and $y = \varepsilon$ in the case of standard gradient algorithms).
In Table 4.1 we see the diminution of π_λ per iteration when three different gradient algorithms have been applied.

Table 4.1. Comparison of different gradient algorithms

Number of iteration	Value of $\pi_\lambda(u^n)$ for different gradient algorithms		
	steepest descent	ZXCGR	BCG
0	2.166	2.166	2.166
1	.426	.418	.935
2	.203	.148	.681
3	.124	.116	.252
4	.110	–	.208
5	.101	–	.144
6	.091	–	.142
7	.090	–	.102
CPU (seconds)	840	181	488

The optimal control found by different gradient algorithms is roughly speaking the same.
In Figures 4.2–4.4 we can see the boundary controls and corresponding temperature distributions obtained by Algorithm 3.2 at time levels $t = .325$, $t = .625$ and for $t = .935$.
From above results we see that gradient algorithms have modified boundary control to the right direction. As the number of boundary nodes is relatively high (i.e. number of unknowns in control problem; 412 in above case) cg-type algorithm is relatively efficient.

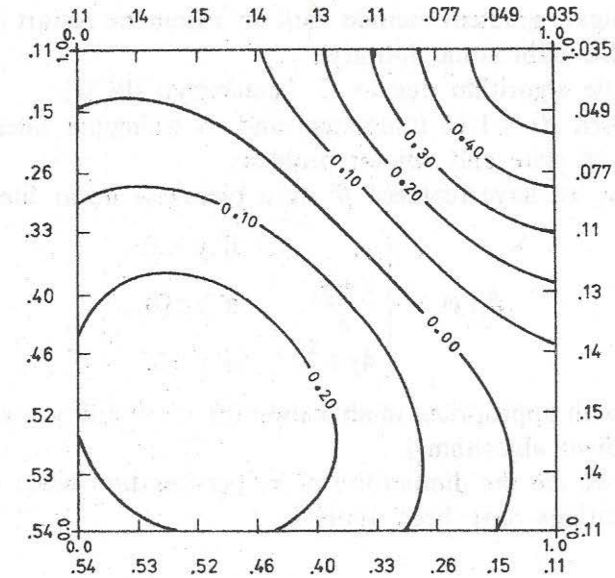

Figure 4.2. u_h and y_h for $t = .325$

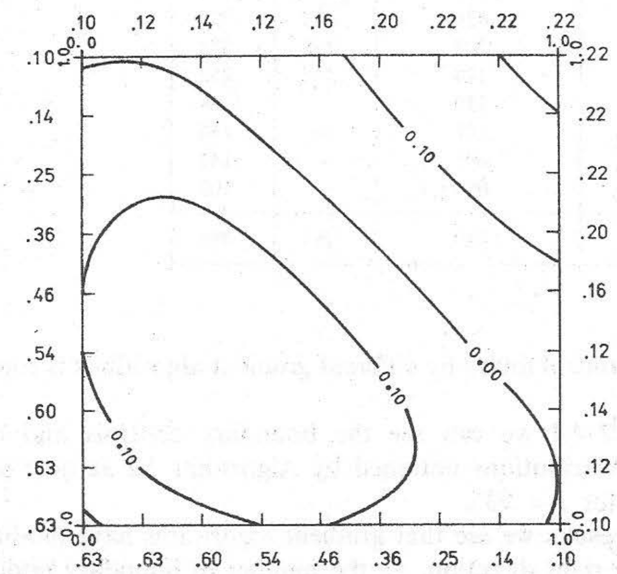

Figure 4.3. u_h and y_h for $t = .625$

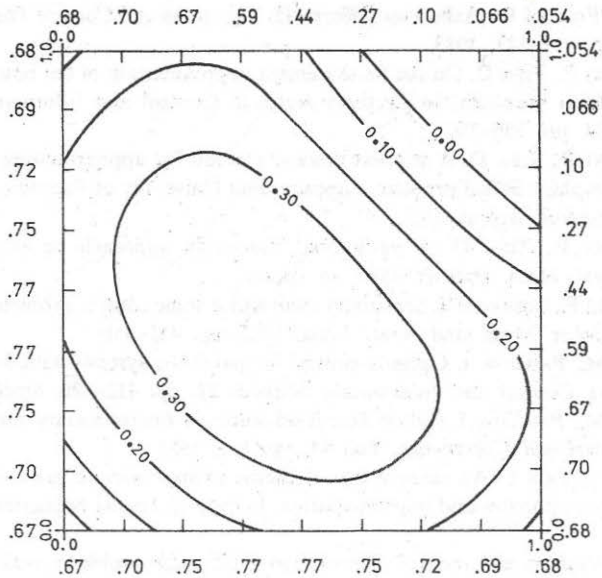

Figure 4.4. u_h and y_h for $t = .935$

As a summary of the above numerical tests we note that the descent property is provided and we have succeeded in improving the given performance as claimed.

5. References

[1] Barbu V. Boundary control problems with nonlinear state equation. *SIAM J., Control Optim.*, **20** (1982), 125–143.

[2] Barbu V. Optimal control of variational inequalities. Research Notes in Mathematics. 100, Boston, Pitman, 1984.

[3] Di Benedetto E., Showalter R. E. Implicit degenerate evolution equations and applications. *SIAM J., Math. Anal.*, **12** (1981), 5.

[4] Bermudez A., Saguez C. Optimal control of variational inequalities. in: [16], 9–30.

[5] Cea J. Optimization – Theory and Algorithms. Tata Inst. of Fund. Research. Bombay, Springer, 1978.

[6] Elliot C. M., Ockendon J. R. Weak and variational methods for moving boundary problems. Research Notes in Mathematics. 59. Boston, Pitman, 1982.

[7] Magenes E. Remarques sur l'approximation des problémes non lineares paraboliques. Pavia, Instituto di Analisi Numerica, Preprint 536, 1986.

[8] Meike Z., Tiba D. Optimal Control for a Stefan problem. in: Lecture Notes in Control and Information Sciences, 44. Springer, 1982, 776–787.

[9] Neittaanmäki P. On the control of the secondary cooling in the continuous casting process, in "Proc. Optimal Control of Partial Differential Equations II", (Eds. K. H. Hoffmann and W. Krabs), Birkhäuser, 1987, pp. 161–178.

[10] Neittaanmäki P., Tiba D. A descent method for the boundary control of a two-phase Stefan

problem, in "Proc. of Workshop on Differential Equations and Control Theory", INCREST, Bucharest, pp. 111–127, 1983.

[11] Neittaanmäki P., Tiba D. On the finite element approximation of the boundary control for two-phase Stefan problems. In Lectures Notes in Control and Information Sciences, 62, Springer, 1984, pp. 356–370.

[12] Neittaanmäki P., Tiba D. A steepest descent method for approximating of the boundary control in two-phase Stefan problem, Lappeenranta University of Technology, Dept. Physics & Math., Research Report 1/85.

[13] Neittaanmäki P., Tiba D. A variational inequality approach to constrained control problems, *Appl. Math. Optimiz.* 1987, to appear.

[14] Neittaanmäki P., Tiihonen T. Sensitivity analysis for some control problems, in Proc. of Int. Confr. on Numer. Meth. and Appl., Sofia, 1985, pp. 451–458.

[15] Niezgódka M., Pawlow I. Optimal control for parabolic systems with free boundaries, in Lecture Notes Control and Information Sciences 22, pp. 412–420, Springer, 1979.

[16] Niezgódka M., Pawlow I. (Eds.); Recent advances in free boundary problems. A special issue of *Control and Cybernetics*, Vol. 14, pp. 1–3, 1985.

[17] Nochetto R., Verdi C. An efficient linear scheme to approximate parabolic free boundary problems: error estimates and implementation, Istituto di Analisi Numerica, Pavia, Preprint 537, 1986.

[18] Pawlow I. Analysis and control of evolution multiphase problems with free boundaries, Prace habilitacyjne, Polska Akademia Nauk, 1987.

[19] Powell M. J. D. Restart procedures for the conjugate gradient method, *Math. Programming* 12, 1977, pp. 241–254.

[20] Saguez C. Contrôle optimal de systemès à frontiere libre, Thèse, Univ. Technologique Compiegne, 1980.

[21] Tiba D. Boundary control for a Stefan problem, ISNM 68, Birkhäuser, Basel, 1984, pp. 229–242.

[22] Tiba D., Tiba M. Regularity of the boundary data and the convergence of the finite element discretization in two-phase Stefan problems, *Int. J. Eng. Sci.* (to appear).

[23] Verdi C., Visintin A. Error estimates for a semi explicit numerical scheme for Stefan-type problems, *Numer. Math.* to appear.

[24] Voller V. R. Implicit finite-difference solutions of the enthalphy formulation of Stefan problems, *IMA J. Numer. Anal.* 5, 1985, pp. 201–214.

[25] White R. E. A nonlinear parallel algorithm with application to the Stefan problem, *SIAM J. Numer. Anal.* 23, 1986, pp. 639–652.

[26] Zlamal M. A finite element solution of the nonlinear heat equation, *R.A.I.R.O. Num. Anal.*, 14, 1980, pp. 203–216.

Received, June 1987.

O aproksymacji sterowań brzegowych w dwufazowych zagadnieniach Stefana

W pracy rozważane są zadania sterowania brzegowego dwufazowych zagadnień Stefana. Konstrukcja rozwiązań optymalnych bazuje na zastosowaniu metody regularyzacji. Przedstawione są wyniki testów numerycznych.

Аппроксимация граничных управлений в двухфазных проблемах Стефана

В работе рассматриваются задачи граничного управления для двухфазных проблем Стефана. Для построения оптимальных решений используются методы регуляризации. Представлены результаты численных экспериментов.