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A binary alloy solidification problem for parabolic differential equations

by

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We solve a one-dimensional free boundary problem which describes solidification of a two-component mixture. We also determine the composition of liquid and the corresponding solid. An existence and uniqueness theorem for the local classical solution is given.

1. Introduction

In the present paper we study the one-dimensional free boundary problem with the free boundary of a solidifying two-component mixture in the segment $0 < x < 1$ as well as the corresponding distributions of temperature and composition to be determined. The temperature at the free boundary changes because of the two-component character of mixture. Dependence of the temperature on the composition is given in chemistry by a so-called phase diagram. Such diagrams are used for molar ratios in the interval (0,1) or for percentage compositions in the interval (0,100). In this paper, the diagrams are given by two functions, describing the dependence of temperature on composition of the liquid and the solid, respectively. We assume in this paper that the diagrams are given by piecewise linear functions. In practice such diagrams are given approximately, so our assumption is reasonable.

The novelty contained in this paper consists in determining the free boundary on which the temperature of solidification is unknown. This temperature will also be calculated. From this we shall find the composition and temperatures of the liquid and the solidified solid as functions of position and time.

The problem including a phase diagram was considered in [15] but in unbounded domain. In real processes one usually considers bounded domains, with boundary conditions strongly affecting the free boundary movement. Therefore, the problem with an unbounded domain is methodologically different from ours. Stefan problems over bounded domains were considered in [4], [6], [8], [9], [10], [12]. An extensive bibliography is offered in [5], [11], [13].

Let us briefly expose a physical origin of the problem under consideration. A detailed presentation is given in [2]. In most papers on the Stefan problem the temperature of solidification is constant, whereas in our paper the solidification process is perturbed by the transport of components in the liquid which induces variation of the solidification temperature on the free boundary according to the phase diagram. In effect we get a system of parabolic equations coupled by relevant boundary conditions.

The diffusion of components in solid is incomparably smaller than in liquid, so we put the thermal diffusivity equal to zero in solid and equal to one in liquid. The diffusivity in the solid could be introduced as a small coefficient and then it would be possible to analyse the limit behaviour of the solution when this coefficient tends to zero. The other one does not change the generality of consideration because there is possibility of transforming the spatial axis in a linear manner to get this coefficient equal to 1.

We shall use the results of [3] in our construction of the integral representation of the solution of our problem.

We would like to point out that the Banach fixed point theorem is used in the proof of the existence and uniqueness of the classical solution.

The correctness of the problem in the Hadamard sense is proved.

2. Statement of the problem

We denote by u the temperature and by c the concentration of one of the components in the mixture.

We consider the problem governed by the following system of equations

$$u_t^i = D_i u_{xx}^i, \quad D_i > 0 \text{ for } i = 1, 2 \text{ are thermal diffusivities,} \tag{1a}$$

$$c_t^1 = c_{xx}^1, \; c_t^2 = 0 \tag{1b}$$

in two sets

$$X^1 = \{(x, t) \in R^2 \colon s(t) > x > 0, \quad 0 < t < \sigma, \; \sigma < \infty\},$$

$$X^2 = \{(x, t) \in R^2 \colon s(t) < x < 1, \quad 0 < t < \sigma, \; \sigma < \infty\}$$

with the initial and boundary conditions

$$u^1(x, 0) = \varphi_u(x), \quad u_x^1(0, t) = 0, \tag{2}$$

$$u^2(1, t) = \psi(t), \tag{3}$$

$$u_x^2(s(t), t) = Au_x^1(s(t), t) + B\frac{ds(t)}{dt}, \quad A = \frac{k_1}{k_2}, \quad B = \frac{\lambda\varrho}{k_2}, \tag{4}$$

where k_1, k_2 are thermal conductivities, λ is the latent heat per unit volume and ϱ is the density.

$$u^1(s(t), t) = u^2(s(t), t), \tag{5}$$

$$c^1(x, 0) = \varphi_c(x), \quad c_x^1(0, t) = 0, \tag{6}$$

$$-c_x^1(s(t), t) = \big(c^1(s(t), t) - c^2(s(t), t)\big)\frac{ds(t)}{dt}, \tag{7}$$

$$c^1(s(t), t) = g\big(u^1(s(t), t)\big), \tag{8}$$

$$c^2(s(t), t) = f\big(u^1(s(t), t)\big), \tag{9}$$

where $\varphi_u(x)$, $\varphi_c(x)$, $\psi(t)$, A, B, D_1, D_2 are given. We assume

(A1) u_{xx}^i, u_t^i, c_{xx}^i, c_t^i are continuous functions for $(x, t) \in X^i$ for $i = 1, 2$,

(A2) $\varphi_u(x), \varphi_c(x) \in C^1([0, 1); (0, 1)), \psi(t) \in C^1((0, \sigma); R^- \cup \{0\})$,

$$\frac{d\psi(t)}{dt} < 0 \text{ for } t \in [0, \sigma), \quad \frac{d\varphi_c(x)}{dx} \geqslant 0 \text{ for } x \in [0, 1], \quad \left.\frac{d\varphi_c(x)}{dx}\right|_{x=1} > 0,$$

$$\frac{d\varphi_u(x)}{dx} \leqslant 0 \text{ and } \frac{d^2\varphi_u(x)}{dx^2} < 0 \text{ for } x \in [0, 1] \text{ and } \left.\frac{d\varphi_u(x)}{dx}\right|_{x=0} = 0,$$

(A3) $s(t) \in C^2((0, \sigma); (0, 1])$,

(A4) f and g are real functions, decreasing and piecewise linear in their domains in R, such that $f < g$ (they define the phase diagram).

We assume that temperature $u^1(x, t)$ of the liquid for $t = 0$ at the point $x = 1$ is the temperature of liquid solidification. We can change the scale of temperature so as to obtain $\varphi_u(1) = 0$, $\varphi_c(1) = 0$, hence $s(0) = 1$.

We say that the functions $u^i(x, t)$, $c^i(x, t)$, $s(t)$ form *a classical solution of problem* (1)–(9) if they satisfy assumptions (A1)–(A4) and equations (1a), (1b) with conditions (2)–(9).

3. The system of integral equations

The solution of problem (1)–(9) can be characterized as satisfying some system of integral equations. This system can be constructed by integrating the following identities

$$D_1\frac{\partial}{\partial\xi}(N^1 u_\xi^1 - N_\xi^1 u^1) = \frac{\partial}{\partial\tau}(N^1 u^1),$$

$$(*)\qquad \frac{\partial}{\partial\xi}(Nc_\xi^1 - N_\xi c^1) = \frac{\partial}{\partial\tau}(Nc^1) \text{ for } 0<\xi<s(\tau),\ 0<\varepsilon<\tau<t-\varepsilon \text{ and}$$

$$D_2\frac{\partial}{\partial\xi}(G^2 u_\xi^2 - G_\xi^2 u^2) = \frac{\partial}{\partial\tau}(G^2 u^2) \text{ for } s(\tau)<\xi<1,\ 0<\varepsilon<\tau<t-\varepsilon,$$

respectively, for the Neumann and Green functions

$$N^1(x, t; \xi, \tau) = K^1(x, t; \xi, \tau) + K^1(-x, t; \xi, \tau),$$

$$G^1(x, t; \xi, \tau) = K^1(x, t; \xi, \tau) - K^1(-x, t; \xi, \tau),$$

$$N^2(x, t; \xi, \tau) = K^2(x-1, t; \xi-1, \tau) + K^2(1-x, t; \xi-1, \tau)$$

$$G^2(x, t; \xi, \tau) = K^2(x-1), t; \xi-1, \tau) - K^2(1-x, t; \xi-1, \tau),$$

$$N(x, t; \xi, \tau) = K(x, t; \xi, \tau) + K(-x, t; \xi, \tau),$$

$$G(x, t; \xi, \tau) = K(x, t; \xi, \tau) - K(-x, t; \xi, \tau),$$

$$K^i(x, t; \xi, \tau) = \frac{1}{2\pi^{\frac{1}{2}} D_i^{\frac{1}{2}} (t-\tau)^{\frac{1}{2}}} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)D_i}\right),$$

$$K(x, t; \xi, \tau) = \frac{1}{2\pi^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right).$$

As in [3], we arrive at the following system of integral equations

$$u^1(x, t) = \int_0^1 N^1(x, t; \xi, 0)\varphi_u(\xi)\,d\xi + D_1\int_0^t N^1(x, t; s(\tau), \tau)u_\xi^1(s(\tau), \tau)\,d\tau + \int_0^t \left[N^1(x, t; s(\tau), \tau)\frac{ds(\tau)}{d\tau} - D_1 N_\xi^1(x, t; s(\tau), \tau)\right]u^1(s(\tau), \tau)\,d\tau, \tag{10}$$

$$u^2(x, t) = \int_0^t \left[D_2 G_\xi^2(x, t; s(\tau), \tau) - G^2(x, t; s(\tau), \tau)\frac{ds(\tau)}{d}\right]u^2(s(\tau), \tau)\,d\tau - -D_2\int_0^t G^2(x, t; s(\tau), \tau)u_\xi^2(s(\tau), \tau)\,d\tau - D_2\int_0^t G_\xi^2(x, t; 1, \tau)\psi(\tau)\,d\tau, \tag{11}$$

$$u_x^1(x, t) = \int_0^1 G^1(x, t; \xi, 0)\frac{d\varphi_u(\xi)}{d\xi}\,d\xi + D_1\int_0^t N_x^1(x, t; s(\tau), \tau)u_\xi^1(s(\tau), \tau)\,d\tau + +D_1\int_0^t N_x^1(x, t; s(\tau), \tau)\left(\frac{ds(\tau)}{d\tau} - D_1\left(\frac{ds(\tau)}{d\tau}\right)^{-2}\frac{d^2 s(\tau)}{d\tau^2}\right)u^1(s(\tau), \tau)\,d\tau + +D_1\int_0^t N_x^1(x, t; s(\tau), \tau)F(s(\tau))\,d\tau, \tag{12}$$

$$u_x^2(x,t) = -D_2\int_0^t N^2(x,t;1,\tau)\dot{\psi}(\tau)\,d\tau - D_2\int_0^t G_x^2(x,t;s(\tau),\tau)u_\xi^2(s(\tau),\tau)\,d\tau +$$

$$+\int_0^t G_x^2(x,t;s(\tau),\tau)\left[D_2\left(\frac{ds(\tau)}{d\tau}\right)^{-2}\frac{d^2s(\tau)}{d\tau^2}-\frac{ds(\tau)}{d\tau}\right]u^2(s(\tau),\tau)\,d\tau -$$

$$-D_2\int_0^t G_x^2(x,t;s(\tau),\tau)F(s(\tau))\,d\tau, \qquad (13)$$

where $F(\xi) = F^i(\xi) = -\frac{d}{d\xi}u^1(\xi, s^{-1}(\xi))$ for $i = 1, 2$. We denote

$$F(s(\tau)) - \left(\frac{ds(\tau)}{d\tau}\right)^{-2}\left(\frac{d^2s(\tau)}{d\tau^2}\right)u^1(s(\tau),\tau) = H(s(\tau)).$$

Further we show that under our assumptions $\frac{ds(t)}{dt} \neq 0$ in the whole time interval of the existence of solution. Then, taking $x \to s(t)-0$ and $x \to s(t)+0$ in (10), (12), (13), respectively, after setting (4) into (13) we have

$$u^1(s(t),t) = 2D_1\int_0^t N^1(s(t),t,s(\tau),\tau)u_\xi^1(s(\tau),\tau)\,d\tau +$$

$$+2\int_0^t N^1(s(t),t;s(\tau),\tau)\frac{ds(\tau)}{d\tau} + D_1 G_x^1(s(t),t;s(\tau),\tau)u^1(s(\tau),\tau)\,d\tau +$$

$$+2\int_0^1 N^1(s(t),t;\xi,0)\varphi_u(\xi)\,d\xi, \qquad (14)$$

$$u_x^1(s(t),t) - H(s(t)) - \frac{1}{D_1}\frac{ds(t)}{dt}u^1(s(t),t) =$$

$$= 2D_1\int_0^t N_x^1(s(t),t;s(\tau),\tau)u_\xi^1(s(\tau),\tau)\,d\tau -$$

$$-2\int_0^t N_x^1(s(t),t;s(\tau),\tau)\frac{ds(\tau)}{d\tau}u^1(s(\tau),\tau)\,d\tau +$$

$$+2D_1\int_0^t N_x^1(s(t),t;s(\tau),\tau)H(s(\tau))\,d\tau +$$

$$+2\int_0^1 G^1(s(t),t;\xi,0)\frac{d\varphi_u(\xi)}{d\xi}\,d\xi, \qquad (15)$$

$$Au_x^1(s,(t),t) - H(s(t)) + 2B\frac{ds(t)}{dt}u^1(s(t),t) =$$

$$= -2D_2\int_0^t G_x^2(s(t),t;s(\tau),\tau)Au_\xi^1(s(\tau),\tau)\,d\tau -$$

$$-2\int_0^t G_x^2(s(t), t; s(\tau), \tau)\frac{ds(\tau)}{d\tau}u^1(s(\tau), \tau)\,d\tau -$$

$$-2D_2\int_0^t G_x^2(s(t), t; s(\tau), \tau)\,H(s(\tau))\,d\tau + 2D_2\int_0^t N^2(s(t), t; 1, \tau)\,\psi(\tau)\,d\tau -$$

$$-2D_2 B\int_0^t G_x^2(s(t), t; s(\tau), \tau)\frac{ds(\tau)}{d\tau}\,d\tau. \tag{16}$$

Proceeding in the same manner as with the function $u^1(x, t)$ in [3], by virtue of (*) we have

$$c_x^1(x, t) = \int_0^1 G(x, t; \xi, 0)\frac{d\varphi_c(\xi)}{d\xi}\,d\xi + \int_0^t N_x(x, t; s(\tau), \tau)\,c_\xi^1(s(\tau), \tau)\,d\tau +$$

$$+\int_0^t N_x(x, t; s(\tau), \tau)\left[\frac{ds(\tau)}{d\tau} - \left(\frac{ds(\tau)}{d\tau}\right)^{-2}\frac{d^2 s(\tau)}{d\tau^2}\right]c^1(s(\tau), \tau)\,d\tau +$$

$$+\int_0^t N_x(x, t; s(\tau), \tau)\,F_c(s(\tau))\,d\tau, \tag{17}$$

where

$$F_c(\xi) = \frac{d}{d\xi}c^1(\xi, s^{-1}(\xi)) = \frac{d}{d\xi}g(u^1(\xi, s^{-1}(\xi)) =$$

$$= \frac{dg}{du^1}\frac{d}{d\xi}u^1(\xi, s^{-1}(\xi)) = \frac{dg}{du^1}F(\xi),$$

$$F_c(s(\tau)) - \left(\frac{ds(\tau)}{d\tau}\right)^{-2}\left(\frac{d^2 s(\tau)}{d\tau^2}\right)c^1(s(\tau), \tau) =$$

$$= \frac{dg}{du^1}F(s(\tau)) - \left(\frac{ds(\tau)}{d\tau}\right)^{-2}\left(\frac{d^2 s(\tau)}{d\tau^2}\right)\frac{g(u^1(s(\tau), \tau))}{u^1(s(\tau), \tau)}u^1(s(\tau), \tau).$$

Recall that f, g are piecewise linear functions. We take a segment which contains the point $(0, 0)$ and define the approximations $g(u^1) = r_1 u^1$, $f(u^1) = \tilde{r}_1 u^1 - b_1$ on this segment. We move $(0, 0)$ to the point $u = (\underline{u}, \bar{u})$ in which the phase diagram is not differentiable. Then, we successively take $g(u^1) = r_i u^1$, $f(u^1) = \tilde{r}_i u^1 - b_i$, to get $\dfrac{dg(u^1)}{du^1} = \dfrac{g(u^1)}{u^1}$. Hence, we obtain

$$F_c(s(\tau)) - \left(\frac{ds(\tau)}{d\tau}\right)^{-2}\frac{d^2 s(\tau)}{d\tau^2}c^1(s(\tau), \tau) =$$

$$= r_1\big(F(s(\tau))\big) - \left(\frac{ds(\tau)}{d\tau}\right)^{-2}\frac{d^2 s(\tau)}{d\tau^2}u^1(s(\tau), \tau) = r_1 H(s(\tau)).$$

From (17), after taking $x \to s(t)-0$ and using (8), we have

$$c_x^1(s(t), t) - r_1 H(s(t)) + r_1 \frac{ds(t)}{dt} u^1(s(t), t) = 2\int_0^1 G(s(t), t; \xi, 0) \frac{d\varphi_c(\xi)}{\xi} d\xi +$$
$$+2\int_0^t N_x(s(t), t; s(\tau), \tau) c_\xi^1(s(\tau), \tau)\, d\tau -$$
$$-2\int_0^t N_x(s(t), t; s(\tau), \tau) \frac{ds(\tau)}{d\tau} d\tau +$$
$$+2r_1 \int_0^t N_x(s(t), t; s(\tau), \tau) H(s(\tau))\, d\tau. \tag{18}$$

To simplify notations, denote the right-hand sides of (14), (15), (16), (18) by $P_1(t)$, $P_2(t)$, $P_3(t)$, $P_4(t)$, respectively.

Denote $\gamma(t) = g(P_1(t)) - f(P_1(t))$. By (7), we have the following system of equations

$$u_x^1(s(t), t) - H(s(t)) + \frac{P_1(t)}{D_1 \gamma(t)} c_x^1(s(t), t) = P_2(t), \tag{19}$$

$$Au_x^1(s(t), t) - H(s(t)) + \left(\frac{P_1(t)}{D_2 \gamma(t)} - \frac{2B}{\gamma(t)}\right) c_x^1(s(t), t) = P_3(t), \tag{20}$$

$$-r_1 H(s(t)) + \left(\frac{-r_1 P_1(t)}{\gamma(t)} + 1\right) c_x^1(s(t), t) = P_4(t). \tag{21}$$

The solution of system (19), (20), (21) is given by

$$u_x^1(s(t), t) = \frac{1}{W(t)\gamma(t)} \left[(-P_2(t) + P_3(t))\gamma(t) + (-P_2(t) + P_3(t))\right] \cdot$$
$$\cdot \Big[-r_1(P_1(t)) + 2BP_4(t) - \frac{1}{D_2} P_1(t) P_4(t) + \frac{1}{D_1} P_1(t) P_4(t) - 2Br_1 P_2(t) +$$
$$+ \frac{1}{D_2} P_1(t) P_2(t) - \frac{r_1}{D_1} P_1(t) P_3(t) \Big], \tag{22}$$

$$H(s(t)) = \frac{1}{W(t)\gamma(t)} \Big[P_3(t)\left(1 - \frac{r_1 P_1(t)}{\gamma(t)}\right) - P_4(t) \frac{P_1(t) - 2BD_2}{\gamma(t)} +$$
$$+ AP_4(t) \frac{P_1(t)}{D_1 \gamma(t)} - AP_2(t)\left(1 - \frac{r_1 P_1(t)}{\gamma(t)}\right) \Big], \tag{23}$$

$$c_x^1(s(t), t) = \frac{1}{W(t)\gamma(t)} [-P_4(t) + r_1 P_3(t) + AP_4(t) - Ar_1 P_2(t)], \tag{24}$$

where the determinant $W(t)$ is given by

$$W(t) = \frac{1}{\gamma(t)}\left[P_1(t) - \frac{b(A-1)-2Br_1}{\frac{Ar_1}{D_1}-\frac{r_1}{D_1}-(A-1)\tilde{r}_1}\right]\left[\frac{Ar_1}{D_1}-\frac{r_1}{D_2}-(A-1)\tilde{r}_1\right]$$

$$\equiv \frac{1}{\gamma(t)}\left[P_1(t)-\frac{E_1}{E_2}\right]E_2 \equiv \frac{1}{\gamma(t)}L(t).$$

It should be pointed out that the system (22)–(24) is only a representation of the solution that merely reflects an iterative procedure resulting from the Banach fixed point theorem. Let us assume that $E_1, E_2 \neq 0$. We are looking for σ such that $W(t) \neq 0$ in the interval $(0, \sigma)$. If $\operatorname{sgn} P_1(t) = \operatorname{sgn}\frac{E_1}{E_2}$, then there exists σ such that

$$|L(t)| \geqslant \frac{1}{2}\left|\left(P_1(0)-\frac{E_1}{E_2}\right)E_2\right| = \frac{|E_1|}{2}.$$

Recall that $P_1(0) = 0$. If $\operatorname{sgn} P_1(t) = -\operatorname{sgn}\frac{E_1}{E_2}$, then for any σ we have

$$|L(t)| \geqslant \frac{1}{2}\left|\left(P_1(0)-\frac{E_1}{E_2}\right)E_2\right| = \frac{|E_1|}{2}.$$

Therefore, there exists σ such that in the interval $(0, \sigma)$

$$|L(t)| \geqslant \frac{E_1}{2}. \tag{**}$$

Equation (7) admits the integral form

$$s(t) = 1 - \int_0^t \frac{1}{\gamma(\tau)} c_x^1(s(\tau), \tau)\, d\tau. \tag{25}$$

4. Monotonicity of $s(t)$

We shall prove the following

Lemma. *If the free boundary $s(t)$ exists in problem (1)–(9) under assumptions (A1)–(A4), then it is strictly decreasing in the whole time interval of its existence.*

Proof. First we shall show that $\frac{ds(t)}{dt} < 0$ in $[0, \sigma)$ for a certain sufficiently small $\sigma > 0$. Indeed, from assumption (A2) we have $c_x^1(x, 0) > 0$ for $x \in [\Delta, 1]$,

$\Delta > 0$. From this, by the compactness of the set $[\Delta, 1] \times \{0\}$ and continuity of c_x^1 we obtain $c_x^1(x, t) > 0$ for $t \in [0, \sigma)$. From (7), (8), (9) we conclude that $\frac{ds(t)}{dt} < 0$ in $[0, \sigma)$.

Now we prove that $\frac{ds(t)}{dt} < 0$ for $t_0 - \delta < t < t_0$, such that t_0 is chosen arbitrarily, $\delta < t_0$ and $0 < \delta < \sigma$. Indeed, u_x is negative in its domain by the maximum principle (see [6], § 3 and [7]). The function $u_t^1 = u_{xx}^1$ is negative for $t \in [0, \sigma)$ because $\frac{\partial^2 \varphi_u}{\partial x^2} < 0$ for $x \in [0, 1]$, and it is continuous in the compact set $[0, 1] \times \{0\}$. Moreover, u_t^1 is negative for $t_0 - \delta < t < t_0$ because we have

$$u_{xx}^1(x, t) = \int_0^1 N^1(x, t; \xi, 0) \frac{d^2 \varphi_u(\xi)}{d\xi^2} d\xi + D_1 \int_0^t N_{xx}^1(x, t; s(\tau), \tau) u_\xi^1(s(\tau), \tau) d\tau +$$

$$+ \int_0^t \left[N_{xx}^1(x, t; s(\tau), \tau) \frac{ds(\tau)}{d\tau} - D_1 N_{xx\xi}^1(x, t; s(\tau), \tau) \right] u^1(s(\tau), \tau) d\tau.$$

The first term is negative and the other ones could be taken sufficiently small in dependence on δ to obtain that u_{xx}^1 is negative. Thus, upon choosing $\delta = \eta$ sufficiently small we obtain $u_t^1 < 0$ for $t_0 - \eta \leqslant t < t_0 + \varepsilon$ at some $\varepsilon > 0$. Therefore u^1 is a decreasing function with respect to x and t in $[t_0 - \eta, t_0 + \varepsilon)$, and $u^1(x, t) - u^1(x + \Delta x, t + \Delta t) = -\frac{\partial u^1}{\partial x} \Delta x - \frac{\partial u^1}{\partial t} < 0$, hence $\frac{\partial u^1}{\partial x} \Delta x + \frac{\partial u^1}{\partial t} \Delta t > 0$ for $\Delta x < 0$ and $\Delta t > 0$. From this u^1 is a decreasing function on the curves $x(t)$ for which $\frac{dx(t)}{dt} < 0$.

But we may see from the definition of f and g (see the phase diagram) that the function c^1 is increasing at those time moments for which u^1 is a decreasing function. Hence, the solid dismisses an excess of the component c into the liquid and thus the solidification process goes on. Therefore we have $\frac{ds(t)}{dt} < 0$ for $t_0 - \eta \leqslant t < t_0 + \varepsilon$. At last, we can cover the whole given time interval by neighbourhoods chosen in the above way, using the Borel theorem. ■

5. The existence and uniqueness theorem

First we shall prove

THEOREM 1. *The problem of finding a solution of problem* (1)–(9) *is equivalent to the problem of finding a continuous solution of the system of integral equations* (14), (15), (16), (18), (25).

Proof. Directly from the construction of equations (14), (15), (16), (18), (25) it follows that each solution $u^i(x, t)$, $c^i(x, t)$, $s(t)$ for $i = 1, 2$, of problem (1)–(9) satisfies integral equations (14), (15), (16), (18), (25), too.

Suppose that for some $\sigma > 0$ the functions $u^1(s(t), t)$, $u_x^1(s(t), t)$, $H(s(t))$, $c_x^1(s(t), t)$, $\frac{ds(t)}{dt}$ form a solution of system (14), (15), (16), (18), (25). Observe that $u^i(x, t)$, $c^i(x, t)$ and $s(t)$ satisfy (1)–(9). It is obvious that they satisfy (1), (2), (3), (6). Conditions (4), (5), (8), (9) are satisfied due the construction of solution by a fixed point argument given in Theorem 2. But c_x^1 (together with u^1, u_x^1, s) have the regularity properties described in § 2, therefore from the calculations in § 3, leading to (25), condition (7) follows, which ends the proof. ■

The correctness of the solution in the Hadamard sense can be derived by a modification of Rubinstein's method [14] which has been applied, for example, by M. Niezgódka [9] to integral equations similar to these of our paper.

THEOREM 2. *There exist some* $\tilde{\sigma} \in (0, \sigma]$ *and a unique solution* $u^i(x, t)$, $c^i(x, t)$, $s(t)$ *of the system of integral equations* (14), (15), (16), (18), (25) *over a time interval* $[0, \tilde{\sigma}]$ *which fulfils conditions* (A1), (A3).

Proof. We shall denote by $X = (C^0)^5$ the space of vectors $v(t) = (v_1(t), \ldots \ldots, v_5(t))$, treated as a Cartesian product with the uniform norm $\|v\| = \max\{\sup_{0 \le \tau \le \sigma} |v_1(\tau)|, \ldots, \sup_{0 \le \tau \le \sigma} |v_5(\tau)|\}$. $C_\sigma^0 = C^0([0, \sigma]; R)$ is the space of real continuous functions in $[0, \sigma]$. Let us define by $X_{\sigma,M}$ the closed ball of $v \in X_{\sigma,M}$ such that $\|v\| \le M$. We introduce the transformation T: $v \in X_{\sigma,M} \to w \in X_{\sigma,M}$, where $v(\tau) = \left(u^1(s(\tau), \tau), u_\xi^1(s(\tau), \tau), H(s(\tau)), c_\xi^1(s(\tau), \tau), \frac{ds(\tau)}{d\tau}\right)$ and $w_i(t) = T_i v(\tau)$ $(i = 1, \ldots, 5)$ are given by (14), (15), (16), (18), (25). First we shall show that T maps $X_{\sigma,M}$ into itself. Let us take $v \in X_{\sigma,M}$ and estimate $P_i(t)$, $i = 1, 2, 3, 4$. We have

$$\sup|v_5(t)| \le \frac{\sup|v_4(t)|}{\mu} \quad \text{for } \mu = \min\gamma(t) > 1,$$

$$\sup|P_2(t)| \le E_{21} M^2 \sigma^{\frac{1}{2}} + E_{22} M^3 \sigma^{\frac{1}{2}} + E_{23} \|\varphi_u\| = h_2(M)\sigma^{\frac{1}{2}} + E_{23} \|\varphi_u\|,$$

$$\sup|P_2(t)| \le E_{21} M^2 \sigma^{\frac{1}{2}} + E_{22} M^3 \sigma^{\frac{1}{2}} + E_{23} \|\dot{\varphi}_u\| = h_2(M)\sigma^{\frac{1}{2}} + E_{23} \|\dot{\varphi}_u\|,$$

$$\sup|P_3(t)| \le E_{31} M^2 \sigma^{\frac{1}{2}} + E_{32} M^3 \sigma^{\frac{1}{2}} + E_{33} \|\dot{\psi}\| \sigma^{\frac{1}{2}} = h_3(M)\sigma^{\frac{1}{2}},$$

$$\sup|P_4(t)| \le E_{41} M^2 \sigma^{\frac{1}{2}} + E_{42} \|\dot{\varphi}_c\| = h_4(M)\sigma^{\frac{1}{2}} + E_{42} \|\dot{\varphi}_c\|,$$

$$\left|\frac{ds(t)}{dt}\right| \le M\frac{1}{\mu} \le M.$$

To estimate w, we transform (22), (23), (24) and we get

$$\sup |u^1(s(t), t)| \leqslant E_{11} M\sigma^{\frac{1}{2}} + E_{12} M^2 \sigma^{\frac{1}{2}} + E_{13} \|\varphi_u\|,$$

$$\sup |u_x^1(s(t), t)| \leqslant \frac{1}{E_0 \|\varphi_u\|}\Big(|\tilde{r}_1 P_1(t) P_2(t)| + |bP_1(t)| + + |bP_3(t)| + |2BD_1| + \frac{1}{D_1}|P_1(t) P_2(t)| + \frac{1}{D_1}|P_1(t) P_4(t)| \Big), \quad (26)$$

$$\sup |H(s(t))| \leqslant \frac{1}{D_1 D_2 E_0 \|\varphi_u\|}(|D_1 D_2 r_1 P_1(t) P_3(t)| + |bD_1 D_2 P_3(t)| + + |P_1(t) P_4(t)| + |2BD_2 P_1(t) P_4(t)| + |AD_1 D_2 P_2(t)(\tilde{r}_1 P_1(t) + b)|), \quad (27)$$

$$\sup |c_x^1(s(t), t)| \leqslant \frac{1}{E_0 \|\varphi_u\|}(|P_4(t)| + |r_1 P_3(t)| + |AP_4(t)| + |Ar_1 P_2(t)|). \quad (28)$$

Let us write out explicitly the terms in products $P_1(t) P_k(t)$ for k = 2, 3, 4 which do not contain $\sigma^{1/2}$ and express the remaining terms as functions of M, $h_{12}(M)$, $h_{13}(M)$, $h_{14}(M)$. We have

$$|P_1(t)|\,|P_2(t)| \leqslant h_{12}(M)\sigma^{\frac{1}{2}} + |E_{13}|\,|E_{23}|\,\|\varphi_u\|\,\|\dot{\varphi}_u\|,$$

$$|P_1(t)|\,|P_3(t)| \leqslant h_{13}(M)\sigma^{\frac{1}{2}},$$

$$|P_1(t)\,|P_4(t)| \leqslant h_{14}(M)\sigma^{\frac{1}{2}} + |E_{13}|\,|E_{42}|\,\|\varphi_u\|\,\|\dot{\varphi}_c\|.$$

We substitute these estimations into (26), (27), (28) to get

$$\sup |u_x^1(s(t), t)| \leqslant \frac{1}{E_0 \|\varphi_u\|}\Big(\sigma^{\frac{1}{2}}\Big(|\tilde{r}_1|\, h_{12}(M) + \frac{1}{D_2} h_{12}(M) + \frac{1}{D_1} h_{14}(M) + + |b|\, h_1(M) + h_3(M)\Big) + |\tilde{r}_1|\, E_{13} E_{23}\, \|\varphi_u\|\,\|\dot{\varphi}_u\| + |b|\, E_{13}\, \|\varphi_u\| + + \frac{1}{D_2} E_{13} E_{23}\, \|\varphi_u\|\,\|\dot{\varphi}_u\| + \frac{1}{D_1} E_{13} E_{42}\, \|\varphi_u\|\,\|\dot{\varphi}_c\| + 2BD_1 \Big),$$

$$\sup |H(s(t))| \leqslant \frac{1}{D_1 D_2 E_0 \mu \|\varphi_u\|}(\sigma^{\frac{1}{2}}(D_1 D_2 |r_1|\, h_{13}(M) + D_1 D_2 |b|\, h_3(M) + + h_{14}(M) + 2BD_2 h_{14}(M) + AD_1 D_2 |\tilde{r}_1|\, h_{12}(M) + AD_1 D_2 |b|\, h_2(M)) + + E_{13} E_{42}\, \|\varphi_u\|\,\|\dot{\varphi}_c\| + 2BD_2 E_{13} E_{42}\, \|\varphi_u\| \dot{\varphi}_c\| + + AD_1 D_2 |\tilde{r}_1|\, E_{13} E_{23}\, \|\varphi_u\|\,\|\dot{\varphi}_u\| + AD_1 D_2 |b|\, E_{23}\, \|\dot{\varphi}_u\|),$$

$$\sup |c_x^1(s(t), t)| \leqslant \frac{1}{E_0 \|\varphi_u\|}(\sigma^{\frac{1}{2}} h_4(M) + |r_1|\, h_3(M) + Ah_4(M) + + A\,|r_1|\, h_2(M) + E_{42}(1 + A)\,\|\dot{\varphi}_c\| + A\,|r_1|\, E_{23}\, \|\dot{\varphi}_u\|).$$

In the above three inequalities, we shall denote by a_1, a_2, a_3 the components which do not contain $\sigma^{\frac{1}{2}}$. From (14) it follows that

$$\sup |(u^1(s(t), t)| \leqslant h_{12}\sigma^{\frac{1}{2}} + E_{13}\|\varphi_u\|.$$

Let us take $M = \max(E_{13}\|\varphi_u\|+1,\ a_1+1,\ a_2+1,\ a_3+1)$. Then we get σ such that

$$(***)\qquad \sup|c_x^1(s(t), t)| \leqslant M,\ \sup|H(s(t))| \leqslant M,\ \sup|u_x^1(s(t), t)| \leqslant M,$$
$$\sup|u^1(s(t), t)| \leqslant M,\ \left|\frac{ds(t)}{dt}\right| \leqslant M.$$

Now we shall show that T is a contraction. Assume that $\|v-v'\| = \varepsilon$. We are going to show that $\|Tv-Tv'\| < \varepsilon$. First we verify it for $s(t)$. We have

$$|s(t)-s'(t)| = \int_0^t \left|\frac{c_\xi(s(\tau), \tau)}{g(P_1(\tau))-f(P_1(\tau))} - \frac{c_\xi(s'(\tau), \tau)}{g(P_1'(\tau))-f(P_1'(\tau))}\right| d\tau \leqslant$$
$$\leqslant \int_0^t \frac{1}{\mu^2}|(r_1-\tilde{r}_1)P_1'(\tau)+b+c_\xi(s'(\tau), \tau)(r_1-\tilde{r}_1)|\, d\tau \leqslant \varepsilon\frac{\mu+M(r_1-\tilde{r}_1)}{\mu^2}t.$$

Thus

$$\left|\frac{ds(t)}{dt} - \frac{ds'(t)}{dt}\right| \leqslant \frac{\mu+M|r_1-\tilde{r}_1|}{\mu^2}$$

and

$$\frac{\mu+M|r_1-\tilde{r}_1|}{\mu^2} < 1 \text{ for } \mu > \frac{1+(1+4|r_1-\tilde{r}_1|M)^2}{2}.$$

Now we consider the difference

$$\sup|u^1(s(t), t)-u^1(s'(t), t)| = \sup|P_1(t)-P_1'(t)|.$$

It contains integrals on the interval $(0, t)$ and an integral on the interval $(0, 1)$. The functions under integrals are lipschitzean, thus the corresponding differences are products of some constants and the differences of arguments. These differences are equal to ε. Thus,

$$|P_1(t)-P_1'(t)| \leqslant c\varepsilon t + \left|\int_0^1 (N^1(s(t), t; \xi, 0) - N^1(s'(t), t; \xi, 0)\varphi(\xi)\, d\xi\right| \leqslant$$
$$\leqslant c\xi t + B\varepsilon\sigma^{\frac{1}{2}}.$$

There exists σ such that

$$|P_1(t)-P_1'(t)| < \varepsilon \text{ so } |u^1(s(t), t)-u^1(s'(t), t)| < \varepsilon.$$

The integrals of such types occur also in $P_2(t)$, $P_3(t)$ and $P_4(t)$, so $|P_j(t)-P_j'(t)| < \varepsilon$ for $j = 2, 3, 4$. From (22), (23), (24) we obtain that $u_x^1(s(t), t)$,

$H(s(t), t)$, $c_x^1(s(t), t)$ are sums of the products $P_1(t)P_k(t)$ ($k = 2, 3, 4$) and $P_i(t)$ ($i = 1, \ldots, 4$). We may see that

$$|P_i(t)P_k(t) - P'_i(t)P'_k(t)| = |P_i(t)P_k(t) - P'_i(t)P_k(t) + P'_i(t)P_k(t) - P'_i(t)P'_k(t)| \leqslant \varepsilon|P'_i(t) + P_k(t)|,$$

where $P'_i(t)$ and $P_k(t)$ are smaller than $c'_i\varepsilon\sigma^{\frac{1}{2}}$ and $c_k\varepsilon\sigma^{\frac{1}{2}}$, respectively. Therefore, the above-mentioned sums of the products are smaller than the product of $\varepsilon\sigma^{\frac{1}{2}}$ and the sum of some constants. We proceed similarly with $cP_i(t)$ ($i = 1, \ldots, 4$). Hence, there exists σ such that the products of $\sigma^{\frac{1}{2}}$ and of the sum of some constants are smaller than 1. So that for such σ we have

(****) $$|u_x^1(s(t), t) - u_x^1(s'(t), t)| < \varepsilon, \quad |H(s(t)) - H(s'(t))| < \varepsilon,$$

$$|c_x^1(s(t), t) - c_x^1(s'(t), t)| < \varepsilon, \text{ therefore } \|Tv - Tv'\| \leqslant \varepsilon.$$

Using the Banach contraction theorem for σ satisfying (**), (***) and (****) we complete the proof of the existence and uniqueness of the local solution. ■

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Paraboliczne równania różniczkowe w zagadnieniach krzepnięcia dwuskładnikowego stopu

W pracy rozważany jest jednowymiarowy problem ze swobodnym brzegiem, występujący w zagadnieniach krzepnięcia dwuskładnikowego stopu. Określana jest zawartość jednego ze składników mieszaniny w cieczy oraz w ciele stałym. Udowodniono istnienie lokalnego rozwiązania i jego jednoznaczność.

Параболические дифференциальные уравнения в проблеме застывания двухкомпонентного сплава

С этой работе рассуждается задача со свободной границей, описывающая застывание двухкомпонентного сплава.

Определяются концентрации жидкого и твёрдого состояний в этом процессе. Доказана теорема о существовании и единственности решений.