Control
and Cybernetics
Vol. 16 (1987) No. 3–4

Stability of linear programming problems in Banach spaces

by

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In the present paper we investigate the stability of general linear programming problems in Banach spaces. We formulate a condition which is sufficient and "almost" necessary for the existence of optimal solutions and for the boundedness of the optimal sets of both linear programming problems, the primal and the dual one. It turns out that the same condition is both necessary and sufficient for the stability of the linear programming problem in the sense that it has optimal solution under small perturbations. Moreover, we show that the optimal value and the optimal set depend continuously on perturbation parameters.

Introduction

For getting stability of mathematical programming problems, that is the existence of optimal solutions under small perturbations, the continuous dependence of the extremal value and the optimal set on perturbation parameters, or also the differential properties of the extremal value function as a function of the perturbation parameter, we need the so-called regularity conditions. These conditions can be either local or global ones. Precisely, in order to show that the problem is stable in some sense, one has to assume either a certain regularity condition is fulfilled at one of its optimal solutions [1, 5, 6, 7], or the system of constraints satisfies another regularity condition under perturbations [3, 4, 10, 2]. In general, these regularity conditions, local or global, are sufficient conditions for stability. For the case of finite-dimensional linear programming problems Robinson [10] has established a regularity condition which is both necessary and sufficient for stability. Precisely, this condition is equivalent to the solvability of two dual problems and the boundedness of their optimal solution sets, and then, to the solvability of the perturbed problems. Moreover, the optimal set map is shown to be "upper Lipschitz semicontinuous". Later, Asmanov [2] has obtained similar results with the regularity condition of other form. It turns out, as we can see at the end of this paper, that the regularity conditions formulated by Robinson and by Asmanov, are equivalent.

In this paper we will analyse stability of the general linear programming problem in Banach spaces. We will formulate two regularity conditions for the systems of constraints of the primal and the dual problem and then we will show that they are both necessary and sufficient for the stability of both problems. Since these conditions are shown to be equivalent to that of Robinson [8, 10], some results for the finite-dimensional case are involved here as a special case.

In section I we will show that the regularity conditions (R_p) and (R_d) imply the existence of optimal solutions for both dual problems and, moreover, the optimal sets are bounded. In Section II we improve lightly one result of Robinson [8] on the stability of the linear system of inequalities. We will state and prove the main result of the paper (Theorem). Moreover, we will prove that the extremal value function is continuous and the optimal set map is upper semicontinuous under the formulated regularity condition.

1. Regularity and boundedness of the optimal solution sets

Let X and Y be two Banach spaces; C and K be two nonempty closed convex cones in X and Y, respectively. We can use C and K to induce two partial order on X and Y, respectively, by defining $x_1 \leqslant_C x_2$ if $x_2 - x_1 \in C$ and $y_1 \leqslant_K y_2$ if $y_2 - y_1 \in K$.

Let A be a continuous linear operator from X into Y, $A \in \mathscr{L}(X, Y)$; c^* be a continuous linear functional on X, $c^* \in x^*$; b be an element of Y. We consider the following general linear program:

$$(\mathscr{P}) \qquad \begin{cases} \text{minimize} \quad \langle c^*, x \rangle & (1.1) \\ \text{subject to} \qquad Ax \leqslant_K b & (1.2) \\ \qquad\qquad\qquad\quad x \geqslant_C 0 & (1.3) \end{cases}$$

and its dual problem:

$$(\mathscr{D}) \qquad \begin{cases} \text{maximize} \quad \langle y^*, b \rangle & (1.4) \\ \text{subject to} \qquad A^* y^* \geqslant_{C^\circ} c^* & (1.5) \\ \qquad\qquad\qquad\quad y^* \geqslant_{K^\circ} 0 & (1.6) \end{cases}$$

where C° and K° are the conjugate cones of C and K in X^* and Y^*, respectively; A^* is the conjugate operator of A, $A^* \in \mathscr{L}(Y^*, X^*)$,

$$C^\circ = \{x^* \in X^* \colon \langle x^*, x \rangle \leqslant 0 \quad \forall x \in C\},$$

$$K^\circ = \{ y^* \in Y^* \colon \langle y^*, y \rangle = 0 \quad \forall y \in K\}.$$

Let us denote by α_0 and β_0 the extremal value of the two problems $(\mathscr{P})$ and $(\mathscr{D})$, respectively. We set $\alpha_0 = +\infty$ $(\beta_0 = -\infty)$ if the problem $(\mathscr{P})$ ($(\mathscr{D})$ respectively) is infeasible. Let P and D denote the optimal solution sets of the problems $(\mathscr{P})$ and $(\mathscr{D})$, respectively.

It is easy to show the following relation: if x is a feasible solution for $(\mathscr{P})$: $Ax \leqslant b$, $x \geqslant 0$ and y^* is a feasible solution for $(\mathscr{D})$: $A^* y^* \geqslant c^*$, $y^* \geqslant 0$ then

$$\langle c^*, x\rangle \geqslant \langle y^*, b\rangle \tag{1.7}$$

We consider also two linear systems of inequalities associated with both problems $(\mathscr{P})$ and $(\mathscr{D})$:

$$(\mathscr{S}_\alpha) \qquad \begin{cases} \langle c^*, x\rangle \leqslant \alpha \\ Ax \leqslant b \\ x \geqslant 0 \end{cases} \quad \text{and } (\mathscr{S}^\beta) \qquad \begin{cases} \langle y^*, b\rangle \geqslant \beta \\ A^* y^* \geqslant c^* \\ y^* \geqslant 0 \end{cases}$$

Let P_α and D^β denote the solution sets of two systems $(\mathscr{S}_\alpha)$ and $(\mathscr{S}^\beta)$, respectively.

When the problem $(\mathscr{P})$ has optimal solutions, its optimal solution set P coincides with the solution set of $(\mathscr{S}_\alpha)$ with $\alpha = \alpha_0$ i.e. $P = P_{\alpha_0}$. Moreover, we have

PROPOSITION 1.1. Suppose that X is reflexive. The problem $(\mathscr{P})$ has optimal solutions, i.e. $P \neq \emptyset$, and the optimal solution set P is bounded if and only if there exists a number α such that the system $(\mathscr{S}_\alpha)$ is solvable and the solution set P_α is bounded.

Proof. By the observation above we need only to prove that the existence of such a number α implies that the problem $(\mathscr{P})$ has optimal solutions and the optimal set P is bounded. The solvability of the system $(\mathscr{S}_\alpha)$ implies $\alpha \geqslant \alpha_0 = \inf\{\langle c^*, x\rangle\colon Ax \leqslant b, x \geqslant 0\}$. Moreover, $\alpha_0 > -\infty$ because, if it were not the case, the set P_α is not bounded. Hence, the problem $(\mathscr{P})$ is equivalent to the following problem:

$$\begin{cases} \text{minimize} \quad \langle c^*, x\rangle \\ \text{subject to } \ x \in P_\alpha \end{cases}$$

It is easy to show that P_α is convex and closed, therefore, it is weakly closed. The weakly continuous linear functional c^* on X need attain a minimum in the weakly closed and bounded set P_α. Since P_α is bounded, the optimal solution set is, of course, bounded. ∎

By the same way we can formulate and prove an analogous proposition for the dual problem $(\mathscr{D})$.

DEFINITION. The system of constraints of the problem $(\mathscr{P})$ is said to be regular if

$$(R_p) \qquad AC + K - R^+ b = Y$$

The system of constraints of the problem $(\mathscr{D})$ is said to be regular if

$$(R_d) \qquad -A^* K^0 + C^0 + R^+ c = X$$

where R^+ is the set of nonnegative numbers.

REMARK. We notice that the regularity condition presented here is similar to that of Zowe and Kurcyusz [11], but the latter is a local condition at some feasible point. Fortunately, we deal with the linear case of the programming problem and, therefore, we can establish a global condition which ensure that the constraints "behave well" at every feasible point.

The following theorem states that the regularity is "almost" necessary and sufficient for the boundedness of the optimal solution sets of the problems $(\mathscr{P})$ and $(\mathscr{D})$.

THEOREM 1.2.

(a) The optimal solution set D of the dual problem $(\mathscr{D})$ is bounded if the condition (R_p) is satisfied.

(b) The optimal solution set P of the problem $(\mathscr{P})$ is bounded if the condition (R_d) is satisfied.

Conversely, if D is bounded then $cl\{AC+K-R^+ b\} = Y$

if P is bounded then $cl\{-A^* K^0 + C^0 + R^+ c\} = X$

Proof. (a) We consider the product space $X \times R$, with $\|(x, \xi)\| = \max\{\|x\|, |\xi|\}$ and define a set-valued map Q from $X \times R$ into Y by setting

$$Q(x, \xi) = \begin{cases} Ax - \xi b + K & \text{if } x \in C,\ \xi \geqslant 0 \\ \varnothing & \text{otherwise.} \end{cases}$$

Q is called the augmented operator associated with the linear system of inequalities (1.2) (1.3). The graph of Q is a closed convex cone, i.e. Q is a closed convex process. The condition (R_p) implies that $0 \in Y$ is an internal point of the range of Q. Hence the map Q is locally surjective at $0 \in Y$: there exists a number $\varrho > 0$ such that $QB_{X \times R} \supset \varrho B_Y$ where $B_{X \times R}$ is the unit ball in $X \times R$ and B_y is the unit ball in Y. This means that for every $y \in Y$ with $\|y\| \leqslant \varrho$ there exist an $x \in C$ with $\|x\| \leqslant 1$ and a number ξ, $0 \leqslant \xi \leqslant 1$ such that $y \in Ax - \xi b + K$ or, equivalently, $y = Ax + \xi b + k$ where $k \in K$. We have $\|k\| \leqslant \|A\| + \|b\| + \varrho$.

To show the boundedness of the optimal solution set D of the problem $(\mathscr{D})$ we rewrite

$$D = D^{\beta_0} = \{y^* \in Y^*\colon\ A^* y^* \geqslant c^*,\ y^* \geqslant 0,\ \langle y^*, b\rangle \geqslant \beta_0\}$$

Fixing $\bar{y}^* \in D$, we will prove that the set $D - \bar{y}^*$ is bounded. In fact, for every $y \in Y$ with $\|y\| \leqslant \varrho$ we have:

$$\begin{aligned} \langle y^* - \bar{y}^*; y\rangle &= \langle y^* - \bar{y}^*, Ax - \xi b + k\rangle \\ &= \langle y^* - \bar{y}^*, Ax\rangle - \xi \langle y^* - \bar{y}^*, b\rangle + \langle y^* - \bar{y}^*, k\rangle \end{aligned}$$

$$= \langle A^* y^*, x\rangle - \langle A^* \bar{y}^*, x\rangle - \xi[\langle y^*, b\rangle - \beta_0]$$
$$+\xi[\langle \bar{y}^*, b\rangle - \beta_0] + \langle y^*, k\rangle - \langle \bar{y}^*, k\rangle$$
$$\leqslant \langle c^* - A^* \bar{y}^*, x\rangle + \xi[\langle \bar{y}^*, b\rangle - \beta_0] - \langle \bar{y}^*, k\rangle$$

where y^* is an arbitrary element of D. We get that by disregarding the nonpositive components in the right-hand side. The right-hand side now does not depend on y^*. This means that the linear functionals of form $y^* - \bar{y}^*$ with $y^* \in D$ are uniformly bounded on the ball ϱB_Y. It implies, by linearity, that $D - \bar{y}^*$ is bounded on the norm of Y and, therefore, D is bounded.

(b) Similarly as in the part (a) we get a number δ such that for any fixed $\bar{x} \in P$ and any $x^* \in X^*$ with $\|x^*\| \leqslant \delta$ the set $\{\langle x^*, x - \bar{x}\rangle : x \in P\}$ is bounded. It implies, by linearity, that the set $\{\langle x^*, x - \bar{x}\rangle : x \in P\}$ is bounded also for any $x^* \in X^*$. Hence, P is weakly bounded and, therefore, is bounded in the norm of X since X is a Banach space.

The rest of the Theorem can be proved by the same reasoning as in [11]. ■

THEOREM 1.3. *Let X and Y be reflexive. Assume that two conditions (R_p) and (R_d) are fulfilled. Then, the problems $(\mathscr{P})$ and $(\mathscr{D})$ have optimal solutions.*

Proof. We observe that the condition (R_p) implies the existence of a feasible point $\bar{x}$ for the primal problem $(\mathscr{P})$ and, analogously, the condition (R_d) implies the existence of a feasible solution $\bar{y}^*$ for the dual problem $(\mathscr{D})$. From the relation (1.7) we have:

$$\langle c^*, \bar{x}\rangle \geqslant \alpha_0 \geqslant \beta_0 \geqslant \langle \bar{y}^*, b\rangle$$

Take a number $\alpha \geqslant \langle c^*, \bar{x}\rangle$ and a number $\beta \leqslant \langle \bar{y}^*, b\rangle$. The systems $(\mathscr{S}_\alpha)$ and $(\mathscr{S}^\beta)$ are solvable. We can show, by the same way as in the proof of Theorem 1.2, that the solution sets P_α and D^β for these systems are bounded. Hence, we can conclude by the force of Proposition 1.1, that both problems $(\mathscr{P})$ and $(\mathscr{D})$ have optimal solutions and, in addition, the optimal sets are bounded.

2. Regularity and stability of linear programming problems

Consider the perturbed problem of $(\mathscr{P})$:

$$(\mathscr{P}_p) \qquad \begin{cases} \text{minimize} \quad \langle c_p^*, x\rangle & (2.1)\\ \text{subject to} \quad A_p x \leqslant b_p & (2.2)\\ \qquad\qquad x \geqslant 0 & (2.3)\end{cases}$$

where p is viewed as the perturbation parameter, which varies in a linear topological space Z and $c_p^* \in X^*$, $A_p \in \mathscr{L}(X, Y)$, $b_p \in Y$ and $c_p^* \to c$, $A_p \to A$, $b_p \to b$ as $p \to 0$.

We define the extremal value function

$$f(p) = \begin{cases} \inf\{\langle c_p^*, x\rangle : A_p x \leqslant b_p, x \geqslant 0\} \\ +\infty \qquad \text{if the problem } (\mathscr{P}_p) \text{ is infeasible} \end{cases}$$

Denoting by $P(p)$ the optimal solution set for the problem $(\mathscr{P}_p)$

$$P(p) = \{x \in X : A_p x \leqslant b_p, x \geqslant 0, \langle c_p^*, x\rangle = f(p)\}$$

and setting $P(0) = P$ we get a set-valued map from Z into X.

Under the term "stability" for the problem $(\mathscr{P})$ we mean the continuity of the extremal value function $f(.)$; the semicontinuity of the set-valued map $P(.)$ at $p = 0$ or, sometimes the weak sense, the existence of optimal solutions for problems $(\mathscr{P}_p)$ with all p belonging to a neighbourhood of $0 \in Z$.

Since stability of the optimization problem strongly depends on the behaviour of its system of constraints, before treating it, we recall some known results on the stability of linear systems of inequalities.

Consider the linear system of inequalities (1.2) (1.3)

$$\text{(I)} \qquad \begin{cases} Ax \leqslant b \\ x \geqslant 0 \end{cases}$$

and the perturbed system (2.2) (2.3)

$$(\mathrm{I}_p) \qquad \begin{cases} A_p x \leqslant b_p \\ x \geqslant 0 \end{cases}$$

Let us denote the solution sets of the systems (I) and (I_p) by F and $F(p)$, respectively. Setting $F(0) = F$, we get a set-valued map $F(.)$ from Z into X.

In [8] Robinson has proved that the condition

$$b \in \operatorname{int}(AC + K) \tag{2.4}$$

is necessary and sufficient for the stability of the system (I) in the sense that the perturbed systems (I_p) are solvable for all p being close enough to $0 \in Z$, and moreover, the distance from any point x to the solution set of (I_p) can be estimated with the help of a "measure of infeasibility".

We notice that the condition (2.4) and the regularity condition (R_p) are equivalent [8, 12].

Assumming, in addition, the boundedness of the solution sets F and $F(p)$ we can improve the result of Robinson.

PROPOSITION 2.1. Assume that the condition (2.4) is fulfilled and the solution sets F and $F(p)$ of the systems (I) and (I_p), respectively, are bounded. Then, the set-valued map $F(.)$: $Z \to X$ is continuous at $p = 0$.

Proof. Let Q and Q_p be the augmented operators associated with the systems (I) and (I_p), respectively; Q^{-1} and Q_p^{-1} be the inverse operators and $\|Q\|$, $\|Q_p^{-1}\|$

... be the norms of Q and Q_p^{-1} considered as the convex process (for the definitions see [8]). Set $\varrho_p(x) = d(b_p - A_p x, K)$. $\varrho_p(x)$ is a measure of the infeasibility of x with respect to the system (I_p). For every $x \in F$, $\varrho_p(x) \leq (\|A_p - A\| + \|b_p - b\|)\max(1, \|x\|)$. Since F is bounded $\varrho_p(x)$ uniformly converges to 0 on F as $p \to 0$. $\|Q_p^{-1}\|$ is uniformly bounded for all p belonging to a neighbourhood of $0 \in Z$ (see [8, Theorem I]). Hence, we can choose a neighbourhood of $0 \in Z$ such that $\|Q_p^{-1}\|\,\varrho_p(x) < 1$ for all p belonging to this neighbourhood and for all $x \in F$. We get now the estimation (see [8, Theorem I])

$$d(x, F(p)) \leq \frac{\|Q_p^{-1}\|\,\varrho_p(x)}{1 - \|Q_p^{-1}\|\,\varrho_p(x)}(1 + \|x\|)$$

for all $x \in F$. Then, $d(F, F(p)) = \sup\limits_{x \in F} d(x, F(p))$ converges to 0 as $p \to 0$.

We exchange now the roles of systems (I) and (I_p). We treat the system (I) as a perturbed system of (I_p). The system (I_p) is regular if p is close enough to $0 \in Z$ (see [8, Theorem I]). Then we have also the estimation

$$d(x, F) \leq \frac{\|Q^{-1}\|\,\varrho(x)}{1 - \|Q^{-1}\|\,\varrho(x)}(1 + \|x\|) \tag{2.5}$$

for all $x \in C$ such that $\|Q^{-1}\|\,\varrho(x) < 1$, where $\varrho(x) = d(b - Ax, K)$.

In order to show that $d(F(p), F) \to 0$ as $p \to 0$ we need to prove the uniform boundedness of the solution sets $F(p)$ for all p belonging to a neighbourhood of $0 \in Z$.

In fact, we have $F \subset rB_x$. We will prove that there exists a neighbourhood of $0 \in Z$ such that for all p belonging to this neighbourhood $F(p) \subset (r+1)B_x$. If it were not the case, we could pick out a sequence $\{p_n\}$, $p_n \to 0$ and a sequence $\{x_n\}$, $x_n \in F(p_n)$ and $\|x_n\| = r+1$. The existence of such sequences follows from the convexity of the sets $F(p_n)$ and the fact that there exists $x_n' \in F(p_n)$ with $\|x_n'\| \leq r + \frac{1}{2}$ because $d(F, F(p)) \to 0$.

We have $\varrho(x_n) \leq \|A - A_{p_n}\|\,\|x_n\| + \|b - b_{p_n}\|$. Hence, $\varrho(x_n)$ converges to zero as $n \to \infty$. We can assume, therefore, that $\|Q^{-1}\|\,\varrho(x_n) < 1$ and get the estimation

$$d(x_n, F) \leq \frac{\|Q^{-1}\|\,\varrho(x_n)}{1 - \|Q^{-1}\|\,\varrho(x_n)}(r+2)$$

for all n large enough. Let $n \to \infty$, the right-hand side converges to zero. It implies, by force, that $d(x_n, F)$ is convergent to zero as $n \to \infty$, which is impossible because $\|x_n\| = r+1$, while $F \subset rB_x$.

Since the sets $F(p)$ are uniformly bounded for all p belonging to a neighbourhood of $0 \in Z$, we can show, similarly as above, that there exists a neighbourhood $\mathscr{U}$ of $0 \in Z$ such that for all $p \in \mathscr{U}$ and for all $x \in F(p)$,

$\|Q^{-1}\|\,\varrho(x) < 1$. Hence, by (2.5) we get

$$d(x, F) \leqslant \frac{\|Q^{-1}\|\,\varrho(x)}{1-\|Q^{-1}\|\,\varrho(x)}(r+2)$$

for all $x \in F(p)$. It implies that $d(F(p), F) \to 0$ as $p \to 0$.

Analogously, similar statements can be made and proved for the dual problem $(\mathscr{D})$ with the perturbation $(\mathscr{D}_p)$. The regularity condition for the system (1.5) (1.6) is the following:

$$c^* \in \operatorname{int}(A^* K^0 - C^0)$$

and this is equivalent to (R_d).

Now we are in position to deal with the stability of the linear programming problem $(\mathscr{P})$.

THEOREM 2.2 *Let X and Y be two reflexive Banach spaces. The dual problems $(\mathscr{P})$ and $(\mathscr{D})$ are stable in the sense that the perturbed problems $(\mathscr{P}_p)$ and $(\mathscr{D}_p)$ have optimal solutions for all p close enough to $0 \in Z$ if and only if two conditions (R_p) and (R_d) are fulfilled.*

Proof. It is easy to see that the conditions (R_p) and (R_d) are necessary for the stability of both problems $(\mathscr{P})$ and $(\mathscr{D})$, because in the case of stability both systems of constraints (1.2) (1.3) and (1.5) (1.6) need be stable and, therefore, the conditions (R_p) and (R_d) hold.

Now we show that they are also sufficient. Indeed, (R_p) and (R_d) imply that the following systems:

$$\begin{cases} Ax \leqslant_K b \\ x \geqslant_C 0 \end{cases} \quad \text{and} \quad \begin{cases} A^* y^* \geqslant_{C^0} c^* \\ y^* \geqslant_{K^0} 0 \end{cases}$$

are regular. By Theorem 1.3, the problems $(\mathscr{P})$ and $(\mathscr{D})$ have optimal solutions. Then, $\alpha_0 > -\infty$ and $\beta_0 < +\infty$. Take $\alpha > \alpha_0$ and $\beta < \beta_0$. The systems $(\mathscr{S}_\alpha)$ and $(\mathscr{S}^\beta)$ are regular ([8, Theorem II]). Hence, the perturbed systems

$$(\mathscr{S}_{\alpha_p}) \begin{cases} \langle c_p^*, x\rangle \leqslant \alpha \\ A_p x \leqslant b_p \\ x \geqslant 0 \end{cases} \quad \text{and} \quad (\mathscr{S}_p^\beta) \begin{cases} \langle y^*, b_p\rangle \geqslant \beta \\ A_p^* y^* \geqslant c^* \\ y^* \geqslant 0 \end{cases}$$

are regular for all p being close enough to $0 \in Z$. Let us denote by $P_\alpha(p)$ and $D^\beta(p)$ the solution sets of these systems, respectively. Similarly as in the proof of Theorem 1.2, we can show that the sets $P_\alpha(p)$ and $D^\beta(p)$ are bounded.

Since the systems $(\mathscr{S}_{\alpha_p})$ and $(\mathscr{S}_p^\beta)$ are solvable and their solution sets are bounded, the perturbed problems $(\mathscr{P}_p)$ and $(\mathscr{D}_p)$ have optimal solutions for all p close enough to $0 \in Z$ (Prop. 1.1). ■

We investigate now the continuity of the functional $f(.)$ and the map $P(.)$.

THEOREM 2.3. *Let X and Y be two reflexive Banach spaces. Assume that two conditions (R_p) and (R_d) are fulfilled. Then, the functional $f(.)$ is continuous at $p=0$ and the set-valued map $P(.)$ is upper semicontinuous at $p=0$.*

Proof. From the proof of Theorem 2.2 we see that the problem $(\mathscr{P})$ is equivalent to the following problem:

$$\begin{cases}\text{minimize} \quad \langle c^*, x\rangle \\ \text{subject to } x \in P_\alpha\end{cases}$$

and the perturbed problem $(\mathscr{P}_p)$ is equivalent to

$$\begin{cases}\text{minimize} \quad \langle c_p^*, x\rangle \\ \text{subject to } x \in P_\alpha(p)\end{cases}$$

Since the system $(\mathscr{S}_\alpha)$ is regular and the solution sets P_α and $P_\alpha(p)$ are bounded, as we have noticed in the proof of Theorem 2.2, the set-valued map $P_\alpha(.)$: $Z \to X$ is continuous at $p=0$ by the force of Proposition 2.1.

Take an arbitrary sequence $\{p_n\}$ converging to zero. We need to show that $\lim_{n\to\infty} f(p_n) = \alpha_0$. Since the problem $(\mathscr{P}_{p_n})$ has optimal solutions for all n large enough, we can find an $x_n \in P(p_n)$ such that

$$\langle c_{p_n}^*, x_n\rangle \leqslant f(p_n)+\varepsilon \tag{2.6}$$

where ε is an arbitrary positive number. Because the sequence $\{x_n\}$ is bounded, we can assume, without loss of generality, that it converges weakly to $\bar{x}$. Then, $\langle c_{p_n}^*, x_n\rangle$ is convergent to $\langle c^*, \bar{x}\rangle$ because we have

$$\begin{aligned}|\langle c_{p_n}^*, x_n\rangle - \langle c^*, \bar{x}\rangle &= |\langle c_{p_n}^*, x_n\rangle - \langle c^*, x_n\rangle + \langle c^*, x_n\rangle - \langle c^*, \bar{x}\rangle| \\ &\leqslant \|c_{p_n}^* - c^*\| \, \|x_n\| + |\langle c^*, x_n\rangle - \langle c^*, \bar{x}\rangle|\end{aligned}$$

and the right-hand side converges to zero as $n \to \infty$.

Now we show that $\bar{x} \in P_\gamma$. We notice, first, that $\bar{x} \geqslant 0$ since $x_n \geqslant 0$ for all n and the cone C, which is closed and convex, is weakly closed in consequence. Further, we verify that $A\bar{x} \leqslant b$. Writing $(A_{p_n} x_n - b_{p_n}) = (A_{p_n} x_n - Ax_n)$ $+(Ax_n - b_{p_n})$ and letting $n \to \infty$ we get that $(A_{p_n} x_n - b_{p_n})$ weakly converges to $A\bar{x}-b$. Hence, $b - A\bar{x} \in K$ or, equivalently, $Ax \leqslant b$, since $(b_{p_n} - A_{p_n} x_n) \in K$ for all n and the cone K, which is closed and convex, is weakly closed.

From (2.6) we get

$$\alpha_0 \leqslant \langle c^*, \bar{x}\rangle \leqslant \liminf_{n\to\infty} f(p_n)+\varepsilon \tag{2.7}$$

Let $\hat{x}$ be an optimal solution for the problem $(\mathscr{P})$, i.e. $\langle c^*, \hat{x}\rangle = \alpha_0$. By the semicontinuity of the set-valued map $P_\alpha(.)$ at $p=0$, we can find a sequence $\{x_n\}$, such that $x_n \in P_\gamma(p_n)$ and $x_n \to \hat{x}$. We have $\langle c_{p_n}^*, x_n\rangle \geqslant f(p_n)$. Letting

$n \to \infty$ we get

$$\alpha_0 = \langle c^*, \hat{x} \rangle \geqslant \limsup_{n \to \infty} f(p_n) \tag{2.7'}$$

Since ε is an arbitrary positive number, it follows from (2.7) and (2.7′) that $\lim_{n \to \infty} f(p_n) = \alpha_0$. This proves the continuity of the functional $f(.)$ at $p = 0$.

Now for showing that the set-valued map $P(.)$ is upper semicontinuous at $p = 0$ we need only repeat the same reasoning as in the proof of the inequality (2.7) but for the sequence $x_n \in P(p_n)$ i.e. x_n is an optimal solution for the perturbed problem $(\mathscr{P}_{p_n})$.

Now we consider, for instance, a special case when the problem $(\mathscr{P})$ is finite-dimensional with $X = R^n$, $Y = R^m$, $C = R^n$, K is the nonnegative cone in R^m and A is a matrix $m \times n$. Precisely, the problem $(\mathscr{P})$ in this case becomes

$$\begin{cases} \text{minimize } \langle c, x \rangle \\ \text{subject to } Ax - b \leqslant 0 \end{cases}$$

The dual problem is

$$\begin{cases} \text{minimize } -\langle b, y \rangle \\ \text{subject to } A^T y - c = 0, \ y \geqslant 0 \end{cases}$$

The condition (R_d) becomes

$$A^T(-K^0) + 0 + R^+ c = R^n$$

It is exactly the condition that the convex cone generated by the vector c and the rows of A coincide with R^n. The condition (R_p) becomes

$$AR^n + K - R^+ b = R^m$$

or, equivalently, cone $\{\pm a_1, \ldots, \pm a_n, e_1, \ldots, e_m, -b\} = R^m$ where $a_1, \ldots, a_n$ are the columms of A; $e_1, \ldots, e_m$ are the unit vectors. We obtain again the regularity conditions, which have been presented in [2].

References

[1] Armacost R. L., Fiacco A. V. Sensitivity analysis of a well-behaved Kuhn–Tucker triple, Surv. Math. Program. Proc. 9th Int. Math. Prog. Symp. Budapest 1976.

[2] Asmanov S. A. The conditions for stability of linear programming problems. *J. Comp. Math. and Math. Phys.* **21** (1981) 6 (in Russian).

[3] Dantzig G. B., Folkman J., Shapiro N. On the continuity of the minimum of a continuous function. *J. of Math. Anal. and App.* **17** (1967).

[4] Evans J. P., Gould F. J. Stability in nonlinear programming, *Oper. Research*, **18** (1970).

[5] Gauvin J., Tolle J. W. Differential stability in nonlinear programming. *SIAM J. on Control*, **15** (1977).

[6] Nguyen D. Hoa Differential stability in parametrized nonlinear programming. *Univ. Jagellonicae Acta Math.* 1985.

[7] Nguyen D. Hoa Tangent cones and differential stability of parametrized nonconvex programming. *Univ. Jagellonicae Acta Math.* 1985.
[8] Robinson S. M. Stability theory for system of inequalities, Part I: Linear system. *SIAM J. on Numer. Anal.* **12** (1975) 5.
[9] Robinson S. M. Regularity and stability for convex multivalued functions *Math. of Oper. Research*, **1** (1976) 2.
[10] Robinson S. M. A characterization on stability in linear programming. *Oper. Research*, **25** (1977) 3.
[11] Zowe J., Kurcyusz S. Regularity and stability for the mathematical programming problem in Banach spaces. *Applied Math. and Optim.* **5** (1979).

Received, September 1986.

Stabilność zadań programowania liniowego w przestrzeniach Banacha

W pracy bada się stabilność ogólnego zadania programowania liniowego w przestrzeniach Banacha. Sformułowano warunek, który jest koniecznym i „prawie" dostatecznym dla istnienia rozwiązań optymalnych i dla ograniczoności zbiorów optymalnych zarówno w zadaniu pierwotnym jak i dualnym. Okazuje się, że ten sam warunek jest zarazem konieczny i wystarczający dla stabilności zadania programowania liniowego w sensie posiadania rozwiązania optymalnego przy małych zaburzeniach. Pokazano także, że wartość funkcji celu w rozwiązaniu i zbiór optymalny zależą w sposób ciągły od parametrów zaburzeń.

Устойчивость задач линейного программирования в банаховых пространствах

В работе исследуется устойчивость обобщенной задачи линейного программирования в банаховых пространствах. Формулируется условие, которое является необходимым и „почти" достаточным для существования оптимальных решений и ограниченности оптимальных множеств, как в исходной так и дуальной задачах. Оказывается, что это же условие является одновременно необходимым и достаточным для устойчивости задачи линейного программирования, в смысле существования оптимального решения при малых возмущениях. Показано также, что значение функции цели в решении и оптимальное множество зависят непрерывным образом от параметров возмущений.